# THE DOUBLE SUSPENSION AND $p$-PRIMARY COMPONENTS OF THE HOMOTOPY GROUPS OF SPHERES

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## Introduction

If $S^n$ denotes the $n$-dimensional sphere, and $p$ is a prime, then the stable $p$-primary components of the homotopy groups of spheres have been computed by H. Cartan in dimensions less than $n + 2p(p - 1)$. The aim of this paper is to compute the non-stable groups over essentially the same range of dimensions. Although the results of Cartan are not published, they have been current for several years now, and that portion of them which it will be convenient to use here will be stated without proof. These results may be obtained by the method of killing homotopy groups ([6], [19]), providing that the homology and cohomology of Eilenberg-MacLane spaces, and some of the relations on Steenrod's reduced $p$th powers are known ([1], [3], [4], [9]).

The method used in this paper will be simply to study the double suspension $E^2:\pi_q(S^{n-1}) \to \pi_{q+2}(S^{n+1})$ in some detail for the case of even $n$. This method has already been exploited to a considerable extent ([13], [16]), and the results obtained here are all derived from the same methods as those used earlier. If $\Omega(S^{n+1})$ denotes the space of loops in $S^{n+1}$, and $\Omega^2(S^{n+1})$ the space of loops in $\Omega(S^{n+1})$, it is well known that $S^{n-1}$ is naturally imbedded in $\Omega^2(S^{n+1})$, and that studying the double suspension is just exactly studying the inclusion homomorphism $i:\pi_q(S^{n-1}) \to \pi_q(\Omega^2(S^{n+1}))$. The new results obtained here are based on extending the known results on the homotopy of the pair $(\Omega^2(S^{n+1}), S^{n-1})$. This is made possible by a computation of the Pontrjagin ring $H*(\Omega^2(S^{n+1}); Z_p)$ where $Z_p$ denotes the integers modulo $p$, and $p$ is an odd prime.

## §1. Spectral sequences

In order to compute $H*(\Omega^2(S^{n+1}); Z_p)$ we need only formal properties of spectral sequences. Aside from the standard properties of spectral sequences ([5], [15]) it is convenient to know what the spectral sequence of the product of two fibre spaces looks like. In particular if $f:E \to B$, and $f':E' \to B'$ are fibre maps ([15]) then with field coefficients the spectral sequence of the fibre maps $f \times f':E \times E' \to B \times B'$ is just the tensor product (over the field) of the spectral sequence of $f:E \to B$ with the spectral sequence of $f':E' \to B'$. A proof of this statement for simplicial singular theory will be summarized, after recalling some of the details in the definition of the simplicial singular homology spectral sequence ([12]).

Suppose that for any space $X$, $C(X)_N$ denotes the normalized singular chain complex of $X$. If $f:E \to B$ is a map, then a singular simplex of $E$ has filtration $m$ either if its dimension is less than or equal to $m$, or if its projection into $B$ may be obtained by applying degeneracy operations ([9]) to a singular simplex of $B$ of dimension less than or equal to $m$. Letting $F_m C(E)_N$ denote the subcomplex of $C(E)_N$ generated by simplexes of filtration $m$, $\{F_m C(E)_N\}$ is a filtration of $C(E)_N$, and there is a resulting spectral sequence $\{E^r(f)\}$ $r \geqq 2$.[1] If $f$ is a fibre map, then $E^2(f)$ is naturally isomorphic with the homology of $B$ with local coefficients in the homology of the fibre $F$ of $f: E \to B$ ([12]).

Let $f: E \to B$ and $f': E' \to B'$ be fibre maps. One defines a filtration on $C(E)_N \otimes C(E')_N$ by setting $F_m(C(E)_N \otimes C(E')_N) = \sum_{i+j=m} F_i C(E)_N \otimes F_j C(E')_N$. Now the natural map $\nabla: C(E)_N \otimes C(E')_N \to C(E \times E')_N$ ([9], [11]) is filtration preserving and induces an isomorphism of spectral sequences. It is now easy to verify that if one uses field coefficients, then $E^r(f) \otimes E^r(f') \simeq E^r(f \times f')$, and that $d^r(x \otimes y) = d^r x \otimes y + (-1)^q x \otimes d^r y$ for $x \in E^r_{m,n}(f)$, $m + n = q$.

