DIFFERENTIABLE NORMS

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1. Introduction

A formal theory of manifolds modeled on general Banach spaces has been developed by S. Lang [6]. A fundamental tool in the study of manifolds modeled on finite dimensional spaces—at least in its geometrical aspects—is the existence of partitions of unity, say of class C^p for $p \geqq 1$. The construction of partitions of unity, say of class C^p for $p \geqq 1$, is still possible in the case of manifolds modeled on separable Hilbert spaces (see [6], p. 30). This construction is based on the existence of an equivalent norm of class C^∞ in a Hilbert space. A systematic study of partitions of unity in infinite dimensional manifolds is to be found in [9]. In this paper we study the problem of existence of equivalent differentiable norms in Banach spaces. The main result is Theorem 3: *A separable Banach space admits an equivalent norm of class C_1 if and only if its dual is separable.* This result has been announced in [10].

2. Differential calculus in Banach spaces

The results stated and proved in this section are to be found in [4]. They are included for ease of reference.

Let (X, α) and (Y, β) be Banach spaces. Let $A \subset X$ be an open set, and let $f:A \to Y$ be continuous. We say that *f is differentiable* at $x_0 \in A$ if there exists a linear map $u(x_0):X \to Y$ such that

$$\lim_{\substack{x \to x_0 \\ x \neq x_0}} \frac{\beta(f(x) - f(x_0) - u(x_0)\cdot(x - x_0))}{\alpha(x - x_0)} = 0.$$

We say that *f is differentiable on A* if f is differentiable at each point $x \in A$. The linear map $u(x):X \to Y$ is called a differential of f at x. If f is differentiable at $x_0 \in A$, we will write $h(x, x_0) = f(x) - f(x_0) - u(x_0)\cdot(x - x_0)$. Thus, if f is differentiable at x_0, $\beta(h(x, x_0))/\alpha(x - x_0) \to 0$ as $x \to x_0$ and

$$f(x) = f(x_0) + u(x_0)\cdot(x - x_0) + h(x, x_0).$$

The definition of differential depends only on the topologies of X and Y.

PROPOSITION 1. *If f is differentiable at x_0, then f has a unique differential, denoted by $f'(x_0)$, given by*

$$f'(x_0)\cdot x = \lim_{\substack{\lambda \to 0 \\ \lambda \neq 0}} \frac{f(x_0 + \lambda x) - f(x_0)}{\lambda}.$$

Proof. $h(x_0 + \lambda x, x_0)/\lambda = [f(x_0 + \lambda_x) - f(x_0) - \lambda\cdot f'(x_0)\cdot(x)]/\lambda$, and $\lim_{\lambda\to 0} [f(x_0 + \lambda x) - f(x_0)]/\lambda = f'(x_0)\cdot x$.

PROPOSITION 2. *If f is differentiable at x_0, then $f'(x_0)$ is a continuous linear map from X into Y.*

Proof. By definition $f'(x_0)$ is linear, so we have to prove only continuity at x_0. From the definition of $h(x, x_0)$ one has $\beta(f'(x_0)\cdot(x) - f'(x_0)\cdot x_0) = \beta(f'(x_0)\cdot(x - x_0)) \leqq \beta(f\cdot x - f\cdot x_0) + \beta(h(x, x_0))$. Let $\epsilon > 0$, and find $0 < \delta(\epsilon) < 1$ such that $\beta(f(x) - f(x_0)) < \epsilon/2$ and $\beta(h(x, x_0)) < \epsilon/2\alpha(x - x_0)$ if $\alpha(x - x_0) < \delta(\epsilon)$. Then $\beta(f'(x_0)\cdot x - f'(x_0)\cdot x_0) < \epsilon$ if $\alpha(x - x_0) < \delta(\epsilon)$. This concludes the proof.

If $(X_1, \alpha_1), \cdots, (X_n, \alpha_n)$ are Banach spaces, then $X_1 \times X_2 \times \cdots \times X_n$ becomes a Banach space if we define $\alpha(x_1, \cdots, x_n) = \text{Sup}\,(\alpha_1(x_1), \cdots, \alpha_n(x_n))$.

