LIMIT SETS AND PERIODIC ORBITS

By J. L. Arraut

1. Introduction

Let X be a metric space and F a continuous flow on X (see §2). With each orbit γ of the flow there is associated a subset $\Lambda^+(\gamma)$, called the positive limit set of γ. It is known that $\Lambda^+(\gamma)$ is a union of orbits. An open problem is to determine the orbital composition of $\Lambda^+(\gamma)$. The problem has been exhaustively studied for the case when the space X is the euclidean plane R^2, and the results related to it form the theory of Poincaré-Bendixon. In the euclidean space R^n, with $n \geqq 3$, or in general in a metric space, the problem becomes very difficult. Here we consider a partial aspect of it, say: Under what conditions does the set $\Lambda^+(\gamma)$ reduce to a periodic orbit? To deal with this partial problem we have introduced the concept of *strong limit point* (see §3). We believe that we have found very general conditions under which the problem is solved; see Theorems 1, 2, and 3. Results related to this problem, as those of Borg (see [3]), can be shown to be contained, in a certain sense, in ours. In §5, we define the *strong flows*, and we show that they behave in a good way; for example, the theorems of Bohr-Fenchel (see [2]) and Poincaré-Bendixon can be applied in these flows. In particular we show that any continuous flow on R^2 is strong.

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2. General concepts

Throughout this paper X will denote a metric space, with distance d, R being the space of real numbers with the usual topology and $R^+(R^-)$, the non-negative (non-positive) reals. The spherical neighborhoods with center at a subset A of X and radius $\delta > 0$, open and closed, are denoted respectively by $S(A, \delta)$ and $B(A, \delta)$. Every time we consider a subset A of X as a space, it should be understood that it possesses the induced topology.

A *continuous flow or dynamical system* is a map $\pi: X \times R \rightarrow X$ which satisfies the two following conditions:

F.1 $\pi(x, 0) = x, \quad x \in X$

F.2 $\pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2), \quad x \in X,\ t_1, t_2 \in R.$

From now on, we write xt in place of $\pi(x, t)$. F.1 and F.2 then read: $x0 = x$ and $(xt_1)t_2 = x(t_1 + t_2)$. The continuity of π will in general be used in the following way: if $\{x_n\}$ is a sequence of points of X such that $x_n \rightarrow x$, and $\{t_n\}$ is a sequence of real numbers such that $t_n \rightarrow t$, then $x_n t_n \rightarrow xt$. The following results easily from the definition of continuous flow: Let x_0 (the initial condition) be a point of X and t_0, a positive number. For each $\epsilon > 0$ there exists $\delta(t) > 0$ such that $d(x, x_0) < \delta$ and $|t| \leqq t_0$ implies $d(xt, x_0t) < \epsilon$.

In a continuous flow F on X we may distinguished two structures, one algebraic and the other geometric. The second is considered our basic object of interest.

Algebraic structure. For each $t \in R$, π induces a map $\pi_t: X \to X$ given by $\pi_t(x) = xt$. Let us consider the collection $\{\pi_t\}_{t \in R}$ together with the operation composition of functions and suppose that $t \neq t'$ implies $\pi_t \neq \pi_{t'}$. Then it is easy to see that the correspondence $R \to \{\pi_t\}_{t \in R}$ given by $t \to \pi_t$ transfers isomorphically onto $\{\pi_t\}_{t \in R}$ the additive group structure of R. It is a direct consequence of this that each π is a homeomorphism of X on X. From now on we will identify the group $\{\pi_t\}_{t \in R}$ with R.

Geometric structure. For each $x \in X$, π induces a map $\pi_x: R \to X$ given by $\pi_x(t) = xt$. Each π_x is called a *movement*, and, coherently with this, the variable $t \in R$ is called *time*. The subset of $X\gamma(x) = \pi_x(R) = \{xt : t \in R\}$ is called the *orbit* of x. It is easy to see that the collection $\{\gamma(x)\}_{x \in X}$ of all the orbits forms a partition of X. Finally, observe that for every point x of a given orbit γ, the movement $\pi_x: R \to X$ is a parametrization of γ.

Let x be a point of X and consider the action of R on it. Two cases are possible:

(i) there exists $t_0 \neq 0$ such that $xt_0 = x$;

(ii) $xt = x$, if and only if $t = 0$.

Analyzing case (i), we will finally arrive at the two following subcases: (i.1) $xt = x$ for all $t \in R$; i.e., $\gamma(x)$ reduces to x. In this situation x is called a *critical* point. (i.2) There exists a minimum $\tau > 0$ such that $x\tau = x$. Here x is called a *periodic point and* τ, the *period.* The orbit $\gamma(x)$ is called *periodic* also, and it is easy to see that $\gamma(x)$ is homeomorphic to the circle.

In the case (ii), we obtain that the map $\pi_x: R \to X$ is one-to-one on $\gamma(x)$. This is properly the general situation. Sometimes the map $\pi_x: R \to \gamma(x)$ is a homeomorphism.