The preceding steps or analogous ones may all be carried through in cubical singular theory. The only problem is to define a filtration so that the map from the tensor product of the chain complexes of the separate fibre spaces into the chain complex of the product fibre space is filtration preserving. This requires some technical changes in the original filtration used in cubical theory ([15]). However, these changes do not affect the spectral sequence at all ([12]).

Suppose that $f: E \to B$ is a fibre map, $E$ and $B$ have an associative multiplication with identity, and $f$ is a homomorphism with respect to this multiplication. We may choose as the fibre $F$ of $f: E \to B$ the counter image of the identity element of $B$. Now $F$ itself also has an associative multiplication with identity. Consider now homology with coefficients in a ring with unit. Then the multiplication in $E$, $B$, or $F$ induces a multiplication in the homology of the respective spaces, and we have the so called Pontrjagin rings $H*(E)$, $H*(B)$, and $H*(F)$ (*e.g.* [3] No. 2). These rings are graded rings with unit, and are anti-commutative if the multiplication in the spaces is commutative up to homotopy. Further we have natural homomorphisms $E^r(f) \otimes E^r(f) \to E^r(f \times f)$, and $E^r(f \times f) \to E^r(f)$ making $E^r(f)$ into a graded (or bigraded) ring with unit such that $d^r$ is an anti-derivation with respect to this multiplication. If $F$ is connected, the local coefficient system $H*(F)$ on $B$ is a simple system of coefficients, and the isomorphism $E^2(f) \simeq H*(B; H*(F))$ is a ring isomorphism ([3], No. 13). This means that if the coefficient ring is a field, then $E^2(f) \simeq H*(B) \otimes H*(F)$, and hence for $E^2(f)$ to be anti-commutative it suffices for $H*(B)$ and $H*(F)$ to be so. Moreover, once one knows that $E^2(f)$ is an anticommutative, the same is clearly true for $E^r(f)$ where $r \geqq 2$.

These algebras have some additional properties derived from the fact that there are diagonal maps $E \to E \times E$, $B \to B \times B$, $F \to F \times F$, these maps are homomorphism, and induce diagonal maps $H*(E) \to H*(E) \otimes H*(E)$, $H*(B) \to H*(B) \otimes H*(B)$, and $H*(F) \to H*(F) \otimes H*(F)$, and also diagonal maps $E^r(f) \to E^r(f) \otimes E^r(f)$. These last diagonal maps commute with $d^r$ in the spectral sequences involved.

[1] Conforming to the notation of [5], the $r$-th term in a homology spectral sequence will be denoted by $E^r$, $E^r = \sum_{m,n} E^r_{m,n}$ where $E^r_{m,n}$ is the term of base degree $m$, and fibre degree $n$. The differential $d^r$ in $E^r$ has the property that $d^r: E^r_{m,n} \to E^r_{m-r,n+r-1}$.

Let $A$ be a graded anti-commutative algebra with unit, and let $A_n$ denote the subspace of $n$-dimensional elements. Such an algebra $A$ together with a homomorphism of algebras $\phi: A \to A \otimes A$ is called a *Hopf algebra* ([2], p. 137) if $A_0$ consists of multiples of the unit, and $\phi(x) = x \otimes 1 + 1 \otimes x + \sum a_i \otimes b_i$ for $x \in A_n$, $n > 0$, and $0 <$ dimension $a_i < n$. An element $x \in A$ is called *primitive* if $\phi(x) = x \otimes 1 + 1 \otimes x$. Suppose that $x$, $y$ are linearly independent primitive elements, then $\phi(xy) = xy \otimes 1 + 1 \otimes xy + x \otimes y + (-1)^{mn} y \otimes x$ when $x \in A_m$, $y \in A_n$. This means $xy$ is not primitive.

Going back now to our fibre space situation, it is immediate that if $E$, $B$, $F$ are connected, and $H*(B)$, $H*(F)$ anti-commutative, then the algebras $H*(B)$, $H*(F)$, and $E^r(f)$, $r \geqq 2$ are Hopf algebras. Further if $x \in E^r(f)$ and $x$ is primitive then $d^r x$ is also primitive since $d^r$ commuted with the diagonal map of $E^r(f)$.

