WEAK ATTRACTORS IN DYNAMICAL SYSTEMS

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J. Auslander, P. Seibert, and the present author have discussed some properties of compact attractors in dynamical systems defined on locally compact, metric spaces [1]. In the present paper we show that most of our results in [1] are in fact applicable to a wider class of compact sets, which we prefer to call here "weak attractors." In the case of dynamical systems defined in a planar region, it is shown that in general the distinction between attractors and weak attractors is in the absence or presence of spiraling trajectories in the region of attraction. Since we show that a stable weak attractor is a stable attractor, we do not go into the properties of stable attractors, as these have been extensively studied elsewhere [1], [2]. Examples of weak attractors are already found in the literature and they are pointed out as such.

As this paper is closely related to [1] and we use the same notations and tools, we do not give proofs of all the theorems.

1. Notation and elementary concepts

X denotes a locally compact metric space, with metric d. R denotes the set of real numbers. R^+ and R^- are the sets of non-negative and non-positive real numbers, respectively.

A continuous map $\pi: X \times R \to X$ of the product space $X \times R$ into X defines a *dynamical system* (or continuous flow) on X if the two following conditions hold:

(I) $\pi(x, 0) = x$, for all $x \in X$,

(II) $\pi(\pi(x, t_1), t_2) = \pi(x, t_1 + t_2)$, for all $t_1, t_2 \in R$, $x \in X$.

If, for each $x \in X$, $\phi(x) \subset X$, then, for any $M \subset X$,

$$\phi(M) = \bigcup_{x \in M} \phi(x).$$

For any given $x \in X$, the orbit, the positive semi-orbit, and the negative semi-orbit are, respectively, the sets $\gamma(x) = \pi(x, R)$, $\gamma^+(x) = \pi(x, R^+)$, and $\gamma^-(x) = \pi(x, R^-)$.

$M \subset X$ is called invariant if $\gamma(M) = M$. It is called positively (negatively) invariant if $\gamma^+(M) = M$ $(\gamma^-(M) = M)$.

The *positive* (or *omega*) *limit set* $\Lambda^+(x)$ of an orbit $\gamma(x)$ is the set of all points $y \in X$, such that there exists a sequence $\{t_n\}$ in R^+, with $t_n \to +\infty$ and $\pi(x, t_n) \to y$.

The *negative* or *alpha limit set* $\Lambda^-(x)$ of an orbit $\gamma(x)$ is defined similarly. It is

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the set of all $y \in X$ such that there exists a sequence $\{t_n\}$ in R^-, with $t_n \to -\infty$ and $\pi(x, t_n) \to y$.

For $x \in X$, $M \subset X$, and $\alpha > 0$, the symbols $d(x, M)$, $S(M, \alpha)$, $B(M, \alpha)$, and $H(M, \alpha)$ stand for, respectively, $\inf \{d(x, y): y \in M\}$, $\{x \in X: d(x, M) < \alpha\}$, $\{x \in X: d(x, M) \leqq \alpha\}$, and $\{x \in X: d(x, M) = \alpha\}$. $\bar{M}$, and ∂M are the closure and boundary of the set M.

The following properties of positive limit sets are well known.

LEMMA 1. *For any* $x \in X$,

(i) $\overline{\gamma^+(x)} = \gamma^+(x) \cup \Lambda^+(x)$;

(ii) $\Lambda^+(x)$ *is closed and invariant*;

(iii) *if* $\Lambda^+(x)$ *is compact, then it is connected.*

Properties (i) and (ii) follow easily from the definitions. (iii) is proved in [4], chapter V.

Remark. Another interesting property of a set $\Lambda^+(x)$ is that if it is not compact, then none of its components is compact. This property was pointed out to us by Professor W. Huebsch. We have not seen it explicitly mentioned anywhere. The proof can be made to depend on the fact that a locally compact, Hausdorff space possesses a one-point compactification by means of an ideal point ∞. The given dynamical system is extended to a dynamical system on the compactified space by definingthe point ∞ to be a rest point (i.e., $\{\infty\} = \gamma(\infty)$). If now $\Lambda^+(x)$ is the positive limit set of any non-compact trajectory $\gamma(x)$ in the given system, then $\Lambda^+(x) \cup \{\infty\}$ is the positive limit set of the same trajectory in the new system. This set is compact and, by property (iii) of Lemma 1, is connected. Now $\Lambda^+(x)$ is an open subset of the Hausdorff continuum $\Lambda^+(x) \cup \{\infty\}$. Any component of $\Lambda^+(x)$, therefore, must have a boundary point in the complement of $\Lambda^+(x)$ in $\Lambda^+(x) \cup \{\infty\}$, i.e., in $\{\infty\}$. Thus no component can be compact (see [7], p. 37).

