ON A THEOREM OF BENDIXSON

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In his mémoire on differential equations (Acta Matematica, 24, pp. 1–88, 1901) Bendixson subjected to a searching investigation analytical systems of the form

$$(*)\qquad \frac{dx}{dt} = X, \quad \frac{dy}{dt} = Y, \qquad X(0, 0) = Y(0, 0) = 0,$$

where X, Y are real and holomorphic at the origin and the origin is an isolated critical point. He showed in particular that by a finite sequence of quadratic transformations one may replace the study of the critical point at the origin by that of a finite set of ordinary points or isolated critical points where at least one characteristic root is non-zero: the so-called Bendixson systems. However in any practical instance (even quite simple) it is not at all easy to obtain the actual quadratic transformations involved.

In the present Note we propose to give a constructive process to reduce the critical point, in a finite number of steps, to ordinary or Bendixson types. Each step is clearly described in terms of the properties of the differential system.

In a paper to appear elsewhere Barocio has made a complete study of analytical critical points. In relation to this paper the present Note provides among other things an explicit method for discovering so-called fans and also *TO*-curves in the "ambiguous" cases of Barocio's paper. (*TO*-curve: a curve tending to or away from the origin in a fixed direction; fan: a continuous collection of *TO*-curves).

Terminology. We shall have to deal repeatedly with power series $f(x, y, \cdots, z)$ in several variables holomorphic at the origin. We shall use the following terminology:

$$f = \begin{cases} \text{a } \textit{unit} \text{ if } f(0, \cdots, 0) \neq 0, \\ \text{a } \textit{non-unit} \text{ if } f(0, \cdots, 0) = 0. \end{cases}$$

Units f such that $f(0, \cdots, 0) = 1$ shall be designated generically by E. A *special polynomial* in x is a polynomial of the form

$$x^r + A_1(y, \cdots, z)x^{r-1} + \cdots + A_r(y, \cdots, z)$$

where the coefficients A_h are non-units.

We recall that the Weierstrass preparation theorem asserts that if $f(x, 0, \cdots, 0) \not\equiv 0$, and r is the last degree of this series in x then

$$f \equiv \alpha\{x^r + A_1(y, \cdots, z)x^{r-1} + \cdots + A_r(y, \cdots, z)\}E(x, y, \cdots, z), \qquad \alpha \neq 0,$$

where the bracket is a special polynomial in x.

1. We first discuss the possible directions of approach of *TO*-curves to the origin. Let the system be

$$\begin{aligned} \frac{dx}{dt} &= X(x, y) = X_n + X_{n+1} + \cdots \\ \frac{dy}{dt} &= Y(x, y) = Y_n + Y_{n+1} + \cdots , \end{aligned} \tag{1.1}$$

where X_h, Y_h are forms of degree h and one of X_n, $Y_n \not\equiv 0$. The number n is the *order* of the system and is independent of the choice of coordinates. Equivalent forms to (1.1) are

$$\text{(a)} \quad \frac{dy}{dx} = \frac{Y}{X}; \qquad \text{(b)} \quad \frac{dx}{dy} = \frac{X}{Y}. \tag{1.2}$$

We will require the important expression

$$\Delta = xY_n - yX_n ,$$

and more generally also

$$\Delta_{h+1} = xY_h - yX_h ,$$

so that $\Delta = \Delta_{n+1}$. A change of coordinates merely multiplies the Δ's by the determinant of the transformation, so that they are fixed up to a constant factor. The system is said to be *regular* if $\Delta \not\equiv 0$ and to be *irregular* otherwise.

Upon applying the transformation

$$x = x, \qquad y = xy_1 \tag{1.3}$$

the *TO*-curves will become curves tending to the y_1 axis and the direction of approach $y = mx$ will correspond to the point $(0, m)$ of the y_1 axis, and conversely.

Applying now (1.3) say to (1.2a) we find

$$\begin{aligned} \frac{dy_1}{dx} &= \frac{Y_n(1, y_1) + xY_{n+1}(1, y_1) + \cdots}{xX_n(1, y_1) + x^2X_{n+1}(1, y_1) + \cdots} - \frac{y_1}{x} \\ &= \frac{\Delta(1, y_1) + x\Delta_{n+2}(1, y_1) + \cdots}{xX_n(1, y_1) + x^2X_{n+1}(1, y_1) + \cdots} . \end{aligned} \tag{1.4}$$

We will now discuss the definite direction of approach $y = mx$, more particularly in relation to the tangents of the curves $X = 0$, $Y = 0$ at the origin. It will be recalled that they are given by $X_n = 0$, $Y_n = 0$, and we will first suppose that $y - mx$ is not a common factor of X_n, Y_n, say that $X_n(1, m) \neq 0$.

We must distinguish now between the regular and the irregular cases.