Finally if the coefficient field is perfect,[2] $H*(B)$, $H*(F)$ are anti-commutative, $H_n(B)$, $H_n(F)$ are finite dimensional vector spaces, and $E$, $B$, $F$ are connected, then the structure theorem of Borel for Hopf algebras ([2], p. 138) is applicable to the algebras $H*(B)$, $H*(F)$, and $E^r(f)$. This theorem says that if $A$ is a Hopf algebra over a perfect field, and $A_n$ is finite dimensional for every $n$, then as an algebra $A$ may be written as a tensor product of algebras with 1-generator. Information is given concerning the generators depending on the characteristic of the field. In particular for characteristic $p$, $p$ an odd prime $x^2 = 0$ for odd dimensional elements, and for even dimensional generators either $x^r \neq 0$ for all $r$, or $x^r = 0$, $x^{r-1} \neq 0$, and $r$ is a power of $p$ called the *height of* $x$.

Having made a summary of the formal properties of spectral sequences of fibre spaces which are needed for our calculations, it will be assumed henceforth that we are working with a singular homology (*e.g.* simplicial or cubical theory) which has these properties, and for which the Hurewicz theorem holds. This means that the results of [13], [15], and [16] apply, and in particular henceforth a knowledge of the Hurewicz Theorem in terms of $C$-theory ([16]), or $p$-primary components ([13]) will be assumed.

## §2. The homology of $\Omega^2(S^{n+1})$

In order to apply all of the preceding paragraph it is first necessary to look a little at loop spaces. We start by defining these. If $X$ is a topological space, a *path* in $X$ is a pair $(f, r)$ where $r$ is a non-negative real number, and $f: [0, r] \to X$ is a map ($[0, r]$ denotes the closed interval from 0 to $r$). A *loop* is a path $(f, r)$ such that $f(0) = f(r)$. Topologize the set of paths in $X$ by using as a subbasis for the topology the sets $W(C, V, U)$ defined as follows: $C$ is a compact subset of $[0, 1]$, $V$ an open subset of the non-negative real numbers, $U$ an open subset of $X$, and $W(C, V, U)$ is the set of paths $(f, r)$ such that $r \in V$, and $f(ry) \in U$ for $y \in C$. Now let $x \in X$, and let $E$ be the space of paths in $X$ which start at $x$, i.e. those paths $(f, r)$ such that $f(0) = x$. Define a map $g: E \to X$ by letting

[2] As is well known, the hypothesis that the field is perfect is not needed when dealing with spaces.

$g(f, r) = f(r)$. It may now be shown that $g$ is a fibre map, and that $E$ is contractible. The fibre $\Omega = g^{-1}(x)$ is the space of loops in $X$ based at $x$. Define a product in $\Omega$ by setting $(f, r)(f', r') = (f'', r + r')$ where $f''(t) = f(t)$ for $0 \leqq t \leqq r$, and $f''(t) = f'(t - r)$ for $r \leqq t \leqq r + r'$. This product is associative and has an identity. Consequently the loop space in any space is a space with an associative multiplication with identity, *i.e.* is a monoid.

Sometimes it is convenient to use the space of normalized paths (or loops), *i.e.* those paths $(f, r)$ such that $r = 1$, instead of the space of all paths. The normalized loop space has the same homotopy type as the full loop space. Therefore it is frequently convenient to fail to distinguish between them.

Now suppose that $X$ is a space with an associative multiplication with identity $e \in X$. Let $E$ be the space of normalized paths in $X$ which start at $e$, and let $g:E \rightarrow X$ be the usual map defined by $g(f, 1) = f(1)$. Define a multiplication in $E$ by setting $(f, 1)(f', 1) = (f'', 1)$ where $f''(t) = f(t)f'(t)$. Evidently $g$ is a homomorphism, and we are in the situation discussed in the preceding paragraph. Further the space $\Omega = p^{-1}(e)$ is commutative up to homotopy, and consequently its Pontrjagin ring is anti-commutative. Therefore, for the Pontrjagin rings $E^r(g)$, $r \geqq 2$, to be anti-commutative it suffices for the Pontrjagin ring of $X$ to be anti-commutative. If $X = \Omega(S^{n+1})$ the loops in an odd dimensional sphere this is indeed the case, and the fibre $\Omega$ is nothing but $\Omega^2(S^{n+1})$.

CONVENTIONS: For the remainder of this paragraph $p$ is an odd prime, all algebras considered are over $Z_p$, and all homology with coefficients in $Z_p$.