2. Weak attractors, attractors, and stable sets

In what follows, M denotes a compact, nonempty subset of X, unless stated to the contrary. The set M will be called

(i) a (*positive*) *weak attractor* if, for some $\delta > 0$, $\Lambda^+(y) \cap M \neq \emptyset$ for each $y \in S(M, \delta)$, $\emptyset$ being the empty set.

(ii) a (*positive*) *attractor* if, for some $\delta > 0$, $\Lambda^+(y) \neq \emptyset$ and $\Lambda^+(y) \subset M$ for each $y \in S(M, \delta)$.

(iii) (*positively*) *stable* if, for each $\alpha > 0$, there is a $\delta > 0$ such that $\gamma^+(S(M, \delta)) \subset S(M, \alpha)$.

Negative weak attractors, negative attractors, and negative stability are similarly defined. (We shall omit the adjective "positive" in our discussion when referring to positive attractors, etc.)

It is clear that (ii) implies (i); i.e., an attractor is a weak attractor. However,

(i) does not imply (ii). As to examples of weak attractors which are not attractors, we refer to Mendelson's geometrical example (see [3] fig. 2) and Digel's analytical example ([5], p. 154). An example on a torus is in the book of Nemytskii and Stepanov ([4], p. 346), and the attractor of Carlos Perello [1] is a negative weak attractor as well.

By the region of attraction $A(M)$ of the set M (which need not be a weak attractor) we mean the set of all $x \in X$ such that $\Lambda^+(x) \cap M \neq \emptyset$. Note that M is a weak attractor if and only if $A(M)$ is a neighborhood of M. Further, in the case of an attractor our definition is easily seen to be equivalent to the one given in [1].

LEMMA 2. *If M is a weak attractor, then $A(M)$ is an open invariant set containing M.*

The proof is the same as that of Lemma 1 in [1] and so is omitted.

3. The prolongation and the prolongational limit set

If $x \in X$, the (*first*) (*positive*) *prolongation* $D^+(x)$ of x, is the set of all $y \in X$ such that there exist sequences $\{x_n\}$ in X and $\{t_n\}$ in R^+, with $x_n \to x$ and $\pi(x_n, t_n) \to y$ (see [1], [2], [6]).

LEMMA 3. (i) *For any compact set $M \subset X$, $D^+(M)$ is closed and positively invariant.* (ii) *The compact set M is stable, if and only if $D^+(M) = M$.*

(i) can easily be proved from the definitions. For the proof of (ii) see [6], page 177.

For $x \in X$, the (*positive*) *prolongational limit set* $\Lambda_D^+(x)$ of x is the set of all points $y \in X$ such that there are sequences $\{x_n\}$ in X and $\{t_n\}$ in R^+, with $x_n \to x$, $t_n \to +\infty$, and $\pi(x_n, t_n) \to y$. The following lemma is analogous to Lemma 1.

LEMMA 4. *For any $x \in X$,*
(i) $D^+(x) = \gamma^+(x) \cup \Lambda_D^+(x)$;
(ii) $\Lambda_D^+(x)$ *is closed and invariant*;
(iii) if $\Lambda_D^+(x)$ *is compact, then it is connected.*

(i) and (ii) easily follow from definitions. Proof of (iii) will be made to depend on the following lemma.

LEMMA 5. *For any $x \in X$, if $w \in \Lambda^+(x)$, then*

$$\Lambda_D^+(x) \subset D^+(w).$$

In particular, if $w \in \Lambda^+(x)$ and $y \in \Lambda_D^+(x)$, then there exist sequences $\{x_n\}$ in X, $\{t_n\}$ and $\{\tau_n\}$ in R^+, with $x_n \to x$, $t_n - \tau_n > 0$ for each n, $t_n \to +\infty$, $\tau_n \to +\infty$, $\pi(x_n, \tau_n) \to w$, and $\pi(x_n, t_n) \to y$.