LEMMA 1. *Let $x \in X$. $\pi_x: R \to \gamma(x)$ is a homeomorphism if and only if for every sequence $\{t_n\}$ such that $xt_n \to x$ we have $t_n \to 0$.*

Proof. The necessity of the condition is evident. To prove sufficiency we have to show that π_x^{-1} exists and is continuous. Let us suppose that $xt = x$; then $t = 0$ follows by applying the hypothesis to the sequence $\{t_n \equiv t\}$. Now let $\{t_n\}$ be a sequence such that $xt_n \to xt_0$. We want to see that $t_n \to t_0$. From the continuity of π it follows that $(xt_n)(-t_0) \to xt_0(-t_0)$; i.e., $x(t_n - t_0) \to x$, thus $t_n \to t_0$.

A subset A of X is called *invariant* (under F) if $x \in A$ implies $\gamma(x) \subset A$. A property defined for points of X is called *orbital* if the set of points which satisfies it is invariant.

The set $\gamma^+(x) = \{xt : t \in R^+\}$ is called the positive *semi-orbit* of x. An orbit $\gamma(x)$ is L^+-stable (positively stable in the sense of Lagrange) if the closure of $\gamma^+(x)$ is compact.

The (positive) limit set $\Lambda^+(x)$ of an orbit $\gamma(x)$ consists of the points $y \in X$ such that there exists a sequence $\{t_n\} \subset R^+$ with $t_n \to +\infty$ and such that $xt_n \to y$. A point $y \in \Lambda^+(x)$ is called a (positive) *limit point* of $\gamma(x)$.

It is known that for any $\gamma(x)$ the limit set $\Lambda^+(x)$ is closed and invariant. Besides, if $\gamma(x)$ is L^+-stable, then $\Lambda^+(x)$ is non-empty, compact, connected, and $\lim_{t\to+\infty} d(xt, \Lambda^+(x)) = 0$.

A point $x \in X$ is (positively) *recurrent* if $x \in \Lambda^+(x)$. Observe that the property of being recurrent is an orbital property. The orbit $\gamma(x)$ is also called recurrent.

COROLLARY (to Lemma 1). *Let $x \in X$. The map $\pi_x: R \to \gamma(x)$ is a homeomorphism if and only if x is neither positively nor negatively recurrent* (*see* [4]).

If $A \subset X$ and $t \in R$, At will denote the set $\{xt : x \in A\}$. A point $x \in X$ is called *non-wandering* if, for every neighborhood $V(x)$ and every $t_0 > 0$, there exists $t \geqq t_0$ such that $V(x)t \cap V(x) = \emptyset$. It is clear that the property of being non-wandering is orbital.

If $A \subset X$, $\gamma(A)$ will denote the set $\bigcup_{x\in A} \gamma(x)$. An orbit $\gamma(x)$ is called (positive) *orbitally stable* if for any $\epsilon > 0$ there exists $\delta(x, \epsilon) > 0$ such that $\gamma^+(S(x, \delta)) \subset S(\gamma^+(x), \epsilon)$.

LEMMA 2. *Let $\gamma(x_0)$ be a L^+-stable, orbitally stable, and non-wandering orbit. Then $\gamma(x_0)$ is recurrent; i.e., $x_0 \in \Lambda^+(x_0)$.*

Proof. Let us suppose that x_0 is not recurrent. Since $\gamma(x_0)$ is L^+-stable, $\Lambda^+(x_0)$ is non-empty and compact. Then $d(x_0, \Lambda^+(x_0)) = \alpha > 0$. For the same reason, if $\epsilon = \alpha/2$ there exists t_0 such that $\gamma^+(x_0t_0) \subset S(\Lambda^+(x_0), \epsilon/2)$. Now, since $\gamma(x_0)$ is orbitally stable for x_0t^*, with $t^* > t_0$, and $\epsilon/2$ given, there exists $\delta = \delta(x_0t^*, \epsilon/2) > 0$ such that $\gamma^+(S(x_0t^*, \delta)) \subset S(\gamma^+(x_0t^*, \epsilon/2)) \subset S(\Lambda^+(x_0), \epsilon)$. Finally, from the continuity of π_{t^*} we can determine a neighborhood $S(x_0, \eta)$ with $\eta < \epsilon/2$ such that $S(x_0, \eta)t^* \subset S(x_0t^*, \delta)$. From this it follows directly that x_0 is wandering.

3. Local sections

For the study of the geometric structure of a continuous flow in the neighborhood of a regular point, the concept of a local section is useful. (The reader is refered to [5], pp. 332–38.)

We will use the open and closed intervals (a, b) and $[a, b]$ in the non-oriented sense: i.e., if $a \leqq b$, then $(a, b) = \{t \in R : a < t < b\}$; if $b \leqq a$, then $(a, b) = \{t \in R : b < t < a\}$; etc. If $x \in X$, the set $x[a, b] = \{xt \in X : x \in [a, b]\}$ is called an *arc* of the orbit $\gamma(x)$ with time length $(b - a)$. Note that the time length of an arc may be a negative number.