NOTATION. Let $E(x, m)$ denote the Grassman algebra with 1-generator $x$ of dimension $m$, $P(y, m)$ the polynomial ring with 1 generator $y$ of dimension $m$, and $P^r(y, m)$ the quotient of $P(y, m)$ by the ideal generated by $y^r$. Sometimes we will also use the notation $P^\infty(y, m)$ for $P(y, m)$.

THEOREM: *If* $n > 0$ *is even, then*

$$H*(\Omega^2(S^{n+1})) \simeq \bigotimes_{k \geqq 0} E(y_k, p^k n - 1) \otimes \bigotimes_{k > 0} P(z_k, p^k n - 2).$$

PROOF: It is well known that $H*(\Omega S^{n+1})) \simeq P(x, n)$, and we have just finished showing that there is a spectral sequence $\{E^r\}$ such that $E^\infty$ is trivial, i.e. $E^\infty_{m,n} = 0$ for $m, n \neq 0$, and $E^\infty_{0,0} \simeq Z_p$, such that $E^2 = P(x, n) \otimes H*(\Omega^2(S^{n+1}))$, and $E^r$ is a Hopf algebra for $r \geqq 2$.

The proof of this theorem consists essentially of showing that this spectral sequence is unique. Just knowing $H*(\Omega^2(S^{n+1}))$ is a Hopf algebra we may write it as $\otimes_{k \geqq 0} E(y_k, m_k) \otimes \otimes_{k>0} P^{r_k}(z_k, n_k)$ where $r_k$ is a power of $p$ or $\infty$. This means that to prove the theorem what we need to compute is $m_k$, $n_k$, and $r_k$ for every $k$.

Let $A^i$ denote $\otimes_{k \geqq i} E(y_k, m_k)$, and let $B^i$ denote $\otimes_{k \geqq i} P^{r_k}(z_k, n_k)$. Now we have that since $d^r$ is a derivation it is zero for $2 \leqq r < n$, and since $E^\infty$ is trivial $d^n: E^n_{n,0} \approx E^n_{0,n-1}$. Consequently we may assume $y_0 = d^n x$, $m_0 = n - 1$. Now $E^{n+1} = H(E(y_0, n - 1) \otimes P(x, n)) \otimes A^1 \otimes B^1$. Letting $w_1 = x^{p-1}y_0$, $x_1 = x^p$

it is not difficult to prove that $H(E(y_0, n-1) \otimes P(x, n)) = E(w_1, pn-1) \otimes P(x_1, pn)$. Notice that the base degree of $w_1$ is $(p-1)n$, and that $w_1$ can never be $d^r$ of any element for its fibre degree is too small.

To find the next non trivial $d^r$ it is necessary only to find where $d^r w_1$ or $d^r x_1$ is non zero. Since both elements $w_1$ and $x_1$ are primitive so are the elements $d^r w_1$ and $d^r x_1$. Therefore, it is convenient to consider the primitive elements of dimension smaller than the dimension of $w_1$ or $x_1$. Such elements are all of base degree zero. This means $d^r = 0$ for $n < r < (p-1)n$, and we may assume $z_1 = d^{(p-1)n} w_1$, $n_1 = pn - 2$. It follows that $E^{(p-1)n+1} = H(P^{r_1}(z_1, pn-2) \otimes E(w_1, pn-1) \otimes P(x_1, pn) \otimes A^1 \otimes B^2$, and letting $v_1 = w_1 z_1^{r_1 - 1}$, we have that

$$H(P^{r_1}(z_1, p_{n-2}) \otimes E(w_1, p_{n-1})) = E(v_1, r_1(pn-1)+1).$$

Now make the following inductive hypotheses:

1) $d^r = 0$ if $r$ is not of the form $p^k(p-1)n$, or $p^k n$,
2) $E^{p^k n+1} = \otimes_{i=1}^{k} E(v_i, r_i(p^i - 2) + 1) \otimes E(w_{k+1}, p^{k+1}n - 1) \otimes P(x_{k+1}, p^{k+1}n) \otimes A^{k+1} \otimes B^{k+1}$,
3) $E^{p^k(p-1)n+1} = \otimes_{i=1}^{k+1} E(v_i, r_i(p^i n - 2) + 1) \otimes P(x_{k+1}, p^{k+1}n) \otimes A^{k+1} \otimes B^{k+2}$, and
4) $x_{k+1} = x_k^p$, $d^{p^k n} x_k = y_k$, $w_{k+1} = x_k^{p-1} y_k$, $v_k = w_k y_k^{r_k - 1}$, and $d^{p^k(p-1)n} w_{k+1} = z_{k+1}$.

