SYMMETRIC FUNCTIONS OF SEVERAL VARIABLES, WHOSE RANGE AND DOMAIN IS A SPHERE

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1. Introduction

Consider a topological n-sphere S^n, where $n \geqq 1$. Let K denote the m-fold cartesian product $S^n \times \cdots \times S^n \times \cdots \times S^n$, and let G be a transitive permutation group of degree m which acts on K by permuting the factors. By a *G-invariant* map I mean a continuous function $f: K \to S^n$ which is invariant under the permutations of G. If $m = 2$, for example, then a G-invariant map is one such that

$$f(x, y) = f(y, x) \qquad (x, y \in S^n).$$

I define the *type* of a G-invariant map f to be the degree of the map of S^n into itself which is obtained from f by fixing all but one of the variables,[1] such as the map $x \to f(x, e, \cdots, e)$, where e is a fixed point. The purpose of this note is to try and determine, for the various cases of n and G, the numbers q such that S^n admits a G-invariant map of type q. Obviously every value of q can be achieved in case $m = 1$. Some results on the problem in case $m = 2$ are contained in [3].

We shall prove

THEOREM 1.1. *Let n be even, $m > 1$. Then S^n admits a G-invariant map of type q if, and only if, $q = 0$.*

Most of the interest, therefore, lies in the case of odd-dimensional spheres. Our main result is

THEOREM 1.2. *Let n and G be given, where n is odd. Then there exists a positive integer k, such that S^n admits a G-invariant map of type q if, and only if, q is a multiple of k. Moreover, none of the prime factors of k is greater than m, the number of variables.*

For example, if $n = 1$ we can represent points of S^n by complex numbers of unit modulus, and then the function

$$f(x_1, \cdots, x_m) = x_1^q \cdot \cdots \cdot x_m^q \text{ (complex multiplication)}$$

provides a G-invariant map of type q, where q is any integer, irrespective of the choice of G. Thus $k = 1$ if $n = 1$.

The cases of chief interest appear to be those in which G is the full symmetric group or the group of cyclic permutations. In the former case we refer to a G-invariant map as a symmetric function of m variables, and in the latter case as a cyclically symmetric function. Clearly, a symmetric function is G-invariant for any permutation group G on the same number of variables; in particular,

[1] Variable, in the present context, means a variable point of S^n.

it is cyclically symmetric. Moreover, by fixing some of the variables we obtain a symmetric function of that many fewer variables. Thus we have immediately

THEOREM 1.3. *To a symmetric function of m variables on S^n there correspond symmetric functions on S^n of the same type for every number of variables less than m.*

In the case of symmetric functions, therefore, it follows that k in 1.2 is a monotone non-decreasing function of m, the number of variables. I do not know whether k ever decreases as n, the dimension, increases. We shall prove

THEOREM 1.4. *Let G, in 1.2, be the cyclic group of order p, where p is prime. Let $n > 1$. Then k is a multiple of p.*

Hence, and from 1.3, we obtain

COROLLARY 1.5. *Let G, in 1.2, be the symmetric group of degree m. Let $n > 1$. Then k is divisible by every prime number which does not exceed m.*

We obtain from 1.5 a lower bound on the value of k in the case of the symmetric group. An upper bound can be obtained by the methods used in [3] for the case of two variables. It seems to be difficult to determine k exactly, even in the case of S^3.

2. The symmetric product

Let K denote the m-fold cartesian product $S^n \times \cdots \times S^n \times \cdots \times S^n$, and let L denote the space which is obtained from K by identifying points which correspond under permutations of the factors, i.e. let L be the m-fold symmetric product of S^n. We embed S^n in L so that $x \to v(x, e, \cdots, e)$, where $x \in S^n$ and $v: K \to L$ is the identification map. The integral homology groups of the symmetric product have been studied by Richardson in [5]. It is found by his methods that $m! H_r(L, S^n) = 0$ if $r \geqq n$. Hence we obtain by means of the universal coefficient theorem for cohomology,

LEMMA 2.1. *Let n be odd, and let G be a finitely generated abelian group. Then $m!\, H^r(L, S^n; G) = 0$ if $r > n$.*

Let L be triangulated as in [5], so that S^n is a subcomplex, and let L^r denote the r-skeleton of L. We use 2.1 to prove

LEMMA 2.2. *Let $r > n$, where n is odd, and let $m > 1$. Suppose that we have a map $h: L^{r-1} \to S^n$. Then there exists a map $h': L^r \to S^n$ whose degree on S^n is equal to the degree of h on S^n multiplied by $2m!$*

For let α denote the characteristic class of h in $H^r(L, S^n; \pi_{r-1}(S^n))$. Consider the endomorphism, u_*, of $\pi_{r-1}(S^n)$ which is induced by a map of degree $2m!$, $u: S^n \to S^n$. Since $m \geqq 2$, $2m!$ is divisible by four, and hence $u_*(\beta) = 2m!\beta$, by 6.7 of [1], where $\beta \in \pi_{r-1}(S^n)$. Therefore $u_\#(\alpha) = 2m!\alpha$, where $u_\#$ denotes the coefficient endomorphism of $H^r(L, S^n; \pi_{r-1}(S^n))$ which is determined by u_*. Hence $u_\#(\alpha) = 0$, by 2.1. But $u_\#(\alpha)$ is the characteristic class of uh, i.e.

the obstruction to extending $uh \mid L^{r-2} \cup S^n$ over L^r. Hence there exists a map $h': L^r \to S^n$ which agrees with uh on S^n. This proves 2.2.