PROPOSITION 3. *Let (X, α) and (Y, β) be Banach spaces. Then,*

1) *if $A \subset X$ is open and $f: A \to Y$ is a constant, $f'(a) = 0$ for each $a \in A$;*

2) *if $f: X \to Y$ is a continuous linear map, f is every where differentiable and $f'(x) = f$ for each $x \in X$.*

3) *if (X_1, α_1), (X_2, α_2) are Banach spaces and $f: X_1 \times X_2 \to Y$ is a continuous bilinear map, f is everywhere differentiable and $f'(x_1, x_2)\cdot(t_1, t_2) = f(x_1, t_2) + f(t_1, x_2)$.*

Proof. The proof follows from Proposition 1.

PROPOSITION 4. *Let (X, Y, Z) be Banach spaces, and let $A \subset X$ and $B \subset Y$ be open sets. Let $f: A \to Y$ be differentiable at $a \in A$, and let g be differentiable at $f(a) \in B$. Then $g\cdot f: A \to Z$ is differentiable at $a \in A$, and $(g\cdot f)'(a) = g'(f(a))\cdot f'(a)$.*

Proof. See [4], page 145.

Let (X, α) and (Y, β) be Banach spaces, and denote by $\mathcal{L}(X, Y)$ the linear space of all continuous linear maps from X into Y. The norms α and β define a norm $\| u \| = \text{Sup}_{\alpha(x) \leqq 1}\, \beta(u\cdot x)$ in $\mathcal{L}(X, Y)$. Denote by $\mathcal{L}_n(X, Y)$ the space of all continuous multilinear maps from $X_1 \times X_2 \times \cdots \times X_n$, where $X_1 = \cdots = X_n = X$, into Y. Then there is a natural isomorphism between $\mathcal{L}_2(X, Y)$ and $\mathcal{L}(X, \mathcal{L}(X, Y))$. Inductively, one could define a natural isomorphism between $\mathcal{L}_n(X, Y)$ and $\mathcal{L}(X_1, \mathcal{L}(X_2, \cdots, \mathcal{L}(X_n, Y))$, where $X_1 = X_2 = \cdots = X_n = X$. We recall that $\mathcal{L}_2(X, Y) \overset{h}{\leftrightarrow} \mathcal{L}(X, \mathcal{L}(X, Y))$ is defined by $(h(u)(x_1))(x_2) = u(x_1, x_2)$.

Let X and Y be Banach spaces; let $A \subset X$ be an open set; and let $f: A \to Y$ be continuous. We say that f *is of class C^1* is $f'(a)$ exists for each $a \in A$ and the map $f': A \to \mathcal{L}(X, Y)$ is continuous. Inductively, we say that f is of class C^p, $p \geqq 1$ if $f^{(p-1)}: A \to \mathcal{L}_{p-1}(X, Y)$ is of class C^1, and we write $f^{(p)} = (f^{(p-1)})'$.

PROPOSITION 5. *Let X and Y be Banach spaces; $A \subset X$, an open set; and $f: A \to Y$, a map of class C^p, $p \geqq 2$. Then the multilinear map $(t_1, t_2, \cdots, t_p) \to f^{(p)}(a)\cdot(t_1, t_2, \cdots, t_p)$ is symmetric for each $s \in A$; that is, $f^{(p)}(a)(t_1, \cdots, t_p) = f^{(p)}(a)(t_{\sigma(1)}, \cdots, t_{\sigma(p)})$ for any permutation σ of the set of indices $\{1, 2, \cdots, p\}$.*

Proof. See [4], pages 176–77.

PROPOSITION 6 (Taylor's formula). *Let X and Y be Banach spaces; $A \subset X$, an open set; $[x, x + t]$, a segment contained in A; and $f:A \to Y$, a map of class C^p, $p \geqq 1$. Then,*

$$f(x+t) = f(x) + \frac{f'(x)}{1!}t + \frac{f^{(2)}(x)}{2!}\cdot t^{(2)} + \cdots + \frac{f^{(p)}}{p!}t^{(p)} + \theta(t),$$

where $t^{(n)} = (t_1, \cdots, t_n)$, $t_1 = \cdots = t_n = t$, and $\theta(t)$ satisfies $\lim_{t\to 0} \theta(t)/\| t \|^p = 0$ ($\| \ \|$ is the norm in X).

Proof. See [6], page 186.