A. Regular system. Thus $\Delta(1, y_1) \not\equiv 0$. Then (1.4) has for path $x = 0$ and so other paths can only tend to critical points on this line. These will occur at places given by $\Delta(1, y_1) = 0$. If $\Delta(1, m) \neq 0$, $y = mx$ is not a direction of approach. If $\Delta(1, m) = 0$ then $X_n(1, m) \neq 0$. It follows that upon applying the change of variables $y_1 - m = y^*$, (1.4) is replaced by a system which, in the notation $[u, v]_q$ = a power series starting with terms of degree $\geqq q$, reads:

$$\frac{dy^*}{dx} = \frac{[x, y^*]_p}{\alpha x + [x, y^*]_2}; \qquad \alpha \neq 0, p \geqq 1. \tag{1.5}$$

If $p = 1$ the point $y_1 = m$, $x = 0$ is either a node, a saddle point or of Bendixson type; if $p > 1$ it is of Bendixson type. In both cases the curves tending to it, hence the TO-curves tangent to $y = mx$ at the origin are known.

B. Irregular system. Here then $\Delta(1, y_1) \equiv 0$. From the assumption $X_n(1, m) \neq 0$ follows that $X_n(1, y_1) \not\equiv 0$. Hence (1.4) becomes

$$\frac{dy_1}{dx} = \frac{x^{h-2}\Delta_{n+h}(1, y_1) + x^{h-1}\Delta_{n+h+1}(1, y_1) + \cdots}{X_n(1, y_1) + xX_{n+1}(1, y_1) + \cdots}.$$

Thus since $X_n(1, m) \neq 0$, the point $x = 0$, $y_1 = m$ is an ordinary point of the y_1 axis, and $y = mx$ is a direction of approach for TO-curves.

Notice that if $l = y - mx$ is a factor of $X_n(x, y)$ but not of $Y_n(x, y)$ the same reasoning would hold with x and y interchanged. Similarly for the factor $l = x$. Thus if $y - l$ is not a factor of both X_n and Y_n there is a TO-curve with the tangent $l = 0$ in the irregular case.

To sum up then if $l = 0$ is not a common tangent to $X = 0$, $Y = 0$ (not a common factor of X_n, Y_n), we know how to determine the TO-curves if any which are tangent to $l = 0$ at the origin. To determine completely the behavior of all TO-curves we are thus led to the study of those which tend to the origin along common tangents to $X = 0$, $Y = 0$ at the origin. This will constitute the core of our investigation.

2. Returning to the system (1.1) we may choose coordinates such that no exceptional direction is one of the axes and furthermore such that X_n and Y_n both contain a term in y^n. Let these terms be ay^n and by^n, $ab \neq 0$. At the cost of changing x, t into $\frac{a}{b}x$, $\frac{a}{b}t$ we may dispose of the situation so that $b = a \neq 0$. Under these circumstances the series

$$Z(x, y) = \lambda X + (1 - \lambda)Y \tag{2.1}$$

will begin with terms of degree n among which there will be the term ay^n. The range of convergency of this series will be a certain complex region Ω of the space of the three complex variables x, y, λ defined by

$$\Omega\colon |x|, |y| < R, \lambda \text{ arbitrary}.$$

Take any particular λ_0 and set $\lambda - \lambda_0 = \mu$. Then $Z(x, y, \lambda_0 + \mu) = Z^*(x, y, \mu)$ is a power series convergent in an obvious region. Applying the Weierstrass factorization theorem we have

$$Z \equiv a(y^n + A_1(x, \mu)y^{n-1} + \cdots + A_n(x, \mu))\, E(x, y, \mu), \qquad E(0, 0, 0) = 1. \tag{2.2}$$

From this follows

$$(2.3)\qquad (y^n + A_1(x, \mu)y^{n-1} + \cdots + A_n(x, \mu)) \equiv Z\frac{E^{-1}}{a} = \frac{ZE_1}{a}.$$

Let

$$Z = Z_0(x, \mu) + yZ_1(x, \mu) + \cdots + ay^n + \cdots,$$
$$E_1 = 1 + yB_1(x,, \mu) + \cdots.$$

Upon identifying the coefficients of y on both sides of (2.3) we find

$$aA_{n-h} = Z_0B_h + Z_1B_{h-1} + \cdots + Z_hB_0, \qquad h < n, \quad B_0 = 1.$$

Since $Z_h(0, \mu) = 0$ identically for $h < n$, it follows that the A_h have the same property. Thus, they are non-units in x, with coefficients power series in μ. We may also think of them as non-units in x with coefficients in the field of the power series in μ with coefficients in the complex field K. We require something else however. Note that the ascending convergent power series in $x^{1/p}$ with complex coefficients, and for all p, form a ring which we write $K[x]$. Since this ring is an integral domain it has a quotient field $K\{x\}$, which is merely the field of all convergent power series with complex coefficients of the form $\alpha x^{s/p}E(x^{1/p})$, s positive negative or zero, and for all p. We shall require both $K[\mu]$ and $K\{\mu\}$. It is known that $K\{\mu\}$ is algebraically closed (Lefschetz: Algebraic Geometry, p. 99).

The equation

$$(2.4)\qquad y^n + A_1(x, \mu)y^{n-1} + \cdots + A_n(x, \mu) = 0,$$

where the A_h are considered as power series in x with coefficients in $K\{\mu\}$ may be solved by the Puiseux process. It is necessary to examine this process a little more closely to bring out a certain property of the solutions.