This lemma is the same as Lemma 4 in [1], the assertion about sequences being a step in the proof there. Therefore we omit the proof.

Proof of Lemma 4 (iii). To be able to apply Lemma 5, we note first that if

$\Lambda_D^+(x)$ is compact and non-empty, then $\Lambda^+(x) \neq \emptyset$. For if we assume the contrary, then there is an $\alpha > 0$ and a $T > 0$ such that $\pi(x, t) \notin B(\Lambda_D^+(x), \alpha)$ for $t \geqq T$. We may thus assume, without loss of generality, that $x \notin \Lambda_D^+(x)$, that $\alpha > 0$ is such that $\gamma^+(x) \cap B(\Lambda_D^+(x), \alpha) = \emptyset$, and that $B(\Lambda_D^+(x), \alpha)$ is compact. Now let $y \in \Lambda_D^+(x)$. Then there is a sequence $\{x_n\}$ in X and a sequence $\{t_n\}$ in R^+, with $x_n \to x$, $t_n \to +\infty$, and $\pi(x_n, t_n) \to y$. We may assume that $x_n \notin B(\Lambda_D^+(x), \alpha)$ and $\pi(x_n, t_n) \in S(\Lambda_D^+(x), \alpha)$ for each n. By continuity of the map π, there exists for each n a t_n', $0 < t_n' < t_n$ such that $\pi(x_n, t_n') \in H(\Lambda_D^+(x), \alpha)$. Since $H(\Lambda_D^+(x), \alpha)$ is compact, the sequence $\{\pi(x_n, t_n')\}$ has a limit point z in $H(\Lambda_D^+(x), \alpha)$. We may assume that $\pi(x_n, t_n') \to z$. Then either $z \in \gamma^+(x)$ or $z \in \Lambda_D^+(x)$. But since $z \in H(\Lambda_D^+(x), \alpha)$, we have $z \notin \Lambda_D^+(x)$. Thus $z \in \gamma^+(x)$. This, however, contradicts the fact that

$$\gamma^+(x) \cap B(\Lambda_D^+(x), \alpha) = \emptyset.$$

We have thus proved that $\Lambda^+(x) \neq \emptyset$. To prove now that $\Lambda_D^+(x)$ is connected, assume the contrary. Then there are non-empty, closed (in this case compact) sets A_1 and A_2 such that $A_1 \cap A_2 = \emptyset$ and $A_1 \cup A_2 = \Lambda_D^+(x)$. Choose now any $w \in \Lambda^+(x)$. As $\Lambda^+(x) \subset \Lambda_D^+(x)$, we have $w \in A_1$ or $w \in A_2$. Suppose $w \in A_1$, and choose $y \in A_2$. By Lemma 5, there are sequences $\{x_n\}$ in X, $\{t_n\}$ and $\{\tau_n\}$ in R^+, with $x_n \to x$, $t_n \to +\infty$, $\tau_n \to +\infty$, $t_n - \tau_n > 0$ for each n, $\pi(x_n, \tau_n) \to w$, and $\pi(x_n, t_n) \to y$. We can choose $\alpha > 0$ such that the sets $B(A_1, \alpha)$ and $B(A_2, \alpha)$ are compact and disjoint. We may also assume, without loss of generality, that $\pi(x_n, \tau_n) \in S(A_1, \alpha)$ and $\pi(x_n, t_n) \in S(A_2, \alpha)$ for each n. But then, by the continuity of the map π, there is for each n a t_n', $t_n > t_n' > \tau_n$, such that $\pi(x_n, t_n') \in H(A_1, \alpha)$. As $H(A_1, \alpha)$ is compact, this implies the existence of a limit point $z \in H(A_1, \alpha)$ of the sequence $\{\pi(x_n, t_n')\}$. As $t_n' \to +\infty$, we have $z \in \Lambda_D^+(x)$; but $z \notin A_1 \cup A_2$—which is a contradiction. Thus $\Lambda_D^+(x)$ is connected.

Remark. $\Lambda_D^+(x)$ too possesses the property mentioned in the remark after Lemma 1. This may be proved by the same method.