3. Differentiable norms

In this section we will always talk about continuous norms in a Banach space X. A norm $\beta:X \to R$ is said to be *differentiable*, or of class C^p, $p \geqq 1$, if $\beta:X - \{0\} \to R$ is differentiable, or of class C^p. In particular, we will sometimes talk about differentiability at a point, or of a norm's being of class C^p at a point. An *inner product norm* is any continuous norm derived from an inner product.

PROPOSITION 7. *Assume that β is an inner product norm. Then β is of class C^∞ and $\beta'(x)\cdot u = (x\cdot u)/\beta(x)$.*

Proof. Let $\beta(x) = (x\cdot x)^{1/2}$ and $f(x) = x\cdot x$; and let $d:X \to X \times X$ be defined by $d(x) = (x, x)$. Clearly d is a continuous linear isomorphism. The map $g(t) = t^{1/2}$ is of class C^∞ except at $t = 0$, so $\beta = g\cdot f\cdot d$ is of class C^∞. A simple argument shows that $\beta'(x)\cdot u = (x\cdot u)/\beta(x)$.

PROPOSITION 8. *Let (X, α) be a separable Banach space. Then there is a continuous norm of class C^∞ defined in X (in general, of course, not equivalent to α).*

Proof. Let $\{f_j\}$ be the sequence in X^* such that $\alpha^*(f_j) = 1$, for all j, and $f_j(x) = 0$, for all j, imply $x = 0$. Define $T:X \to l^2$ by $T\cdot x = (f_j\cdot x)$. Then $p(x) = \| Tx \|_{l^2}$ is a continuous norm of class C^∞, since it is the composition of a linear map T and an inner product norm $\| \ \|_{l^2}$.

PROPOSITION 9. *If a norm β in (X, α) is differentiable at each point $x \in S_\beta$, then β is differentiable and $\beta'(\lambda x) = \beta'(x)$ for each real $\lambda > 0$.*

Proof. Since $\lambda > 0$, then

$$\frac{\beta(\lambda x + u) - \beta(\lambda x) - \beta'(x)\cdot(u)}{\alpha(u)} = \frac{\beta\left(x + \frac{u}{\lambda}\right) - \beta(x) - \beta'(x):\left(\frac{u}{\lambda}\right)}{\alpha\left(\frac{u}{\lambda}\right)}.$$

Since the second term of the equality approaches zero as $u \to 0$, it follows that $\beta'(x) = \beta'(\lambda x)$.

4. Smooth norms

A linear space with a topology defined by a norm α will be written (X, α) and the dual space, (X^*, α^*). Let β be a continuous norm in X, and let $S_\beta = \{x \mid \beta(x) = 1\}$.

i) $f \in X^*$ is a normalized *support functional* at $x \in S_\beta$ if $\alpha^*(f) = 1$ and $\sup_{\beta(y)=1} f\cdot y = f(x)$.

ii) β is *smooth* if there is a unique normalized support functional at each $x \in S_\beta$.

iii) β is *rotund* if S_β contains no line segments.

The following proposition is well known.

PROPOSITION 10. *Let (X, α) be a normed space, and let β be a norm equivalent to α.* Then,

a) if β^* is rotund, β is smooth;

b) if β^* is smooth, β is rotund.

Proof. a) Assume that β is not smooth. Then, for some $x_0 \in S_\alpha$, there are two normalized support functionals f_1 and f_2 at x_0, and $(f_1 + f_2)/2$ would also be a normalized support functional at x_0. This is a contradiction because β^* is rotund.

b) Assume that β is not rotund. Then, for some x_0, $x_1 \in S_\beta$, $x_t = tx_1 + (1 - t)x_0 \in S_\beta$, $0 \leqq t \leqq 1$. By the Hahn-Banach theorem, there is a normalized support functional f such that $f\cdot x_t = 1$ for all $0 \leqq t \leqq 1$. But then x_t would be a normalized support functional at f for all $0 \leqq t \leqq 1$. This is a contradiction, since β^* is smooth.

PROPOSITION 11. *a) A continuous norm β is smooth if and only if β is smooth in any planar section through the origin.*

b) A continuous norm β is rotund if and only if, for any x_1, x_2 such that $\beta(x_1) = \beta(x_2) = 1$, one has $\beta(x_1 + x_2)/2 < 1$ if $x_1 \neq x_2$.