One begins by selecting by means of the Newton polygon an admissible approximate solution $a_1x^{n_1}$, where $n_1 = p_1/q_1$ is a positive irreducible rational fraction and a_1 satisfies an algebraic equation

$$D_0(\mu)a_1^{r_1} + \cdots + D_{r_1}(\mu) = 0$$

where the D_j are polynomials in a finite number of the coefficients of the series A_h in x. Hence $a_1 \in K\{\mu\}$. One sets then $y = (y_1 + a_1)x_1^{p_1}$, $x = x_1^{q_1}$ and obtains a similar system in x_1, y_1 whose coefficients are now polynomials in the A_h and a_1, etc. This process in known to terminate with an equation in x_k, y_k with coefficients polynomials in the A_j and $a_1, \cdots, a_k$ and will have a term of the first degree in y_k alone. In particular with a certain x^* such that $x^{*q} = x$, we will have

$$y = a_1x^{*r_1} + \cdots + a_{k-1}x^{*r_{k-1}} + (a_k + y_k)x^{*r_k},$$
$$r_1 < r_2 < \cdots < r_k.$$

Thus the y_k equation will have the form

$$(2.5) \qquad F \cdot y_k + G(x^*, y_k) = 0$$

where $F \epsilon K\{\mu\}$ and G is a power series in its two variables with coefficients polynomials in the $a_1, \cdots, a_k$ and the coefficients of the series A_j in x. Moreover G will contain no term of degree one in y_k alone.

It follows that we can solve (2.5) for y_k as a power series in x^* with coefficients power series in some $\mu^{1/\sigma}$ divided by powers of $\varphi(\mu) = F$. If $\varphi(0) = 0$ we can always choose an interval of μ, hence of $\lambda: 0 < \gamma \leqq \lambda \leqq \delta$, in which $F \neq 0$ and the formal solution will be an actual solution as follows from the Puiseux theory. As a consequence we shall obtain a power series solution

$$(2.6) \qquad y(x, \lambda) = b_1(\lambda)x^{*r_1/q} + b_2(\lambda)x^{*r_2/q} + \cdots$$

valid in the same λ interval. Moreover we can choose this interval so that the n roots of (2.4) have a representation (2.6) and this with the same q.

For $|x| < R$ and $\lambda \epsilon [\gamma, \delta]$, the n Puiseux series solutions of $Z = 0$ in y will converge absolutely and uniformly and so they will be in a certain region $|y| < S$.

3. We will now prove a generalization of a special case of the well known Bertini lemma of algebraic geometry.

(3.1) LEMMA. *For $|x| < R$ and all but a finite number of values of $\lambda \epsilon [\gamma, \delta]$ the n Puiseux series solutions $y_j(x, \lambda)$, $j = 1, \cdots, n$, $|y_j| < S$, of $Z = 0$, will all be distinct.*

Suppose that there is a multiple root, which we write $y(x, \lambda)$, depending actually on λ. Then

$$\frac{\partial Z(x, y(x, \lambda), \lambda)}{\partial \lambda} = \left(\lambda \frac{\partial x}{\partial y} + (1 - \lambda) \frac{\partial Y}{\partial y}\right) \frac{\partial y}{\partial \lambda} + X(x, y(x, \lambda)) - Y(x, y(x, \lambda)) = 0.$$

Since $y(x, \lambda)$ is a multiple root of $\lambda X + (1 - \lambda) Y$ the coefficient of $\partial y/\partial \lambda$ vanishes and so

$$X(x, y(x, \lambda)) - Y(x, y(x, \lambda)) = 0.$$

However since $y(x, \lambda)$ annuls Z we have for $y = y(x, \lambda)$:

$$\lambda(X - Y) + Y = 0$$

and hence $Y(x, y(x, \lambda)) = 0 = X(x, y(x, \lambda))$. Thus $y(x, \lambda)$ is a root of Z independent of λ and common to X and Y. But by hypothesis no such root exists. Hence for arbitrary λ the n roots are distinct.

Now from equating any two roots there will result a certain number of analytic relations in λ. Since the number of their solutions on $[\gamma, \delta]$ must be finite the lemma follows.

As a consequence of the lemma there is an interval $[\gamma_1, \delta_1] \subset [\gamma, \delta]$ such that for $\lambda \in [\gamma_1, \delta_1]$ the n solutions of the lemma are all distinct.

4. Corresponding to any $\lambda \in [\gamma_1, \delta_1]$ among the n roots considered a certain number $\nu(\lambda) \leqq n$ will be complex. Let λ_0 be a λ for which ν has its maximum value n_1. Since the roots are continuous in λ for x fixed, there will be n_1 complex roots for λ near λ_0. Hence there is an $[\alpha, \beta] \subset [\gamma_1, \delta_1]$, such that for every $\lambda \in [\alpha, \beta]$ there will be the same fixed number n_1 of complex roots and hence the same fixed number n_0 of real roots, and all these roots will be distinct.