We use 2.2 to prove

THEOREM 2.3. *Let n be odd, $m > 1$. Then there exists a symmetric function $f: K \to S^n$ whose type, q, is a positive integer which has no prime factors greater than m.*

For let $g_n: L^n \to S^n$ be an extension of the identity map on S^n. Then it follows by induction from 2.2 that there exists a sequence of maps $g_n, g_{n+1}, \cdots, g_r, \cdots$, where $g_r: L^r \to S^n$, such that the degree of g_r on S^n is a positive integer with no prime factors greater than m. Let $f = g_{mn}v$, where $v: K \to L$ is the identification map. Then f has the stated properties.

3. The cyclic product

In this section we prove

THEOREM 3.1. *Let $n > 1$ and let $m = p$, where p is prime. Suppose that we have a cyclically symmetric function $f: K \to S^n$ of type q. Then q is divisible by p.*

Let M denote the space which is obtained from K, the p-fold cartesian product, by identifying points which correspond under cyclic permutation of the factors, i.e. let M be the p-fold cyclic product of S^n. We embed S^n in M so that

$$x \to w(x, e, \cdots, e),$$

where $x \in S^n$ and $w: K \to M$ is the identification map. Since f is cyclically symmetric, a map $g: M \to S^n$ is determined by $f = gw$, and g maps S^n with degree q.

In cohomology modulo p, consider the homomorphisms

$$H^r(S^n; Z_p) \xrightarrow{g^*} H^r(M; Z_p) \xrightarrow{i^*} H^r(S^n; Z_p)$$

which are induced by g and the inclusion map. Let α generate $H^n(S^n; Z_p)$. Then $i^*g^*(\alpha) = q\alpha$. If $g^*(\alpha) \neq 0$ then $\mathcal{P}^1 g^*(\alpha) \neq 0$, by 6.2 of [4], where $\mathcal{P}^1$ denotes the cyclic reduced power homomorphism of Steenrod. But this is impossible, since $\mathcal{P}^1 g^*(\alpha) = g^*\mathcal{P}^1(\alpha)$, by naturality, and $\mathcal{P}^1(\alpha) \in H^{n+2p-2}(S^n; Z_p) = 0$. Hence $g^*(\alpha) = 0$, and therefore $q\alpha = 0$. Hence q is divisible by p.

4. Proof of the theorems of the Introduction

We begin by recalling two theorems of Hopf from [2]. The *type* of a map

$$g: S^n \times S^n \to S^n$$

is the pair of integers (t, t'), where t, t' are the degrees of the maps

$$x \to g(x, e), \qquad x \to g(e, x),$$

respectively, $(x \in S^n)$.

THEOREM 4.1. *Let n be even. Then there exists a map $g: S^n \times S^n \to S^n$ of type (t, t') if, and only if, t or t' is zero.*

THEOREM 4.2. *Let n be odd. Then there exists a map $g:S^n \times S^n \to S^n$ of type (t, t') if either t or t' is even.*

Let n be even and $m \geqq 2$. Let $f:K \to S^n$ be a G-invariant map of type q, where G is arbitrary. Then a map $g:S^n \times S^n \to S^n$ is defined by

$$g(x, y) = f(x, y, e, \cdots, e) \qquad (x, y \in S^n).$$

Since g is of type (q, q), it follows from 4.1 that $q = 0$. Moreover, there exist G-invariant maps of type zero, such as the constant map. This proves 1.1.

We shall now prove[2] 1.2. If $m = 1$, 1.2 is obvious, so let $m \geqq 2$, and let n be odd. Let A denote the set of integers q such that S^n admits a G-invariant map of type q. It follows at once from 2.3 that A contains non-zero integers. We have

LEMMA 4.3. *Let t and t' be integers, one of which is even. If $a, a' \in A$ then*

$$at + a't' \in A.$$

For let $f, f':K \to S^n$ be G-invariant maps of type a, a', respectively. Since n is odd and since t or t' is even, there exists a map $g:S^n \times S^n \to S^n$ of type (t, t'), by 4.2. Let $h:K \to S^n$ be the map which is defined by

$$h(z) = g(f(z), f'(z)) \qquad (z \in K).$$

Then h is G-invariant, and it follows easily from considerations of homology that h is of type $at + a't'$. This proves 4.3.

Let k be the highest common factor of the non-zero integers in A, and let a be the least positive integer in A. I say that $a = k$. For otherwise A contains an integer a' which is not a multiple of a, and so there exists a positive integer b, less than a, such that either $a' + b$ or $a' - b$ is an even multiple of a. But then it follows from 4.3 that $b \in A$, which contradicts the definition of a. Therefore $a = k$, hence $k \in A$, and it follows that A consists of the multiples of k. This proves the first assertion of 1.2. The second part follows at once from 2.3 and the first part.

Finally, 1.4 follows immediately from 1.2 and 3.1. This completes the proof of the theorems stated in the introduction.

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[2] The following proof is much simpler than the proof in [3] for the case $m = 2$.