Proof. The proof is indicated in [3]; we give it here for completeness.

a) Assume that α is smooth. If α is not differentiable in some planar section P, then $P \cap S_\alpha$ is a convex curve in a plane having two tangents, say y_1 and y_2, at some point $x \in P \cap S_\alpha$ (assume that $\alpha(y_1) = \alpha(y_2) = 1$). Define $f_i: P \to R$ by $f_i(y_i) = 0$, $f_i(x) = 1$, and $f_i(\lambda y_i + \mu x) = \mu f_i(x)$, $i = 1, 2$. By the Hahn-Banach extension theorem, f_i has an extension g_i such that $\alpha^*(f_i) = \alpha^*(g_i)$, $i = 1, 2$. If $z = \lambda y_i + \mu\cdot x$ and $\alpha(z) = 1$, then $|\mu| \leqq 1$; so $|f_i\cdot z| = |\mu| \leqq 1$ and $\alpha^*(f_i) = 1$ $i = 1, 2$. Therefore g_1, g_2 are two different support functionals at x. This is a contradiction. The other part of the proof is trivial.

b) Assume that α is rotund. If $\alpha(x_1 + x_2)/2 = 1$, then $(x_1 + x_2)/2$ would be a boundary point of B_α. Since α is rotund, $\alpha(tx_1 + (1 - t)x_2) < 1$, for at least one $0 < t < 1$ (say $\frac{1}{2} < t < 1$). Thus $z = tx_1 + (1 - t)x_2$ is an interior point; so a small neighborhood U about z is contained in B_α. The cone $(x_1, U]$ is contained in B_α by convexity, and $(x_1 + x_2)/2$ is an interior point of it; therefore,

$(x_1 + x_2)/2$ is an interior point of B_α. This is a contradiction. The other part of the proof is trivial.

PROPOSITION 12. *Let (X, α) be a normed space, and assume that β is a norm which is continuous and smooth (perhaps not equivalent to α). Then the map $\nu: S_\beta \to S_{\beta*}$ which assigns to each $x \in S_\alpha$ the unique normalized support functional $\nu(x)$ is continuous if the β-topology is considered in S_β and the w^*-topology is considered in $S_{\beta*}$.*

Proof. Let $x \in S_\beta$, and let (x_n) (where n is in a directed set D) be a net converging to x. Then $(\nu(x_n))$ is a net in $\bar{B}_{\beta*}$ which converges to $\nu(x)$, as we will show next. If we assume that $(\nu(x_n))$ does not converge to $\nu(x) = f$, then there is a neighborhood $U(f)$ (in the w^*-topology) such that for each $m \in D$ there is some $m' \in D$ with the property $m' \geqq m$ and $\nu(x_{m'}) \notin U$. The subnet $(\nu(x_{m'}))$ has a subnet (still denoted by $(\nu(x_{m'}))$) which converges to some $g \in \bar{B}_{\beta*}$, because $\bar{B}_{\beta*}$ is w^*-compact, and $g \notin U$. Now $| \nu(x_{m'}) \cdot (x_{m'} - x) | = | \nu(x_{m'}) \cdot x_{m'} - \nu(x_{m'}) \cdot x | = | 1 - \nu(x_{m'}) \cdot x \leqq \beta^*(\nu(x_m')) \cdot \beta(x_{m'} - x) = \beta(x_{m'} - x) \to 0$; therefore, $\lim \nu(x_{m'}) \cdot x = g \cdot x = 1$, and g is a normalized support functional at x different from $\nu(x)$. This is a contradiction because β is smooth.

Remark. Since the β-topology is weaker than the α-topology, ν is also continuous if we consider the α-topology in X.

THEOREM 1. (*Klee*, [5]). *Assume that both (X, α) and (X^*, α^*) are separable Banach spaces. Then there exists a conjugate norm β^* equivalent to α^* such that,*
1) *β^* is rotund;*
2) *if f_n converges to f in the w^*-topology and $\beta^*(f_n) \to \beta^*(f)$, then $\beta^*(f_n - f) \to 0$.*

5. Differentiability and smoothness

PROPOSITION 13. *Let (X, α) be a Banach space, and let β be a differentiable norm (perhaps not equivalent to α). Then β is smooth and, for any $x \in S_\beta$, $\beta'(x)$ is a support functional.*

Proof. Let $x \varepsilon S_\beta$, and let P be any plane containing x and the origin. Then the equation of the curve $P \cap S_\beta$ near x is of the form $h(t) = u(t)/\beta(u(t))$, where $u(t) = t \cdot x_1 + (1 - t)x_0$, $\beta(x_1) = \beta(x_0) = 1$, and $h(t) = x$, for some $0 < t < 1$. Therefore β is smooth in any planar section through the origin, and β is smooth by Proposition 11.