If we choose two distinct values $\lambda, \mu \in [\alpha, \beta]$ we shall be assured that the series $X_1 = \lambda X + (1 - \lambda)Y$, $Y_1 = \mu X + (1 - \mu)Y$ will each have n distinct roots in $|y| < S$ for $|x| < R$, and that among these roots n_1 will be complex and n_0 real. Thus $X_1 = 0$, $Y_1 = 0$ will each have n_0 real and n_1 complex branches initiating at the origin. Moreover the two sets will have no common branches. We can change coordinates to x_1, y_1 so that (1.1) becomes

$$\frac{dx_1}{dt} = X_1, \qquad \frac{dy_1}{dt} = Y_1.$$

Writing now $x, \cdots, Y$ for $x_1, \cdots, Y_1$ we may say that we are dealing with a system (1.1) such that among the two sets of branches $X = 0$, $Y = 0$ there are the same numbers n_0, n_1 of real and complex branches.

5. Suppose now that the branches of $X = 0$ and $Y = 0$ have a common tangent L. Under our assumptions L is not an axis. Hence it has an equation $y = mx$ with $m \neq 0$.

Suppose that k of the (real or complex) branches of $Z = \lambda X + (1 - \lambda)Y$, $\lambda \in [\alpha, \beta]$ are tangent to L. They will then have representations

$$y = mx + a_{q+1}x^{1+1/q} + \cdots. \tag{5.1}$$

Among the coefficients let a_{p-1} be the last independent of λ. Thus the particular representation (5.1) has the "fixed" part

$$f(x) = mx + a_{q+1}x^{1+1/q} + \cdots + a_{p-1}x^{(p-1)/q}.$$

We may decompose $f(x)$ into a sum

$$f = f_1 + \cdots + f_s,$$

and we will set

$$g_h = f_1 + \cdots + f_h$$

so that $g_s = f$. The properties assumed, with $k > h$, are: (a) g_h is the fixed part of some branch of $Z = 0$; (b) f_h and f_k have no common terms; (c) the lowest degree term in f_k is of higher degree than the highest in f_h. Notice that (a) implies that there are fewer branches with the fixed part g_k than there are with the fixed part g_h. We shall also say that g_k *contains* g_h: $g_h \subset g_k$.

One may represent schematically the situation by a graph (a tree) each arm of which such as CD corresponds to an f_h—here f_3. The terminal dotted lines correspond to the various branches which separate at E, that is to say in whose representation the first term beyond f depends on λ. Suppose that this first term is $a(\lambda)x^{(p-1)/q}$. Then there may be further branches in which the fixed part contains a term $bx^{(p-1)/q}$, where b is independent of λ, and this fixed part is represented by EK, $EL \cdots$ in the graph.

6. Let us set $x^{1/q} = u$, and $f(x^{1/q}) = \varphi(u)$. The representations of EF, EG, $\cdots$, will be of the form $a(\lambda)u^r + \cdots$, $r = p/q$, and those of EK, $\cdots$, of the form $b(\lambda)u^r + \cdots$. If we set

$$y - \varphi(u) = y_1 u^r$$

we will have

$$Z(x, y) = Z(u^q, y) = u^l W(u, y_1)$$

where $W(0, y_1) \neq 0$ and $W(0, y_1) = 0$ has for roots the various $a(\lambda)$, b say $a_1(\lambda)$, $\cdots$, $a_\nu(\lambda)$, b_1, $\cdots$, b_ρ. Applying now the direct lemma of Bertini we learn that for all but a finite number of values of $\lambda \in [\alpha, \beta]$ the ν values $a_i(\lambda)$ are distinct and they will represent ν distinct branches of $Z = 0$ related to f. The total number of exceptional λ values for all possible choices of f is finite. Hence one may replace $[\alpha, \beta]$ by a subinterval, still called $[\alpha, \beta]$, deprived of exceptional values of λ. Thus for any $\lambda \in [\alpha, \beta]$ there will be the same fixed number of distinct values such as $a_i(\lambda)$ and this for all possible combinations such as f.

Take in particular $\lambda_1 = \alpha$, $\lambda_2 = \beta$ and let X^*, Y^* denote the corresponding functions Z. The branches represented by $X^* = 0$, $Y^* = 0$, will all be distinct and the same number ν will be attached to $f(x)$.