Let x_0 be a point of X and δ and α, two positive numbers. The set

$$T = T(x_0, \delta, \alpha) = B(x_0, \delta)[-\alpha, \alpha]$$

is called a *tube* of time length 2α over $B(x_0, \delta)$. *A* tube T is a closed subset of X. In fact, let $\{y_n\}$ be a sequence of points of T such that $y_n \to y^* \in X$. From the definition of T we can write $y_n = x_nt_n$, where $x_n \in B(x_0, \delta)$ and $-\alpha \leqq t_n \leqq \alpha$. The sequence $\{t_n\}$, being contained in $[-\alpha, \alpha]$, has at least one limit point t^*, with $-\alpha \leqq t^* \leqq \alpha$, to which a subsequence converges. Without loss of generality, suppose that $t_n \to t^*$. Then, by the continuity of π, $y_n(-t_n) \to y^*(-t^*)$; i.e., $x_n \to y^*(-t^*)$. Now, since $B(x_0, \delta)$ is closed, we know that $y^*(-t^*) \in B(x_0, \delta)$ and therefore $y^* \in T$.

We say that $\mathcal{S} \subset T$ is a *local section*, or for short a *section*, of T if the following conditions are satisfied:

1) $\mathcal{S}$ is a closed subset of X;

2) to each point $x \in T$ there corresponds a unique number t_x, with $| t_x | \leqq 2\alpha$, such that $xt_x \in \mathcal{S}$ and $x[0, t_x] \subset T$.

The two following theorems (see [5], pp. 333–35 for the proofs) ensure the existence of tubes with a section.

THEOREM I. *Let $x_0 \in X$ be a regular point. Then, for a sufficiently small $\alpha > 0$, there exists $\delta > 0$ such that the tube $T(x_0, \delta, \alpha)$ has a section $\mathcal{S}$ which contains x_0.*

THEOREM II. *Let $x_0 \in X$ be a regular point. If α is any positive number and $\alpha < \tau/4$ when x_0 is periodic with period τ, then there exists $\delta > 0$ such that the tube $T(x_0, \delta, \alpha)$ has a section $\mathcal{S}$ which contains x_0.*

Let T be a tube with a section $\mathcal{S}$. The relation defined for points of $T: x \sim y$, if and only if $xt_x = yt_y$, is clearly an equivalence relation. It is easy to see that each element of the associated partition is an arc which has a unique point in common with $\mathcal{S}$ and whose time length is a number in the interval $[-4\alpha, 4\alpha]$. We say that $x \in \mathcal{S}$ is *interior* to $\mathcal{S}$ if x is an interior point of T. It is also easy to see that if $x \in S(x_0, \delta)$, then xt_x is interior to $\mathcal{S}$.

LEMMA 3. *If $\mathcal{S}$ is a section of the tube $T(x_0, \delta, \alpha)$ and $x[a, b]$ is any arc, then there are only a finite number of values of t,* in the interval $[a, b]$, *such that $xt \in \mathcal{S}$.*

Proof. In fact, $x[a, b] \cap T$ is a disjoint union of arcs contained in T whose time lengths, with two possible exceptions, are contained in the interval $[-4\alpha, 4\alpha]$. Since each one of these arcs has exactly one point in common with $\mathcal{S}$, the lemma follows.

COROLLARY. *A periodic orbit intersects a section $\mathcal{S}$ of a tube T in only a finite number of distinct points.*

LEMMA 4. *Let $\mathcal{S}$ be a section of a tube $T(x_0, \delta, \alpha)$. Then:*

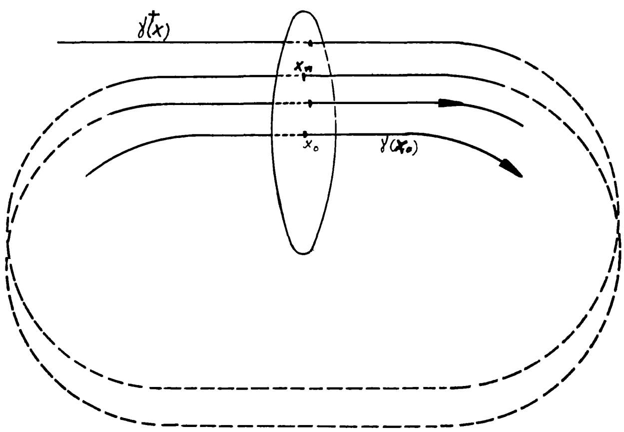
i) the function $T \to [-2\alpha, 2\alpha]$, which associates to each $x \in T$ the unique number t_x such that $xt_x \in \mathcal{S}$ and $x[0, t_x] \subset T$, is continuous;

ii) the function $T \to \mathcal{S}$, which associates to each $x \in T$ the point $xt_x \in \mathcal{S}$, is continuous.

Proof. Let $\{x_n\}$ be a sequence of points of T convergent to $x^* \in T$. We are going to prove that the sequence $t_n = t_{x_n}\}$ converges to $t^* = t_{x^*}$. Since $\{t_n\} \subset [-2\alpha, 2\alpha]$, we know that every limit point of $\{t_n\}$ is contained in $[-2\alpha, 2\alpha]$. Let t' be a limit point of $\{t_n\}$ and $\{t_{n_k}\}$, the corresponding subsequence. The continuity of π implies that $x_{n_k}t_{n_k} \to x^*t'$. Since $\mathcal{S}$ is closed, then $x^*t' \in \mathcal{S}$. Besides, it is clear that $x^*[0, t'] \subset T'$ Thus t' must be equal to t^*. This proves that $t_n \to t^*$. The proof of ii) follows easily from the continuity of t_x and that of π.