The second part of the proposition follows from the fact that, for any u with $\beta(u) \leqq 1$,

$$\beta'(x) \cdot u = \lim_{\lambda \to 0} \frac{\beta(x + \lambda u) - \beta(x)}{\lambda} \leqq \lim_{\lambda \to 0} \frac{\beta(xu)}{\lambda} = \beta(u) \leqq 1$$

and $\beta'(x) \cdot x = 1$.

PROPOSITION 14. *Let (X, α) be a Banach space, and let β be any continuous norm in X (perhaps not equivalent to α). Assume that*

a) *β is a smooth norm;*

b)
$$\lim_{\substack{y \to x_0 \\ \beta(y)=1}} \frac{|\nu(x_0)\cdot(y - x_0)|}{\beta(y - x_0)} = 0,$$

where $\nu(x_0)$ is a support functional at x_0 such that $\beta^*(\nu(x_0)) = 1$. Then β is differentiable at x_0 and $\beta'(x_0) = \nu(x_0)$.

Proof. We have to prove that

$$\lim_{\substack{y \to x_0 \\ y \neq x_0}} \frac{|h(y, x_0)|}{\alpha(y - x_0)} = 0,$$

where $h(y, x_0) = \beta(y) - \beta(x_0) - \nu(x_0)\cdot(y - x_0)$. We observe first that we have to prove only that $\lim_{y\to x_0} | h(y, x_0) |/\beta(y - x_0) = 0$, since $| h(y, x_0) |/\alpha(y - x_0) = [| h(y, x_0) |/\beta(y - x_0)]\cdot[\beta(y - x_0)/\alpha(y - x_0)]$, and $\beta(y - x_0)/\alpha(y - x_0) \leqq M$ for some positive constant M. Let $f = \nu(x_0)$, and consider the following three cases.

(a) *Case* $f(y) \geqq 1$. Let $r(y) = y + (1 - f(y))x_0$ be the projection of y on $f^{-1}(1)$; then $\beta(r(y) - y) = f(y) - 1 = f(y - x_0)$. Let $z(y) = y/\beta(y)$ and $q'(y) = y/f(y)$; then $\beta(y) - \beta(x_0) = \beta(y - z(y)) \geqq \beta(y - z(y)) \geqq \beta(y - q(y)) \geqq \beta(y - r(y))$. Thus $| \beta(y) - \beta(x_0) - f\cdot(y - x_0) | = \beta(y - z(y)) - \beta(y - r(y))$. If y is restricted to a small neighborhood around x_0, then $p(y) = y + \lambda\cdot x_0$ satisfies $\beta(p(y) = 1$, for some real λ. Then, for any y in such neighborhood, $\beta(y - z(y)) \leqq \beta(y - p(y))$ and $| \beta(y) - \beta(x_0) - f\cdot(y - x_0) | \leqq \beta(y - p(y)) - \beta(y - r(y)) \leqq \beta(p(y) - r(y))$. Thus

$$(1) \qquad \frac{|h(y, x_0)|}{\beta(y - x_0)} \leqq \frac{\beta(p(y) - r(y))}{\beta(r(y) - x_0)} \cdot \frac{\beta(r(y) - x_0)}{\beta(y - x_0)}.$$

Since $\beta(y - r(y)) \leqq \beta(y - x_0)$, it follows that $\beta(r(y) - x_0) - \beta(y - x_0) \leqq \beta(y - r(y)) \leqq \beta(y - x_0)$, and

$$(2) \qquad \frac{\beta(r(y) - x_0)}{\beta(y - x_0)} \leqq 2.$$

On the other hand $\beta(p(y) - r(y))/\beta(r(y) - x_0) \leqq \beta(p(y) - r(y))/| \beta(p(y) - x_0) - \beta(p(y) - r(y)) | = \theta(y)/| 1 - \theta(y) |$, where $\theta(y) = \beta(p(y) - r(y)) /\beta(p(y) - x_0)$ and $\beta(p(y) - r(y)) = | f\cdot(p(y) - x_0) |$. The map $y \to p(y)$ is continuous and thus, by condition (2) of the theorem $\theta(y) \to 0$ as $y \to x_0$. Therefore, $\beta(p(y) - r(y))/\beta(r(y) - x_0) \to 0$ as $y \to x_0$. The result now follows from (1) and (2).