Suppose these hypotheses have been proved for $k \leqq m$. Then $E^{p^m(p-1)n+1}$ is of the desired form, and for $d^r$ to be non zero it is necessary that $d^r v_i$ be non zero, or $d^r x_{m+1}$ be non zero. Here we are assuming $r > p^m(p-1)n$. The base degree of $v_i$ is $p^{i-1}(p-1)n$, and for $d^r v_i$ to be non zero we must have $p^{i-1}(p-1)n - r \geqq 0$. Since this is clearly impossible, in order for $d^r$ to be non zero it is necessary that $d^r x_{m+1}$ be non zero. For this to happen we must have a primitive element whose dimension is smaller than the dimension of $x_{m+1}$. Such elements have base degree 0, $(p-1)n, \cdots$, or $p^m(p-1)n$. One might suppose that the base degree could be a sum of such base degrees. However, since $E_{s,t}^r = 0$ for $r > \sup\{s, t+1\}$ we have that the fibre degree of $v_i$ is greater than or equal to $p^m(p-1)n$, and the dimension of $v_i v_j$ is greater than $p^{m+1}n$. Similarly the product of $v_i$ and a positive dimensional element of base degree zero has dimension greater than $p^{m+1}n$. Therefore, if the base degree of $d^r x_{m+1}$ is $p^i(p-1)n$ we may assume $d^r x_{m+1} = v_i$. This means that $p^{m+1}n - 1 = r_i(p^i n - 2) + 1$, and since $r_i$ is a power of $p$ and $p$ is an odd prime this is impossible. Consequently, we may assume $r = p^{m+1}n$, and $d^r x_{m+1} = y_{m+1}$. Now we have that $E^{p^{m+1}n+1}$ is of the desired form.

To find the next non trivial $d^r$ we must have either $d^r x_{m+2} \neq 0$, or $d^r w_{m+2} \neq 0$. Since these elements are primitive we again need to find primitive elements with smaller dimension. By an easy argument the next non trivial $d^r$ is for $r = p^{m+1}(p-1)n$, and we may assume $d^r w_{m+2} = z_{m+2}$ finishing the necessary verification to prove the inductive hypothesis.

It now follows that $r_i = \infty$ for all $i$, and that actually $E^{p^k n+1} =$

$E(w_{k+1}, p^{k+1}n - 1) \otimes P(x_{k+1}, p^{k+1}n) \otimes A^{k+1} \otimes B^{k+1}$ and $E^{p^k(p-1)n+1} = P(x_{k+1}, p^{k+1}n) \otimes A^{k+1} \otimes B^{k+2}$. Thus the proof of the theorem is complete.

Actually it is not difficult to show that if $\partial*: H*(\Omega^2(S^{n+1})) \to H*(\Omega^2(S^{n+1}))$ is the Bockstein homomorphism originating from the coefficient sequence $0 \to Z_p \to Z_{p^2} \to Z_p \to 0$, then $\partial* y_k = z_k$ for $k > 0$. This may be done by looking at the natural map of the integer spectral sequence into the one we have been considering.

## §3. Some preliminary considerations relating homotopy and homology

We have seen in the last paragraph that for $n$ even $H_q(\Omega^2(S^{n+1}); Z_p) \simeq Z_p$ for $q = 0, n - 1, pn - 2$, and is zero otherwise for $q < pn - 2$. Letting $Q$ denote the field of rational numbers, it is well known that $H_q(\Omega^2(S^{n+1}); Q) \simeq Q$ for $q = 0, n - 1$ and is zero otherwise. This means that the $p$-primary component of the integral homology group $H_{pn-2}(\Omega^2(S^{n+1}); Z)$ is isomorphic with $Z_p$ ([13], [16]). The question now arises as to when this group is generated by a spherical homology class. At present it is not possible to settle this question completely. However a good deal is known, as is shown by the next proposition.