4. The strong limit point

Let $\gamma(x)$ be an orbit, and suppose there exists a regular point $x_0 \in \Lambda^+(x)$. We then know that there exists a tube $T(x_0, \delta, \alpha)$, with a section $\mathcal{S}$, which contains x_0. Our purpose is to obtain information about the orbit $\gamma(x_0)$ by means of the study of the set $\gamma^+(x) \cap \mathcal{S}$ (see fig. 1).



Lemma 5. *Let x_0 be a regular point of the limit set $\Lambda^+(x)$ of a certain orbit $\gamma(x)$, and let $\mathcal{S}$ be a section through x_0. Then we have:*

i) $\gamma^+(x) \cap \mathcal{S} = \{x_n\}$, where $x_n = xt_n$ with $\{t_n\} \subset R^+$ and $t_n \to \infty$ monotonically.

ii] there exists a subsequence of $\{x_n\}$ which converges to x_0. Besides, if x' is any other point of $\Lambda^+(x)$ interior to $\mathcal{S}$, then there also exists a subsequence of $\{x_n\}$ which converges to x'.

Proof. Let $\mathcal{S}$ be the section of $T(x_0, \delta, \alpha)$ through x_0. Since $x_0 \in \Lambda^+(x)$, for every $t_1 > 0$ there exists $t > t_1$ such that $xt \in S(x_0, \delta)$. Each one of the points $xt \in S(x_0, \delta)$ determines a point $xt' \in \mathcal{S}$. Thus there exists an unbounded, infinite set of positive numbers t' such that $xt' \in \mathcal{S}$. Considering $\gamma^+(x)$ as a denumerable union of arcs with time length l, we see that, because of Lemma 4, the collection $\{t'\}$ can be ordered in such a way that the resulting sequence $\{t_n\}$tends monotonically to $+\infty$ and $\gamma^+(x) \cap \mathcal{S}$ is exactly the sequence $\{x_n = xt_n\}$. Part ii) of the lemma follows from the continuity of the function defined in part ii) of Lemma 4.

Definition 1. *We say that a regular point $x_0 \in X$ is a (positive) strong limit point of $\gamma(x)$ if the following two conditions are satisfied:*

1) x_0 is a (positive) limit point of $\gamma(x)$ (i.e., $x_0 \in \Lambda^+(x)$);

2) there exists a section $\mathcal{S}$ through x_0 such that the sequence $\{x_n\} = \gamma^+(x) \cap \mathcal{S}$ converges to x_0.

Remark. If x_0 is a strong limit point of $\gamma(x)$, then we have associated with x_0

two sequences $\{x_n\}$ and $\{t_n\}$ which, we recall, are such that $\{x_n\} = \gamma^+(x) \cap \mathcal{S}$ and $x_n = xt_n$, with $t_n \to +\infty$ monotonically, and $x_n \to x_0$. From the sequence $\{t_n\}$ we derive the sequence $\{\tau_n = t_{n+1} - t_n\}$ of positive numbers, where τ_n is the time needed to go from x_n to x_{n+1} through the orbit $\gamma(x)$; i.e., $x_n\tau_n = x_{n+1}$. The sequences $\{x_n\}$ and $\{\tau_n\}$ play a fundamental role in our study.

Proposition 1. *If x_0 is a strong limit point of $\gamma(x)$, then $\gamma(x_0)$ is an open subset of* $\Lambda^+(X)$.

Proof. Let $\mathcal{S} \subset T(x_0, \delta, \alpha)$ be the section through x_0 that by hypothesis exists. Let us consider the unique arc of T, say $x_0[a, b]$, that contains x_0. We state that $S(x_0, \delta) \cap \Lambda^+(x) = x_0[a, b]$. If this is not the case, then we have a point y in $S(x_0, \delta) \cap \Lambda^+(x)$ but not in $x_0[a, b]$. Now, $y \in S(x_0, \delta)$ implies that yt_y is interior to $\mathcal{S}$; moreover, $yt_y \neq x_0$ and $yt_y \in \Lambda^+(x)$. Thus, by Lemma 5, part ii), there exists a subsequence of $\{x_n\}$ which converges to yt_y. But this contradicts the fact that x_0 is a strong limit point of $\gamma(x)$ and proves that x_0 is an interior point of $\gamma(x_0) \subset \Lambda^+(x)$. By continuity in the initial condition, it follows easily that each point of $\gamma(x_0)$ is interior, i.e., that $\gamma(x_0)$ is an open subset of $\Lambda^+(x)$.

Proposition 2. *If x_0 is a strong limit point of $\gamma(x)$ and $\gamma(x_0)$ is not periodic, then the function $\pi_{x_0}: R \to \gamma(x_0)$ is a homeomorphism.*

Proof. Since the regular point x_0 is not a periodic one, then π_{x_0} is one-to-one. It remains only to show that $\pi_{x_0}^{-1}$ is continuous. Let $\{t_n\} \subset R$ be a sequence such that $x_0t_n \to x_0$; then there exists N(positive integer) such that $n \geqq N$ implies $x_0t_n \in x_0[a, b]$ (see the proof of Proposition 1) and, therefore, $\{t_n\} \subset [a, b]$. By the continuity of π it follows easily that $t_n \to 0$. Applying now Lemma 1, we obtain that π_{x_0} is a homeomorphism.