(b) *Case* $f\cdot y \leqq 1$ *and* $\beta(y) \geqq 1$. We keep the notation used in (a). The conditions of (b) imply $\beta(r(y) - y) = 1 - f(y) = -f(y - x_0)$. Thus $| \beta(y) - \beta(x_0) - f\cdot(y - x_0) | = \beta(y) - \beta(x_0) + \beta(y - r(y)) = \beta(y - z(y)) +$

$\beta(y - r(y)) \leqq \beta(y - p(y)) + \beta(y - r(y)) = \beta(r(y) - p(y))$. From here on the proof proceeds as in Case (a).

(c) *Case* $f \cdot y \leqq 1$ *and* $\beta(y) \leqq 1$. Again we keep the notation of Case (a). Then $\beta(y) - \beta(x_0) = -\beta(y - z(y))$, and $\beta(y - r(y)) = -f(y - x_0)$; moreover, $\beta(y - z(y)) = -\beta(y) + \beta(p(y)) \leqq \beta(y - p(y)) \leqq \beta(y - r(y))$. Thus $|\beta(y) - \beta(x_0) - f \cdot (y - x_0)| = -\beta(y - z(y)) + \beta(y - r(y))$. Let $s(y) = z(y) + (1 - f \cdot z(y))x_0$; then $\beta(s(y) - z(y)) = 1 - f \cdot y/\beta(y)$ and $\beta(y - r(y)) - \beta(y - z(y)) = \beta(y) - f(y) \leqq \beta(s(y) - z(y))$. From now on the proof proceeds as in Case (a). $|\beta(y) - \beta(x_0) - f \cdot (y - x_0)|/\beta(y - x_0) \leqq \beta(s(y) - z(y))/\beta(y - x_0) = [\beta(s(y) - z(y))/\beta(z(y) - x_0)] \cdot [\beta(z(y) - x_0)/\beta(y - x_0)]$. Now, $\beta(z(y) - x_0)/\beta(y - x_0) \leqq [\beta(y - x_0) + \beta(y - z'y))]/\beta(y - x_0) = 1 + [\beta(x_0) - \beta(y)/\beta(y - x_0)] \leqq 2$, for all $y \neq x_0$; moreover, $\beta(s(y) - z(y)) = |f(z(y) - x_0)|$, and the map $y \rightarrow z(y)$ is continuous. Therefore, by condition (2) of the theorem,

$$\lim_{y \to x_0} \frac{|\beta(y) - \beta(x_0) - f \cdot (y - x_0)|}{\beta(y - x_0)} \leqq 2 \quad \text{and} \quad \lim_{y \to x_0} \frac{\beta(s(y) - z(y))}{\beta(z(y) - x_0)} = 0.$$

From the considerations in (a), (b) and (c), it follows that $\lim_{y \to x_0} |h(y, x_0)|/\beta(y - x_0) = 0$. This concludes the proof.

Let (X, α) be a Banach space, and let β be a norm equivalent to α. For each $x \in S_\beta$, $\nu(x)$ is a support functional at x for which $\beta^*(\nu(x)) = 1$.

THEOREM 2. *Let (X, α) be a Banach space, and let β be a smooth norm equivalent to α. Then*

a) *if $\nu: S_\beta \rightarrow S_{\beta *}$ is continuous (in S_β we consider the α-topology and in $S_{\beta *}$ the α^*-topology y), then β is differentiable;*

b)† *if β is differentiable, then ν is continuous.*

Proof. a) Let $x_0 \in S_\beta$, and let $\epsilon > 0$. Then there exists $\delta = \delta(\epsilon, x_0)$ such that

$$\beta^*(\nu(x) - \nu(x_0)) < \epsilon$$

whenever

$$(1) \qquad \beta(x - x_0) < \delta, \quad \text{and} \quad x, x_0 \in S_\beta.$$