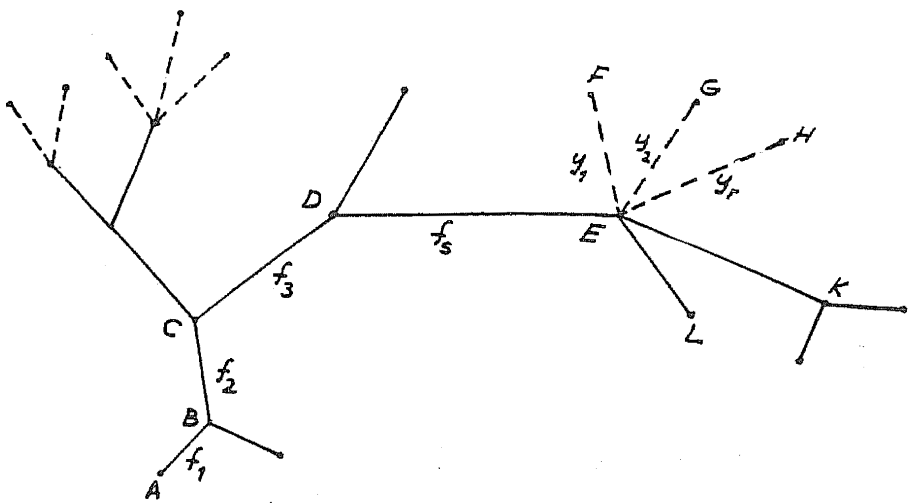

FIG. 1

Let us now make the change of variables

$$x^* = \alpha x + (1 - \alpha)y$$
$$y^* = \beta x + (1 - \beta)y.$$

The determinant is $\alpha - \beta \neq 0$, so that this is a non-singular change of coordinates. It is an elementary matter to show that the new variable $u^* = x^{*1/q}$ is regularly related to $u: u \to u^*$ is a regular transformation. As a consequence if we express the φ's in terms of u^*, the earlier situation will be preserved. The branches corresponding to φ_h will simply become the branches corresponding to φ_h^*.

Returning now to the designations $x, \cdots, y$ instead of $x^*, \cdots, y^*$ we have the following result:

(6.1) Lemma. *The coordinates may be so chosen that the basic equation in the form* (1.1) *is such that* $X = 0$, $Y = 0$ *have the same number* ν *of branches attached to* φ, *and this for every* φ.

In the sequel we shall assume that the situation described in this lemma already prevails. Furthermore we only discuss what happens for $x > 0$. One takes care of $x < 0$ by applying the change of variable $x \to -x$ to (1.1).

7. We are now ready for the analysis of the local phase-portrait of the fundamental system (1.1) at the origin. We first put it in the form

$$\frac{dy}{dx} = \frac{\delta E(x, y) Y(x, y)}{X(x, y)}, \tag{7.1}$$

where under our general choice of coordinates both X and Y are special polynomials of degree n in y and δ is a constant.

Let us fix our attention upon a certain $\varphi(x)$ corresponding say to a point between C and D on the graph. Let

$$\varphi(x) = mx + x^{(q-1)/q} + \cdots + \sigma x^{(p-1)/q},$$

where we assume that the first $\psi \supset \varphi$, $\psi \neq \varphi$, is of the form

$$\psi = \varphi + a x^{p/q}$$

We seek the *TO*-curves between φ and ψ and so we apply the transformation

$$y = \varphi(x) + x^{\mu} y_1, \tag{7.2}$$

where $1 \leqq (p - 1)/q < \mu < p/q$. Now the factors of X are of the form $y - \psi \cdots$, where either $\psi \subset \varphi$ or else $\psi \supset \varphi$ and $\psi \neq \varphi$. If $\psi \subset \varphi$ (7.2) replaces the factor by $x^r (x^{\mu - r} y_1 - \alpha \cdots)$, $r < \mu$, $\alpha \neq 0$. If $\psi \supset \varphi$ the factor is replaced by $x^{\mu}(y_1 - \alpha x^{r-\mu} \cdots)$, $r > \mu$. The same thing holds for Y and the same number of factors of each type with the same exponents r is found in each. Hence under

(7.2) and since $\mu > 1$, (7.1) is replaced by

$$\frac{dy_1}{dx} = \frac{\delta E(x, y_1) Y_1(x, y_1)}{x^\mu X_1(x, y_1)}, \tag{7.3}$$

which it is more convenient to write as

$$\frac{dx}{dy_1} = \frac{x^\mu X(x, y_1)}{\delta E(x, y_1) Y_1(x, y_1)}. \tag{7.4}$$

Observe here that X_1, Y_1, E are power series in x, x^μ, y_1 and that $X_1(y_1, 0) = Y_1(y_1, 0) = y_1^k$. As a consequence (7.4) satisfies a Lipschitz condition in x in the whole set $\Omega: x \geqq 0$, origin excluded. It follows that the whole existence theorem of Cauchy-Lipschitz, uniqueness included, is applicable throughout Ω. In particular through a point P of the y_1 axis other than the origin, there passes a unique path in Ω. Since the half of the y_1 axis through P is already such a path it is the only one reaching P. Hence the image of a TO-curve of (7.1) can only terminate at the origin on the y_1 axis or else tend to infinity along that axis. Thus in the process either these TO-curves have already appeared earlier, i.e. for some $\psi \subset \varphi$, or else they will appear later. In our systematic "sweeping out" process for TO-curves we are only concerned with the curves that appear later and for these we must take a φ corresponding to a vertex of the graph and $\mu = p/q$ rational.