Proposition 3. *Let x_0 be a strong limit point of $\gamma(x)$, and suppose that $\gamma(x)$ is L^+-stable. Then, if $\gamma(x_0)$ is periodic, $\Lambda^+(x) \equiv \gamma(x_0)$.*

We recall that if $\gamma(x)$ is L^+-stable, then $\Lambda^+(x)$ is connected. Now, since $\gamma(x_0)$ is periodic, we know that $\gamma(x_0)$ is a closed subset of $\Lambda^+(x)$. Besides, by Proposition 1, $\gamma(x_0)$ is an open subset of $\Lambda^+(x)$. But this is possible if and only if $\Lambda^+(x) \equiv \gamma(x_0)$.

Remark. Let $\gamma(x)$ be an L^+-stable orbit, and suppose x_0 is a strong limit point of $\gamma(x)$. Then Propositions 2 and 3 tell us that there are, topologically speaking, two possibilities only for $\gamma(x_0)$: either $\gamma(x_0)$ is homeomorphic to R (the homeomorphism being given by π_{x_0}), or $\gamma(x_0)$ is homeomorphic to the circle (is periodic) and, in this case, $\Lambda^+(x) \equiv \gamma(x_0)$. We are interested in this second case. The following theorems refer to it.

Theorem 1. *Let x_0 be a strong limit point of $\gamma(x)$, and suppose that $\gamma(x)$ is L^+-stable. Then the following statements are equivalent:*

A) *$\gamma(x_0)$ is a periodic orbit and $\Lambda^+(x) \equiv \gamma(x_0)$;*

B) *the sequence* $\{\tau_n = t_{n+1} - t_n\}$ *is bounded*;
C) x_0 *is (positively) recurrent (i.e.,* $x_0 \in \Lambda^+(x)$*).*

Proof. Since x_0 is a strong limit point of $\gamma(x)$, there exists a section $\mathcal{S}$ through x_0 of certain tube $T(x_0, \delta, \alpha)$, such that $\gamma^+(x) \cap \mathcal{S} = \{x_n\}$, where $x_n = xt_n$, $t_n \to +\infty$ monotonically, and $x_n \to x_0$. $A \Rightarrow C$ is obvious.

$B \Rightarrow A$. Since by hypothesis the sequence $\{\tau_n\}$ is bounded, then it has at least a limit point τ (clearly $\tau > 0$) to which a subsequence $\{\tau_{n_k}\}$ converges. Now, by the continuity of π, we have that $x_{n_k}\tau_{n_k} \to x_0$. But $x_{n_k}\tau_{n_k} = x_{n_k+1}$, and $x_{n_k+1} \to x_0$. Thus $x_0\tau = x_0$; i.e., $\gamma(x_0)$ is a periodic orbit. Moreover, it follows from Proposition 3 that $\Lambda^+(x) \equiv \gamma(x_0)$.

$C \Rightarrow A$. Since x_0 is (positively) recurrent, given $t_1 \in R^+$, there exists $t_1' > t_1$ such that $x_0t_1' \in S(x_0, \delta)$. To the point x_0t_1' there corresponds under the function t_x a point x_0t_0 interior to $\mathcal{S}$. It follows that $x_0t_0 = x_0$, since otherwise x_0 would not be a strong limit point of $\gamma(x)$ (see part ii of Lemma 5). Moreover it follows from propositihon 3 that $\Lambda^+(x) \equiv \gamma(x_0)$.

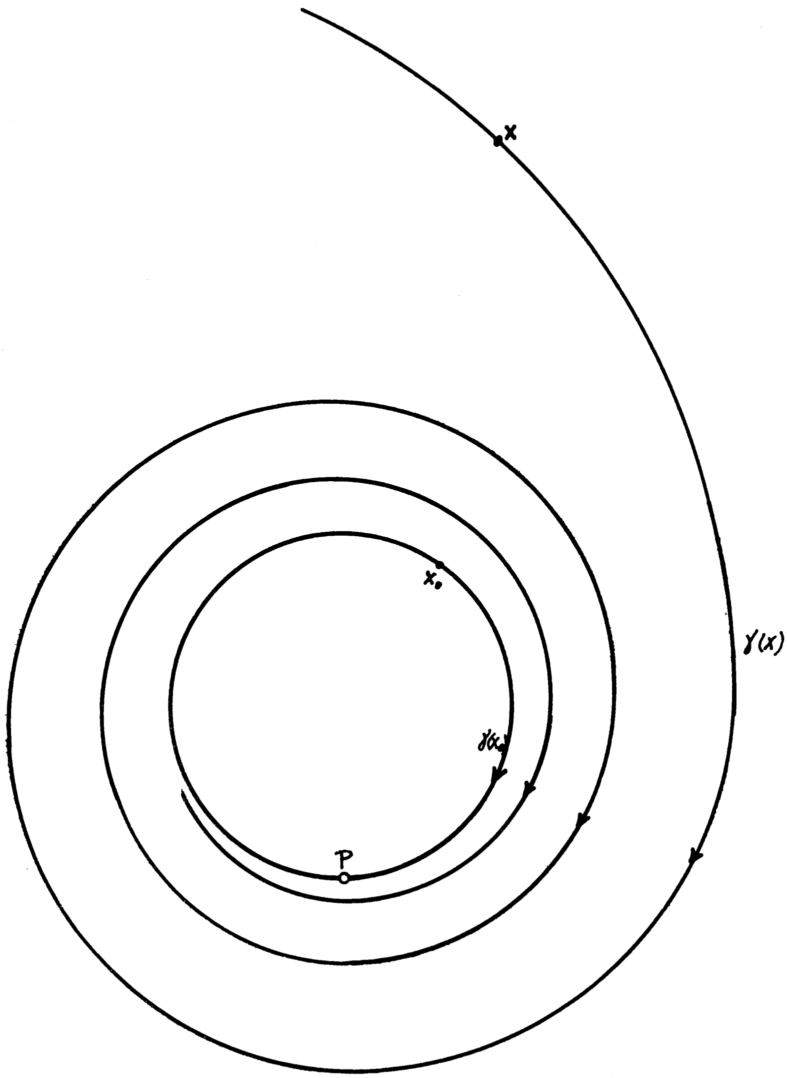
$A \Rightarrow B$. Let τ be the period of $\gamma(x_0)$. Given $\epsilon > 0 (\epsilon < \delta)$, there exists $\eta < \epsilon$ such that $d(x, x_0) < \eta$ implies $d(x\tau, x_0\tau) = d(x\tau, x_0) < \epsilon$ (this follows from the continuity of π). Since $x_n \to x_0$, from certain N (positive integer) on, $x_n \in S(x_0, \eta)$ and, therefore, $d(x_n\tau, x_0) < \epsilon$. The map $T \to \mathcal{S}$ associates with $x_n\tau$ a point $x_n\tau_n' \in \mathcal{S}$, where clearly $\tau_n' - \tau \to 0$ and, therefore, $\{\tau_n'\}$ is bounded. On the other hand since $\gamma^+(x) \cap \mathcal{S}$ is exactly the sequence $\{x_n\}$, we must have $\tau_n \leqq \tau_n'$ for each $n \geqq N$. This proves that $\{\tau_n\}$ is bounded.