Let us construct a ball $B_\beta(x_0, r)$ small enough that if $y_0 \in B_\beta(x_0, r) \cap S_\beta$ and $\theta(t) = x_0(1 - t) + ty_0$, $0 \leqq t \leqq 1$, then the curve $y(t) = \theta(t)/\beta(\theta(t))$ is contained in $B_\beta(x_0, \delta) \cap S_\beta$. Since β is continuous, there is some $0 < \eta < \delta/2$ such that $|\beta(z) - \beta(x_0)| < \delta/2$ if $\beta(z - x_0) < \eta$. Let $U = \{z \mid \beta(z - x_0) < \eta$ and $\nu(x_0) \cdot z = 1\}$, and consider the cone $C(U) = \{tz \mid t > 0$ and $z \in U\}$. Then $C(U) \cap S_\beta = \{[z/\beta(z)] \mid z \in U\}$; and for any $y = z/\beta(z)$ one has $\beta(x_0 - y) \leqq \beta(x_0 - z) + \beta(z - y) = \beta(x_0 - z) + |\beta(z) - \beta(x_0)| = (\delta/2) + (\delta/2) = \delta$. Since $C(U)$ is open and contains x_0 as an interior point, there is a ball $B_\beta(x_0, r) \subset C(U)$. This ball satisfies our requirements.

† This is due to R. Phelps.

For any $y_0 \in B_\beta(x_0, r)$, the segment $\theta(t) = (1 - t)x_0 + t \cdot y_0$ is contained in $B_\beta(x_0, r)$. By the construction above, the curve $y(t) = \theta(t)/\beta(\theta(t))$ is contained in $B_\beta(x_0, r) \cap S_\beta$; and, since β is smooth, $y(t)$ is a differentiable curve (see Proposition 11) in the plane determined by x_0, y_0 and the origin. Thus, by the mean-value theorem, there is a point $y(t_0)$ such that $y'(t_0) = \lambda(y_0 - x_0)$ for some scalar $\lambda \neq 0$. Now, by the Hahn-Banach extension theorem, there is a support functional g at $y(t_0)$ such that $g(y'(t_0)) = 0$; and, since β is smooth, $g = \nu(y(t_0))$.

We can now finish the proof. Let $y_0 \in B_\beta(x_0, r) \cap S_\beta$. Then, by (1), $| (\nu(x_0) - \nu(y(t_0)) \cdot [y_0 - x_0/\beta(y_0 - x_0)] | < \epsilon$; and, by the remarks in the previous paragraph, $\nu(y(t_0)) \cdot [y_0 - x_0/\beta(y_0 - x_0)] = 0$. Therefore, $| \nu(x_0) \cdot (y_0 - x_0) | / \beta(y_0 - x_0) < \epsilon$ whenever $\beta(y_0 - x_0) < r$, $y_0 \in S_\beta$; so

$$\lim_{\substack{y \to x_0 \\ \beta(y)=1}} \frac{| \nu(x_0) \cdot (y - x_0)|}{\beta(y - x_0)} = 0.$$

It is clear, then, that a) follows from Proposition 13.

b) Assume that β is differentiable (this proof is due to Phelps). If ν is not continuous at some point $x_0 \in S_\beta$, then there is a net $\{x_n\}$ (where n is in some directed set D) such that $\lim_{n \in D} x_n = x_0$ and $\nu(x_n)$ does not converge to $\nu(x_0)$. Thus, for some $\epsilon > 0$, there is a subnet (still denoted by $\{x_n\}$) such that $\beta^*(\nu(x_n) - \nu(x_0)) > 2\epsilon$ and $\lim_{n \in D} x_n = x_0$. But this means that, for each n, there is some $y_n \in S_\beta$ such that $| (\nu(x_n) - \nu(x_0))y_n | \geqq 2\epsilon$. Let $z_n = [(1 - \nu(x_n) \cdot x_0)/\epsilon]y_n$, and observe that $\beta(z_n) \to 0$ (see Proposition 12). Now $\beta(x_0 + z_n) - \beta(x_0) - \nu(x_0) \cdot z_n \geqq \nu(x_n) \cdot (x_0 + z_n) - 1 - \nu(x_0) \cdot z_n = (\nu(x_n) - \nu(x_0)) \cdot z_n + (x_0) \cdot x_0 - 1 \geqq [2\epsilon \cdot (1 - (x_n) \cdot x_0)/\epsilon] + \nu(x_n) \cdot x_0 - 1 = 1 - \nu(x_n) \cdot x_0 \geqq 0$. Thus $| \beta(x_0 + z_n) - \beta(x_0) - (x_0) \cdot z_n | / \beta(z_n) \geqq | \nu(x_n) \cdot x_0 - 1 | / \beta(z_n) = \epsilon$. This is a contradiction because β is differentiable.