PROPOSITION: *For $n$ even and greater than zero, if the $p$-primary component of $H_{pn-2}(\Omega^2(S^{n+1}); Z)$ is spherical, then $n = 2p^k$ for some integer $k$.*

PROOF: Suppose the given group is spherical. Then there is a map $f: S^{pn-2} \to \Omega^2(S^{n+1})$ which sends the fundamental class of $H_{p_n-2}(S^{p_n-2}; Z)$ into a generator of the $p$-primary component of $H_{pn-2}(\Omega^2(S^{n+1}); Z)$, and a corresponding map $f': S^{p_n-1} \to \Omega(S^{n+1})$ so that if $S:\pi_q(\Omega^2(S^{n+1})) \to \pi_{q+1}(\Omega(S^{n+1}))$ denotes the suspension homomorphism, then $S[f] = [f']$ (for a map $g: S^m \to X$, $[g]$ is its homotopy class). Now recall that $\Omega(S^{n+1})$ or at least something of the same homotopy type may be obtained by attaching cells $e^q$ so that we have $S^n \cup e^{2n} \cup e^{3n} \cup \cdots$. The map $f'$ may be considered as a map into $K^{(p-1)} = S^n \cup \cdots \cup e^{(p-1)n}$, and a new space $Y$ may be formed by attaching a cell $e^{pn}$ to $K^{(p-1)}$ by means of $f'$.

The space $Y$ space has the property that its $n$-dimensional cohomology group $H^n(Y; Z_p)$ is isomorphic with $Z_p$, and letting $\alpha$ be a generator we have $\alpha^p \neq 0$. However, $\alpha^p$ is just $\mathcal{P}^{n/2}\alpha$, the Steenrod $p$th power of $\alpha$.

Now suspend $Y$, and denote the resulting space by $s(Y)$. Then $s(Y) = s(K^{(p-1)}) \cup e^{pn+1}$, and $s(K^{(p-1)})$, the suspension of $K^{(p-1)}$, has the homotopy type of the bouquet of spheres $S^{n+1} \vee \cdots \vee S^{(p-1)n+1}$. Form a new space $W$ by collapsing the subspace $S^{2n+1} \vee \cdots \vee S^{(p-1)n+1}$ of $s(Y)$ to a point. Then $W = S^{n+1} \cup e^{pn+1}$, and if $\alpha'$ is a generator of $H^{n+1}(W; Z_p)$, we have that $P^{n/2}(\alpha')$ is a generator of $H^{n+1}(W; Z_p)$. Using the relations on Steenrod powers ([1], [4]) this is easily seen to be impossible if $n$ is not of the form $2p^k$ for some integer $k$.

The preceding proposition is also valid for $p = 2$, and in this case is equivalent to the theorem of Adem asserting that if $S^{2n-1} \to S^n$ is a map of Hopf invariant 1, then $n$ is a power of 2.

LEMMA: *If $n$ is even and greater than 2, then with coefficients in $Z_p$, $p$ an odd prime, the ring $H*(\Omega^3(S^{n+1}))$ is isomorphic with the ring $P(y_0, n - 2) \otimes E(z_1, pn - 3) \otimes P(y_1, pn - 2)$ in dimensions less than $p(pn - 2) - 2$.*

The details of the proof of this lemma are an exercise in the use of spectral sequences very similar to the arguments already used, and will therefore be omitted.

Let $U^n$ denote the space of paths in $\Omega^2(S^{n+1})$ which start at the identity, and end in $S^{n-1}$.

LEMMA: *If $n$ is even and greater than 2, then with coefficients in $Z_p$, $p$ an odd prime, $H_q(U^n) \simeq (E(z_1, pn - 3) \otimes P(y_1, pn - 2))_q$ for $q < p(pn - 2) - 2$.*

To prove this lemma one notices that $U^n$ is a fibre space over $S^{n-1}$ with fibre $\Omega^3(S^{n+1})$. Then using the Wang sequence ([15]) the result follows immediately from the preceding lemma. In this application of the Wang sequence it is necessary to know that the homomorphism $H_q(\Omega^3(S^{n+1})) \to H_{q+n-2}(\Omega^3(S^{n+1}))$ is nothing but multiplication by $y_0$, where $y_0$ is the generator of $H_{n-2}(\Omega^3(S^{n+1}))$ of the preceding lemma.

Let $g: S^{pn-3} \to S^{pn-3}$ be a map of degree $p$, $X_g$ the mapping cylinder of $g$.