8. This time then we apply the more complete transformation

$$x = u^q, \qquad y = \varphi(u) + u^p y_1, \qquad \varphi(u) = mu^q + \cdots + \sigma u^{p-1}, \tag{8.1}$$

and we note also that $p > q \geqq 1$. The same factors as before will appear, of one of the two forms

$$u^r(u^{p-r} y_1 - \alpha \cdots), \qquad r < p, \quad \alpha \neq 0,$$

$$u^r(y_1 - \alpha u^{r-p} \cdots), \qquad r > p.$$

In addition however there may appear a third type with $r = p$:

$$u^p(y_1 - \alpha \cdots), \qquad \alpha \neq 0.$$

Here again the number in each type will be the same for X and Y. Hence this time we have an equation

$$\frac{dy_1}{du} = \frac{E(u, y_1) F^* G^* H^*}{u^s FGH}, \qquad s = p - q + 1 \geqq 2, \tag{8.2}$$

where the pairs (F, F^*), (G, G^*), (H, H^*) correspond to the three types of factors of X and Y and

$F(0, y_1)$, $F^*(0, y_1)$ are constants;

$G(0, y_1)$, $G^*(0, y_1)$ are polynomials of the same degree whose common and only possible multiple roots are the values b_1, b_2, $\cdots$, corresponding to fixed parts $\supset \varphi$;

$$H(0, y_1) = H^*(0, y_1) = y_1^k .$$

Notice that the degrees of GH and G^*H^* in y_1 do not exceed the degree n of X, Y in y.

The images of TO-curves of interest now are those if any, which reach the y_1 axis at points other than the b_i. They are the real roots $a_1, \cdots, a_\nu$ of $G^*(0, y_1) = 0$ other than the b_i, and their number $\nu \leqq n$, none being a root of $G(0, y_1) = 0$. That is to say they are the real roots of $G^*(0, y_1)/G(0, y_1) = 0$. Let c be such a root and set $y_1 - c = y_1^*$. The (y_1^*, u) equation will thus have the form

$$\frac{dy_1^*}{du} = E(u, y_1^*) \cdot \delta \cdot \frac{K(u, y_1^*)}{u^s}, \tag{8.3}$$

$$K(u, y_1^*) = y_1^{*\pi} + A_1(u) y_1^{*\pi-1} + \cdots + A_\pi(u),$$

$$\pi \leqq \nu, A_j(0) = 0, \qquad \delta = \text{constant}$$

The relation $K = 0$ will represent a certain number of branches through the (y^*, u) origin. Upon writing then the representations of these branches in fractional powers of u we can construct a graph such as that of fig. 1, corresponding to the common parts of these representations. Let us select a definite branch Γ. In the new graph it will correspond to an arc going from the starting point to the end. Let Γ have a representation in terms of a suitable v, $v^k = u$,

$$y_1^* = \varphi(v) + a_\rho v^\rho + a_{\rho+1} v^{\rho+1} + \cdots,$$

where φ is the part common to other branches as well. Upon introducing a new variable by $y_1^* - \varphi(v) = v^\rho y_2$, we obtain a relation

$$\frac{dy_2}{dv} = \frac{\varepsilon L(v, y_2) M(v, y_2) N(v, y_2) E(v, y_2)}{v^\sigma},$$

quite similar to (8.2). The properties of L, M, N are essentially the same as those of F, G, H before. In particular their degrees in y_2 are $\leqq \nu$. The process is repeated until we reach an equation in terms of new variables y_k, w_k analogous to y_2, v of the form

$$\frac{dy_k}{dw_k} = E(y_k, w_k) \cdot \varepsilon \cdot \frac{(y_k - \varphi(w_k))^r}{w_k^s}. \tag{8.4}$$

Here ε is a constant and φ is a power series. If $r = 1$ this equation is of Bendixson type and its phase-portrait is completely known. If $r > 1$ and $\varphi \neq 0$ we apply the further transformation

$$w_k = w_k, \qquad y_k = \varphi(w_k) + z_k,$$

which replaces (8.4) by

$$\frac{dz_k}{dw_k} = \varepsilon E(z_k, w_k)\left\{\frac{z_k^r - \psi(w_k)}{w_k^s}\right\}, \tag{8.5}$$

$$\frac{\varepsilon E(0, w_k)\psi(w_k)}{w_k^s} = \varphi'(w_k) \neq 0.$$

Let $\psi(w_k) = \alpha w_k^\rho + \cdots$. If $\alpha^{1/r}$ is complex then there is no *TO*-curve of our system of type $z = \gamma w_k^{p/r}$. Let $\beta = \alpha^{1/r}$ be real. The transformation

$$w_k = w^{*r}, \qquad z_k = y^* w^{*\rho}$$

reduces (8.5) to the form

$$\frac{dy^*}{dw^*} = \eta w^{*(\rho+1)(r-1)-\rho s}(y^{*r} - \alpha - \rho w^* - \cdots) - \frac{\rho y^*}{w^*} \tag{8.6}$$

$$\eta = r\varepsilon.$$

We must now distinguish several cases. One must bear in mind that only the *TO*-curves for which $w^* > 0$ do matter.

A. $(\rho + 1)(r - 1) - \rho s \geqq 0$. The system (8.6) is now equivalent to

$$\frac{dy^*}{dt} = -\rho y^* + \lambda w^* + \cdots, \quad \frac{dw^*}{dt} = w^*. \tag{8.7}$$

The only critical point on the w^* axis is the origin. It is an ordinary critical point with characteristic roots $-\rho$, 1 and so it is a saddle point. It has one *TO*-curve other than $w^* = 0$ and it corresponds to a single *TO*-curve in the initial x, y system.