THEOREM 2. *Let $\gamma(x)$ be an L^+-stable orbit. Then $\Lambda^+(x)$ consists of a unique orbit which is periodic if and only if each point of $\Lambda^+(x)$ is a strong limit point of $\gamma(x)$.*

Proof. Suppose that $\Lambda^+(x) \equiv \gamma_0$ and that γ_0 is periodic with period τ. Let x_0 be any point of γ_0. We want to see that x_0 is a strong limit point of $\gamma(x)$. By the Corollary to Lemma 3, we know that if $\mathcal{S}$ is a section through x_0, then $\gamma(x_0) \cap \mathcal{S} = \Lambda^+(x) \cap \mathcal{S}$ consists of a finite number of distinct points and, because of Theorem II (of §3), we can select the section $\mathcal{S}$ in such a way that $\gamma(x_0) \cap \mathcal{S}$ is exactly the point x_0. Let us consider now the sequence $\{x_n\} = \gamma^+(x) \cap \mathcal{S}$. We state that $x_n \to x_0$. If this is not the case, then, since $\gamma(x)$ is L^+-stable and $\mathcal{S}$ is closed, $\{x_n\}$ has a limit point $x_0' \neq x_0$, with $x_0' \in \Lambda^+(x) \cap \mathcal{S}$. This proves that x_0 is a strong limit point.

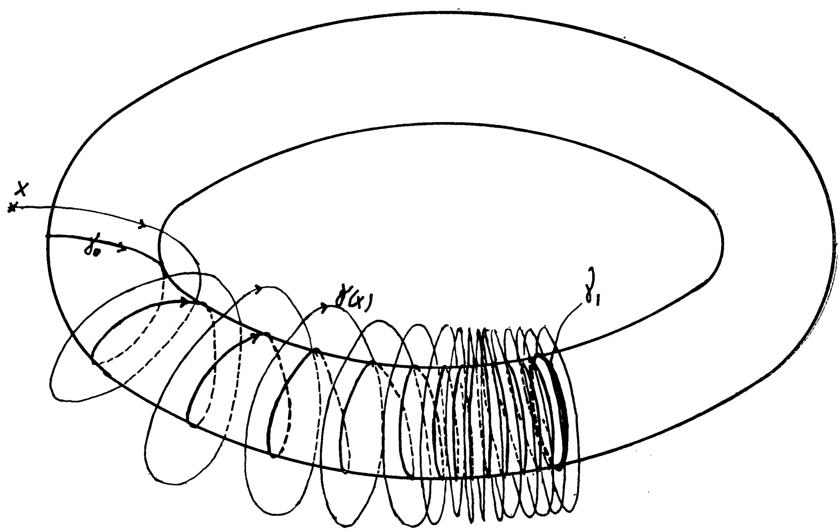
Suppose now that each point of $\Lambda^+(x)$ is a strong limit point of $\gamma(x)$. Let $x_0 \in \Lambda^+(x)$. By Proposition 1 we know that $\gamma(x_0)$ is an open subset of $\Lambda^+(x)$. We are going to show that $\gamma(x_0)$ is also a closed subset of $\Lambda^+(x)$. In fact, if this is not the case, we have in $\Lambda^+(x)$ a limit point x_0' of $\gamma(x_0)$ with $x_0' \notin \gamma(x_0)$. But now any section $\mathcal{S}$ through x_0' contains in its interior points of $\gamma(x_0)$, and this contradicts the fact that x_0' is a strong limit point of $\gamma(x)$. Thus $\gamma(x_0)$ is closed in $\Lambda^+(x)$. Now, since $\Lambda^+(x)$ is connected, $\Lambda^+(x) \equiv \gamma(x_0)$ and $\gamma(x_0)$ is periodic as well.