LEMMA: *The $p$-primary component of $\pi_q(X_g, S^{pn-3})$ is isomorphic with that of $\pi_q(U^n)$ for $q < p(pn - 2) - 3$.*

We already know that the $p$-primary component of $\pi_{pn-3}(U^n)$ is cyclic of order $p$. Therefore, there is a map of $S^{pn-3} \to U^n$ which represents a generator of the $p$-primary component of $\pi_{pn-3}(U^n)$. This induces a map of $X_g$ into $U^n$, and it may be assumed that for this map $S^{pn-3}$ considered as a subspace of $X_g$ goes into the base point of $U^n$. Letting $L$ denote the space of paths in $X_g$ which start at the base point and end in $S^{pn-3}$, there is a map of $L \to \Omega(U^n)$ induced by the preceding. The homology of $L$ is known ([14]) and with coefficients in $Z_p$ is isomorphic with $P(z_1, pn - 4) \otimes E(y_1, pn - 3)$. Using the preceding lemma, one calculates the homology of $\Omega(U^n)$ with coefficients in $Z_p$, and sees that it is isomorphic with $P(z_1, pn - 4) \otimes E(y_1, pn - 3)$ in dimensions less than $p(pn - 2) - 2$. Now it is not difficult to see that the map $L \to \Omega(U^n)$ induces an isomorphism of the homology groups with coefficients in $Z_p$ for dimensions less than $p(pn - 2) - 2$, and therefore the $p$-primary components of $\pi_q(L)$ and $\pi_q(\Omega(U^n))$ are isomorphic for $q < p(pn - 2) - 4$. This last statement implies the lemma.

NOTATION: If $G$ is an abelian group, let $G_p$ denote the quotient of $G$ by the sum of its $q$-primary components for $q \neq p$. Notice that if $G$ is a torsion group $G_p$ is just the $p$-primary component of $G$.

PROPOSITION: *If $n$ is even and greater than 2, and $q < p(pn - 2) - 3$ there is an exact sequence*

$$0 \to \pi_q(S^{pn-3}) \otimes Z_p \to \pi_{q+1}(\Omega^2(S^{n+1}), S^{n-1})_p \to Tor(\pi_{q-1}(S^{pn-3}), Z_p) \to 0.$$

This proposition follows immediately from the preceding lemma, and the fact since $pn - 3$ is odd the homomorphism of $\pi_q(S^{pn-3}) \to \pi_q(S^{pn-3})$ induced by a map of degree $p$ is multiplication by $p$ modulo the 2-primary component.

## §4. Homotopy groups of spheres

We will now start to collect the data of the earlier sections to apply to homotopy groups of spheres. First we state a result of H. Cartan on the stable homotopy groups of spheres. This result has also been proved by H. Toda.

THEOREM: *If $n$ is even and greater than $2p$, then for $k < 2(p+1)(p-1) - 2\pi_{n+k}(S^{n+1})_p$ is zero except for $k = 1, 2(p-1), \cdots, 2(p-1)(p-1), 2p(p-1)$, and $2p(p-1) - 1$, $\pi_{n+k}(S^{n+1})_p$ is isomorphic with $Z_p$ for $k = 2i(p-1)$ $i, = 1, \cdots, p-1$, $\pi_{n+2p(p-1)}(S^{n+1})_p \simeq Z_{p^p}$, and $\pi_{n+2p(p-1)-1}(S^{n+1})_p \simeq Z_{p^{p-1}}$.*

COROLLARY 1: *If $n$ is even and greater than 2, then for $k < 2p(p-1) - 2$ the group $\pi_{pn-2+k}(\Omega^2(S^{n+1}), S^{n-1})_p$ is zero except for $k = 2i(p-1)$, $i = 0, \cdots, p-1$, and $k = 2i(p-1) - 1$, $i = 1, \cdots, p-1$, and in these exceptional cases is isomorphic with $Z_p$.*

The corollary follows immediately from the preceding theorem, and from the last proposition of the preceding section.

COROLLARY 2: *The group $\pi_{2+k}(S^3)_p$ is isomorphic with $Z_p$ for $k = 2i(p-1)$, $i = 1, \cdots, p$, $k = 2i(p-1) - 1$, $i = 2, \cdots, p$, or for $k = 2(p+1)(p-1) - 2$, and is zero otherwise for $k < 2(p+1)(p-1) - 2$.*

It was shown in [14], that $\pi_q(S^3)_p \simeq \pi_{q+1}(X_f, S^{2p+1})$ for $q < 2p^2 - 1$, where $f: S^{2p+1} \to S^{2p+1}$ is a map of degree $p$, and $X_f$ is the mapping cylinder of $f$. Consequently for $1 < k < 2(p+1)(p-1) - 1$ we have $\pi_{2+k}(S^3)_p \simeq \pi_{2+k+1}(S^{2p+1}) \otimes Z_p + Tor(\pi_{2+k}(S^{2p+1}), Z_p)$; and the results follows.