B. $\rho s - (\rho + 1)(r - 1) = 1$. The equation for the critical points on the y^* axis is

$$\eta(y^{*r} - \alpha) - \rho y^* = 0. \tag{8.8}$$

If this equation has no real roots there are no corresponding TO-curves in the initial system. Let γ be a real root and let it be simple. Setting $\bar{y} = y^* - \gamma$, our equation becomes

$$\frac{d\bar{y}}{dw^*} = \frac{\zeta\bar{y} + \lambda w^* + \cdots}{w^*},$$

where ζ is a known constant $\neq 0$. Hence $(\gamma, 0)$ is an ordinary critical point: a node if $\zeta > 0$, a saddle point if $\zeta < 0$. In the second case there is a single *TO*-curve, in the first a fan in the initial system. If γ is a multiple root one has a Bendixson critical point.

C. $\rho s - (\rho + 1)(r - 1) = k > 1$. This time (8.6) reads

$$\frac{dy^*}{dw} = \frac{\eta E(y^{*r} - \alpha - \beta w - \cdots) - \rho y^* w^{k-1}}{w^{*k}}.$$

The critical point equation is

$$y^{*r} - \alpha = 0.$$

It has one real root $\alpha^{1/r}$ if r is odd, two real roots $\pm\alpha^{1/r}$ if r is even. The same change of variables as before yields a Bendixson critical point for each root, and at least one corresponding TO-curve for each root.

9. There remains (8.4) with $\varphi = 0$, or the general type

$$\frac{dy}{dx} = \frac{\varepsilon y^r}{x^s} E(x, y), \qquad \varepsilon \text{ a constant} \neq 0. \tag{9.1}$$

If r or s, or both $= 1$, we have a Bendixson or an ordinary critical point, so that we may assume both >1.

Suppose first $r = s$. Thus (9.1) is

$$\frac{dy}{dx} = \frac{\varepsilon y^r}{x^r} E(x, y). \tag{9.2}$$

To determine the possible directions of approach of TO-curves we make the change of variables $y = xy_1$ which yields

$$\frac{dy_1}{dx} = \frac{\varepsilon y_1^r E(x, xy_1) - y_1}{x}. \tag{9.3}$$

The x and y_1 axes are solutions and the other directions of approach correspond to the critical points on the y_1 axis other than the origin of (9.3). They are given by the roots of

$$y_1^{r-1} = \frac{1}{\varepsilon}. \tag{9.4}$$

This relation may have: (a) no real roots, hence there are no TO-curves other than along the axes, or else: (b) one or two real roots. Let ξ be one of these. The change of variables $y_1 = y^* + \xi$ reduces (9.3) to the form

$$\frac{dy^*}{dx} = \frac{\eta x + (r - 1)y^* + \cdots}{x}.$$

Thus the point $y_1 = \xi$ is a node and we may have another node if (9.4) has two roots, at $y_1 = \pm\,\xi$. This completes the treatment of the present case. Each node yields a fan for (9.1).

10. We return now to (9.1) and assume $r \neq s$ and both >1. The change of variables $y = xy_1$ yields this time

$$\frac{dy_1}{dx} = \frac{\varepsilon_1 y_1^r E(x, xy_1) - y_1 x^{s-r}}{x^{s-r+1}}. \tag{10.1}$$

If $s > r$ the only critical point is at the origin and so the only direction of approach for TO-curves is along Ox. If $s < r$ we study dx/dy and the situation is then the same with x and y interchanged.

Suppose then $s > r$. Thus along the TO-curves if any are present y is of order >1 in x for x small. Let us make the transformation of variables

$$x = u^q, \qquad y = u^p y_1, \qquad p > q \tag{10.2}$$

or $y = x^\mu y_1$, $\mu > 1$. Thus we endeavor to "capture" the possible TO-curves of (9.1) as it were. We find

$$\begin{aligned} \frac{dy_1}{du} &= \frac{\varepsilon q y_1^r E(u, uy_1)}{u \cdot u^{(s-1)q-(r-1)p}} - \frac{py_1}{u} \\ &= \frac{\varepsilon q y_1^r E(u, uy_1) - py_1 u^{(s-1)q-(r-1)p}}{u \cdot u^{(s-1)q-(r-1)p}}. \end{aligned} \tag{10.3}$$

This shows that if

$$\mu = \frac{p}{q} < \frac{s-1}{r-1} = \sigma,$$

the (u, y_1) origin is the only critical point and so along possible TO-curves y is of order $>\mu$ in x. On the other hand if $\mu > \sigma$ the origin is merely a saddle point, and has the x axis as only TO-curve (outside of the y_1 axis itself). This means that the same order $<\sigma$. Hence σ is the only possible order for $y(x)$ along a TO-curve. Taking then $\mu = \sigma$ we have in place of (10.3):

$$\frac{dy_1}{du} = \frac{\varepsilon q y_1^r E(u, uy_1) - py_1}{u}. \tag{10.4}$$

The critical points other than the origin correspond now to the real roots of

$$y_1^{r-1} = \frac{\mu}{\varepsilon}$$

and the situation is the same as discussed before: either no critical point and hence no TO-curves or one or two nodes, and hence one or two fans for (9.1), and therefore also for the initial system (1.1). This also completes the treatment of (9.1) and hence of (1.1).