Example 1. Let F be a continuous flow in the plane R^2. Let $P \in R^2$ be a critical point and $\gamma(x_0)$, an orbit such that $\Lambda^+(x_0) = \Lambda^-(x_0) = \{P\}$. Finally, let $\gamma(x)$ be a spiral orbit such that $\Lambda^+(x) = \gamma(x_0) \cup \{P\}$ (see fig. 2).



In this example the orbit $\gamma(x)$ is L^+-stable, and x_0 is a strong limit point of $\gamma(x)$. The orbit $\gamma(x_0)$ is not periodic, and therefore it is homeomorphic with R (cf. Prop. 2). Observe that none of the statements A, B, or C of Theorem 1 is satisfied. Finally, as P is critical it can not be a strong limit point of $\gamma(x)$ (cf. Th. 2).

Example 2. Let F be a continuous flow in the space R^3. Suppose that there exists an invariant torus M which contains a periodic orbit γ_1 and is such that, for every point $x \in M$, $\Lambda^+(x) = \Lambda^-(x) = \gamma_1$. Finally, suppose that there exists $x \in M$ such that $\Lambda^+(x) = \gamma_0 \cup \gamma_1$ (see fig. 3).



Here, $\gamma(x)$ is L^+-stable and each point x_0 of γ_0 is a strong limit point of $\gamma(x)$. The orbit $\gamma(x_0)$ is not periodic and therefore is homeomorphic with R (cf. Prop. 2). None of the statements of Theorem 1 is satisfied. Finally, no point of γ_1 is a strong limit point of $\gamma(x)$ (cf. Th. 2).

THEOREM 3. *Let $\gamma(x)$ be an L^+-stable orbit. Suppose that there exists $x_0 \in \Lambda^+(x)$ which is a strong limit point of $\gamma(x)$ and is such that $\gamma(x_0)$ is (positively) orbitally stable. Then $\gamma(x_0)$ is periodic, and $\Lambda^+(x) \equiv \gamma(x_0)$.*

Proof. Let $\mathcal{S}$ be a section through x_0 such that $\gamma^+(x) \cap \mathcal{S} = \{x_n\}$, where $x_n = xt_n$, $t_n \to +\infty$ monotonically, and $x_n \to x_0$ (see Definition 1). Let $\{t_{n_k}\}$ be a subsequence of $\{t_n\}$ constructed by the property that $t_{n_k} - t_k > k$. The sequence $\{\tau_{n_k} = t_{n_k} - t_k\}$ is, by construction, such that $\tau_{n_k} \to +\infty$. Now since $x_k \to x_0$, $\tau_{n_k} \to +\infty$, and $x_k\tau_{n_k} = x_{n_k} \to x_0$, we have that x_0 is non-wandering. Applying now Lemma 2 we obtain that x_0 is positively recurrent. The theorem follows from $C \Rightarrow A$ in Theorem 1.

5. Strong flows

In §4 we defined the (positive) strong limit point of an orbit. It is clear that we can analogously define the (negative) strong limit point of an orbit. Let us now define a more restrictive concept which will permit us to define a special kind of flows.

DEFINITION 2. *Let F be a continuous flow on X. We say that a regular point $x_0 \in X$ is a strong limit point (with respect to F) if it is the limit point of some orbits* *and if, for every orbit* $\gamma(x)$ *such that* $x_0 \in \Lambda^+(x)$ $(x_0 \in \Lambda^-(x))$, *we have that* x_0 *is a positive (negative) strong limit point of* $\gamma(x)$.

DEFINITION 3. *We say that a continuous flow* F *on* X *is* strong *if each limit point which is not critical is a strong limit point (with respect to* F*).*

The three following theorems refer to a strong continuous flow F on X. They will follow easily from Theorems 1, 2, and 3 of §4.

THEOREM 4. *Let* F *be a strong continuous flow on* X. *If* $x_0 \in X$ *is (positively) recurrent, then* x_0 *is periodic.*

Proof. By hypothesis, $x_0 \in \Lambda^+(x_0)$ and, therefore, x_0 is a (positive) strong limit point of $\gamma(x)$. The proof follows now as in part $C \Rightarrow A$ of Theorem 1.

THEOREM 5. *Let* F *be a strong continuous flow on* X, *and let* $\gamma(x)$ *be an* L^+*-stable orbit. If* $\Lambda^+(x)$ *does not contain critical points, then* $\Lambda^+(x)$ *consists of a unique orbit* γ *which is periodic.*

Proof. Since, by hypothesis, $\Lambda^+(x)$ does not contain critical points, then each point of $\Lambda^+(x)$ is a (positive) strong limit point of $\gamma(x)$. The proof follows now as that of the sufficiency of Theorem 2.

THEOREM 6. *Let* F *be a strong continuous flow on* X. *Suppose that* $\gamma(x)$ *is an* L^+*-stable orbit and that there exists a regular point* $x_0 \in \Lambda^+(x)$ *such that* $\gamma(x_0)$ *is (positive) orbitally stable. Then* $\gamma(x_0)$ *is periodic and* $\Lambda^+(x) \equiv \gamma(x_0)$.