Using the isomorphism between $\pi_q(S^3)_p$ and $\pi_{q+1}(S^{2p+1}) \otimes Z_p + Tor(\pi_q(S^{2p+1}), Z_p)$ one may obtain a homomorphism $\lambda: \pi_q(S^{2p+1})_p \to \pi_q(S^3)_p$. This may be combined with the iterated suspension homomorphism $E^{2(p-1)}: \pi_q(S^3) \to \pi_{q+2(p-1)}(S^{2p+1})$ to obtain a homomorphism

$$\xi = \lambda \circ E^{2(p-1)}: \pi_q(S^3)_p \to \pi_{q+2(p-1)}(S^3)_p .$$

PROPOSITION: *If $\alpha$ is a generator of $\pi_{2+k}(S^3)_p$, $k = 2i(p-1)$ and $0 < i < p$ then $\xi(\alpha)$ is a generator of $\pi_{2+k+2(p-1)}(S^3)_p$.*

To prove this proposition it suffices to show that $E^{2(p-1)}: \pi_{2+k}(S^3)_p \overset{\approx}{\to} \pi_{2p+k}(S^{2p+1})_p$. However since we already know both of these groups are cyclic of order $p$, it suffices to show that $\pi_{n+k}(S^{n+1})_p \to \pi_{n+k}(\Omega^2(S^{n+3}))_p$ is a monomorphism for $n$ even, $2 < n < 2(p-1)$. The kernel of this preceding homomorphism is an image of the group $\pi_{n+k+1}(\Omega^2(S^{n+3}), S^{n+1})_p$, but this group is zero by corollary 1, and the result follows.

THEOREM: *If $0 < n < p$, then $\pi_{2n+k}(S^{2n+1})_p$ is isomorphic with $Z_p$ for $k = 2i(p-1)$, $i = 1, \cdots, p-1$, $k = 2i(p-1) - 1$, $i = n+1, \cdots, p-1$,* *and is isomorphic with* $Z_{p^n}$ *for* $k = 2p(p-1)$, *or* $2p(p-1) - 1$. *For other values of* $k$ *such that* $1 < k < (p+1)(p-1) - 2$, $\pi_{2n+k}(S^{2n+1})_p$ *is trivial.*

These results are summarized in table 1, with a slight amount of additional information. The proof of the results consists in filling in the table by using the preceding theorem, corollaries, and proposition of this chapter. This can be done most easily by working from right to left. Letting $n$ be even and greater than 2, notice that $E^2: \pi_{n-2+k}(S^{n-1})_p \to \pi_{n+k}(S^{n+1})_p$ is an isomorphism if $k = 2i(p-1)$, $i = 1, \cdots, p-1$, is zero if $k = 2i(p-1) - 1$, $i = n, \cdots, p-1$, and is a monomorphism for $k = 2p(p-1)$, or $2p(p-1) - 1$.

TABLE 1

| $k$ | $\pi_{2+k}(S^3)_p$ | $\pi_{4+k}(S^5)_p$ $\cdots$ | $\pi_{2(p-1)+k}(S^{2p-1})_p$ | $\pi_{2p+k}(S^{2p+1})_p$ |
|---|---|---|---|---|
| $2(p-1)$ | $Z_p$ | $Z_p$ | $Z_p$ | $Z_p$ |
| $4(p-1)-1$ | $Z_p$ | | | |
| $4(p-1)$ | $Z_p$ | $Z_p$ | $Z_p$ | $Z_p$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |
| $2p(p-1)-1$ | $Z_p$ | $Z_{p^2}$ | $Z_{p^{p-1}}$ | $Z_{p^{p-1}}$ |
| $2p(p-1)$ | $Z_p$ | $Z_{p^2}$ | $Z_{p^{p-1}}$ | $Z_{p^p}$ |
| $2(p+1)(p-1)-2$ | $Z_p$ | $Z_p$ | $Z_p$ | $Z_p$ |

To fill in the bottom line in table 1 requires only a slight additional computation, and we have then listed all the non-zero $p$-primary components of the homotopy groups $\pi_{n+k}(S^{n+1})$ for $n$ even and $k < (p+1)(p-1) - 1$.

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