11. Concerning the application of the method of the Newton polygon. This process was applied to differential equations for the first time by Briot and Bouquet (see notably: Ince, Ordinary differential equations, p. 297). It is used to determine the order in x of $y(x)$ on a TO-curve. The basic assumption is that such an order exists. That is to say that for such a curve there is a $\mu > 0$ such that $|y| / |x^\mu|$ tends to a finite limit d as $x \to 0$. We extend the definition to include μ defined as follows: $|y| / |x^{\mu+\varepsilon}| \to 0$, $|y| / |x^{\mu-\varepsilon}| \to \infty$ for arbitrary positive ε.

12. If a TO-curve tends to the origin along $y = mx$ with $m \neq 0$, then clearly $\mu = 1$. Let the curve tend to the origin along the x axis. Then at all events $|y| / |x| \to 0$ and so if μ exists it is >1. Notice that approach along Oy would be dealt with by interchanging x and y: if $|x| / |y^\nu| \to d \neq 0$ then $|y| / |x^{1/\nu}| \to d^{-\nu}$ and so the result is the same. Thus we may confine our attention to tangency to Ox, and clearly we only need to consider TO-curves to the right of Oy. This means that we may replace $|x|$ by x throughout.

Generally speaking if our TO-curve arises as one of the dotted lines in the graph of fig. 1, at the end of a definite path in that graph then it has a representation

$$y = \varphi(u) + y_1 u^s, \qquad u^q = x$$

$$\varphi(u) = \alpha u^p + \cdots + \beta u^{s-1}, \qquad \alpha \neq 0.$$

Hence on the TO-curve we find that $|y| / x^{p/q}$ tends to a limit as $x \to 0$. In a certain plane u, y_1 this limit will be a "critical value" of y_1.

13. There remains then the case when $\varphi = 0$ and approach along the direction of the x axis or the y axis. The latter may be taken care of by interchanging x and y, so that it is only necessary to consider tangential approach to the x axis, say on the positive side. Approach on the negative side is disposed of by the change of variables $x \to -x$.

Since we have dealt with the "φ case", we may suppose that X_n, Y_n (notations of 1) are not both tangent to the x axis. Now to have it be among the directions of approach $xY_n - yX_n$ must be divisible by y. Hence y must not be a factor of X_n and so $X_n \not\equiv 0$. We will then have

$$X_n = x^p X^*_{n-p}(x, y), \qquad Y_n = y^q x^r Y^*_{n-q-r}(x, y),$$

where X^*, Y^* do not have x, y as factors and certainly $X^* \not\equiv 0$. Applying now the transformation $y = xy_1$, we find

$$\frac{dy_1}{dx} = \frac{\{y_1^s + xW(x, y_1)\}E(x, y_1)}{x}, \qquad s \geqq 1, \tag{13.1}$$

or else

$$\frac{dy_1}{dx} = U(x, y_1) \tag{13.2}$$

where U is a non-unit.

The only point of interest is the new origin 0_1. In the case of (13.2) it is an ordinary point and so only one TO-curve has an image through it. Since that curve is an ordinary analytical branch we have on it: $y_1 = ax^\nu + \cdots$, hence y_1 is of order ν and therefore y of order $\nu + 1$ on the TO-curve.

Consider now (13.1). There are several possibilities. Let us write it as

$$\frac{dy_1}{dx} = \frac{N}{x}. \tag{13.3}$$

(a) *N has terms of the first degree of form $\alpha x + \beta y_1$ with $\beta \neq 0$.* Then 0_1 is a simple critical point. On its TO-curves y_1 is of a definite order ν in x, hence on the original TO-curves y is of order $\nu + 1$.

(b) *N has terms of the first degree of form αx with $\alpha \neq 0$.* Then 0_1 is a Bendixson point. Its analysis brings out readily that it has two hyperbolic sectors to the right of Oy_1 with separatrices S_1 and S_2 along the y_1 axis and S_3 tangent to $y_1 = \alpha x$. The first two do not yield TO-curves in the (x, y) plane. The separatrix S_3 does yield such a curve say S_3'. On S_3 y_1 is of order one, hence on S_3' y is of order two.

(c) *N has no terms of the first degree.* The geometric analysis is really the same as in (b) save that this time S_3 is tangent to the x axis. To have the corresponding order we apply the transformation $y_1 = xy_2$ which yields

$$\frac{dy_2}{dx} = \frac{-y_2 + \cdots}{x} \tag{13.4}$$

so that the new origin O_2 is a saddle point. It has a single separatrix S_3^* not tangent to the y_2 axis and corresponding to S_3. On S_3^* y_2 is analytical in x and therefore of a definite order ν in x. To S_3^* there corresponds a TO-curve in the (x, y) plane on which y is of order $\nu + 2$ in x.

To sum up we have proved our order assertion on TO-curves and this justifies the Briot-Bouquet process.

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