Proof. The point x_0 is regular, and so it is a (positive) strong limit point of $\gamma(x)$. The proof is now the same of that of Theorem 3.

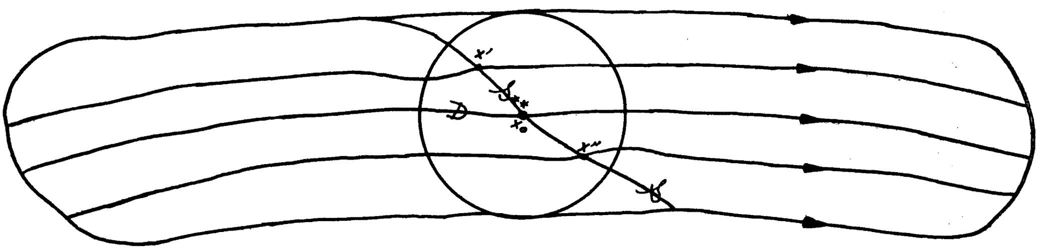
Remark. If F is a strong continuous flow on the plane R^2, then the Theorems 4 and 5 are known as theorems of Bohr-Fenchel (see [2]) and Poincaré-Bendixon, respectively. We can say then that, for a strong continuous flow on a metric space X, the theorems of Bohr-Fenchel, Poincaré-Bendixon, and Theorem 6 as well apply.

In relation to the plane R^2, we are going now to prove that any continuous flow on it, is strong. Let F be a continuous flow on R^2. In order to prove that F is strong, it is enough to prove that if $x_0 \in R^2$ is a positive (negative) limit point of $\gamma(x)$ and x_0 is not critical, then x_0 is a positive (negative) strong limit point of $\gamma(x)$. Suppose for a moment that we have shown the following: for every regular point $x_0 \in R^2$ there exists a section $\mathcal{S}$ through x_0 such that $\mathcal{S}$ is a simple arc, i.e., the topological image of the closed interval [0, 1]. Then, by using classical arguments of the Poincaré-Bendixon theory, it is easy to prove that if x_0 (regular point) is a (positive) limit point of $\gamma(x)$, then the sequence $\{x_n\} = \gamma^+(x) \cap \mathcal{S}$, where $\mathcal{S}$ is a simple arc, converges to x_0 ; i.e., x_0 is a (positive) strong limit point of $\gamma(x)$. Thus in order to prove that every continuous flow on the plane is strong we have to establish the existence of sections which are simple arcs.

Let $A \subset R^2$ be a connected set. $x \in A$ is a *cut point* if $A - \{x\}$ is not connected. $A \subset R^2$ is a continuum if it is compact, connected, and contains at least two points. A subset L of $B(x_0, \delta) \subset R^2$ is a *crosscut* of $B(x_0, \delta)$ if L is a simple arc such that all its points, with the exception of the two ends which belong to the frontier, are contained in $S(x_0, \delta)$. It is easy to see that a crosscut decomposes $B(x_0, \delta)$ in two connected components, each one of which is a 2-cell. The following theorem characterizes a simple arc (see [6], Chap. IV):

THEOREM. *A continuum $A \subset R^2$, with the property that all its points, with two possible exceptions, are cutpoints, is a simple arc.*

PROPOSITION 4. *Let F be a continuous flow on R^2. Then, for any regular point $x_0 \in R^2$, there exists a section $\mathcal{S}$ which is a simple arc (see fig. 4).*



Proof. Let x_0 be a regular point and $\mathcal{S}$, a section of a certain tube $T(x_0, \delta, \alpha)$. Since the function $T \to \mathcal{S}$ is continuous and the image of $B(x_0, \delta)$ is $\mathcal{S}$, we know that $\mathcal{S}$ is a continuum. Let $\mathcal{S}_1$ be the connected component of $\mathcal{S} \cap B(x_0, \delta)$ which contains x_0. We are going to show that certain subset $\mathcal{S}^*$ of $\mathcal{S}_1$ is a simple arc. It is easy to see that every point $x \in S(x_0, \delta)$ determines a crosscut L_x of $B(x_0, \delta)$. Let B' and B'' be the two components of $B(x_0, \delta)$ given by the crosscut L_{x_0}, and let $\mathcal{S}_1'$, $\mathcal{S}_1''$ be the intersections of $\mathcal{S}_1$ with B' and B'' respectively. It is clear that $\mathcal{S}_1'$ and $\mathcal{S}_1''$ are non-empty. Let us take $x' \in \mathcal{S}'_1$ and $x'' \in \mathcal{S}_1''$. The two crosscuts $L_{x'}$ and $L_{x''}$ determine a 2-cell D whose frontier consists of $L_{x'}$ and $L_{x''}$ and two simple arcs contained in the frontier of $B(x_0, \delta)$. Let $\mathcal{S}^* = \mathcal{S}_1 \cap D$. If x is a point of $\mathcal{S}^*$ different from x' and x'', it is clear that the crosscut L_x disconnects D and, therefore, also $\mathcal{S}^*$. This proves that $\mathcal{S}^*$ is a simple arc.

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