6. The main theorem

THEOREM 3. *A separable Banach space (X, α) admits a norm β equivalent to α of class C' if and only if (X^*, α^*) is also separable.*

Proof. a) Assume that X^* is separable, and let β^* be the norm of Klee's theorem (Theorem 1). Then β is smooth (Proposition 10), so therefore the map $\nu: S_\beta \to S_{\beta*}$ which assigns to each $x \in S_\beta$ the normalized support functional at x is continuous if the β-topology is used in S_β and the w^*-topology, in $S_{\beta*}$ (Proposition 12). Let $x_0 \in S_\beta$, and let $x_n \to x_0$, $x_n \in S_\beta$. Then $\nu(x_n) \to \nu(x_0)$, in the w^*-topology, and $\beta^*(\nu(x_n)) \to \beta^*(\nu(x_0))$; so, by Klee's theorem (Theorem 1), $\beta^*(\nu(x_n) - \nu(x_0)) \to 0$. Therefore, ν is continuous in the norm topologies, and β is of class C' (Theorem 2).

b) Assume there is a norm β equivalent to α of class C'. Extend the map $\beta': X - \{0\} \to S_{\beta*}$ to a continuous map $\mu: X - \{0\} \to X^*$ defined by $\mu(x) = \beta(x)\beta'(x)$. Then the image of μ is the set of all support functionals to $\{x \mid \beta(x) \leqq 1\}$. ($f$ is a support functional if $\text{Sup}_{\beta(x) \leqq 1} f \cdot x = f(y)$ for some y with

$\beta(y) = 1$.) Let $\{x_n\}$ be a countable dense sequence in X; let $f \in X^*$; and let $U(f)$ be a neighborhood of f. Since the set of support functionals is dense in X^* ([1], Cor. 4, p. 31), there is some support functional g in $U(f)$ of the form $g = \mu(x)$. Thus $x \in \mu^{-1}(U)$, by continuity, and some $x_n \in \mu^{-1}(U)$; so some $\mu(x_n) \in U$. Therefore the sequence $(\mu(x_n))$ is dense in X^*, and X^* is separable.

Remark 1. Let C_0 be the space of all sequences $\{x_n\}$ of real numbers such that $x_n \to 0$ with $\| x \| = \text{Sup}_n \mid x_n \mid$; let l_1 be the space of all sequences $\{x_n\}$ such that $\sum \mid x_n \mid < \infty$, with $\| x \| = \sum \mid x_n \mid$; let $C[0, 1]$ be the space of continuous functions with $\| x \| = \text{Sup}_{0 \leqq t \leqq 1} \mid x(t) \mid$. Then, by Theorem 3, the topologies in l_1 and $C[0, 1]$ cannot be defined by any norm of class C_1. On the other hand, the topology in C_0 can be defined by a norm of class C_1. (Phelps has constructed an equivalent norm of class C_1 in C_0.)

Remark 2. It follows from Theorem 3 that we cannot drop the hypothesis of (X^*, α^*) being separable in Klee's theorem (Theorem 1). Thus, we can not construct any norm in l_∞ satisfying the conditions of Klee's theorem.

THEOREM 4. *Let (X, α) be a separable Banach space. Then, if both α and α^* are of class C', (X,α) is reflexive.*

Proof. Let $\mu:X \to X^*$ and $\mu^*:X^* \to X^{**}$ be defined by $\mu(x) = \alpha(x)\alpha'(x)$ and $\mu^*(f) = \alpha^*(f)(\alpha^*)'(f)$. Now $j:X \to X^{**}$ (defined by $j(X)\cdot f = f(X)$) is an isometry of X into X^{**}, and $j = \mu_0{}^*\mu$. Thus, by Theorem 3, $j(X)$ is dense in X^{**}. Therefore $j(X) = X^{**}$, and X is reflexive.

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