Another interesting observation is that $\overline{\gamma^+(x)}$ is always connected, whereas $D^+(x)$ is in general not connected. For example, see the case of an improper saddle point at infinity ([4], *p.* 411). This example shows further that if $\Lambda_D^+(x)$ is not compact, then $\Lambda^+(x)$ may be empty, even though $\Lambda_D^+(x)$ is connected. We have, however, the following:

LEMMA 6. *Let $M \subset X$ be connected; then if $D^+(M)$ is compact, it is connected.*

Proof. If $D^+(M)$ is not connected, then there are non-empty compact sets A_1 and A_2 such that $A_1 \cup A_2 = D^+(M)$ and $A_1 \cap A_2 = \emptyset$. Since M is connected and $M \subset D^+(M)$, we have $M \subset A_1$ or $M \subset A_2$. Let $M \subset A_1$, and choose $y \in A_2$. There are then sequences $\{x_n\}$ in X and $\{t_n\} \in R^+$, with $x_n \to x \in M \subset A_1$ and $\pi(x_n, t_n) \to y \in A_2$. We may choose $\alpha > 0$ such that the sets $B(A_1, \alpha)$ and $B(A_2, \alpha)$ are compact and disjoint. Further, we may assume that $x_n \in S(A_1, \alpha)$ and $\pi(x_n, t_n) \in S(A_2, \alpha)$ for each n. Then, by continuity of the map π, there exists for each n a t_n', $0 \leqq t_n' < t_n$ such that $\pi(x_n, t_n') \in H(A_1, \alpha)$.

Since $H(A_1, \alpha)$ is compact, this implies the existence of a point z of $D^+(x)$ in the set $H(A_1, \alpha)$. This is ruled out, as $D^+(x) \subset D^+(M) = A_1 \cup A_2$ and $H(A_1, \alpha) \cap (A_1 \cup A_2) = \emptyset$. This contradiction proves that $D^+(M)$ is connected.

4. Theorems on weak attractors

THEOREM 1. *Let the set M be stable. Then M is an attractor if and only if M is a weak attractor.*

Before proving this theorem, we give

LEMMA 7. *For any set $M \subset X$, $x \in A(M)$ implies $\Lambda^+(x) \subset D^+(M)$.*

Proof. As $\Lambda^+(x) \cap M \neq \emptyset$, choose $w \in \Lambda^+(x) \cap M$. Let now $y \in \Lambda^+(x)$ be arbitrary. There are sequences $\{t_n\}$ and $\{\tau_n\}$ in R^+, with $\tau_n \to +\infty$, $t_n \to +\infty$, $\pi(x, t_n) \to y$, and $\pi(x, \tau_n) \to w$. We may assume, if necessary by choosing a subsequence, that $t_n - \tau_n > 0$ for each n. As $\pi(x, t_n) = \pi(\pi(x, \tau_n), t_n - \tau_n)$, we see that $y \in D^+(w)$. This proves the lemma.

Proof of Theorem 1. Let M be a weak attractor. Let $x \in A(M) \setminus M$. As $\Lambda^+(x) \cap M \neq \emptyset$, choose $w \in \Lambda^+(x) \cap M$. Then by Lemma 7, $\Lambda^+(x) \subset D^+(M)$. However, M is stable; so $D^+(M) = M$ by Lemma 3 (ii). We have thus proved that $\Lambda^+(x) \subset M$ for each $x \in A(M)$; i.e., M is an attractor. The converse is trivial; thus the theorem holds.

THEOREM 2. *Let the set M be a weak attractor. Then $D^+(M)$ is a compact set which is a stable attractor, its region of attraction, $A(D^+(M))$, coinciding with $A(M)$. Moreover, $D^+(M)$ is the smallest stable attractor containing M.*

The proof of this theorem differs from that of Theorem 1 in [1] only in insignificant detail. It is omitted. We have in fact the following stronger version of the Lemma 5 in [1] whose proof is also exactly the same.

LEMMA 8. *Let the set M be a weak attractor, and let $\alpha > 0$. Then there is a $T > 0$ such that*

$$D^+(M) \subset \pi(B(M,\alpha), [0,T]).$$

The following theorem is a stronger version of Theorem 2 in [1].

THEOREM 3. *Let M be invariant. If the set M is a weak attractor and $y \in D^+(M)$, then $\Lambda^-(y) \cap M \neq \emptyset$.*

Proof. If $y \in M$, then $\Lambda^-(y) \subset M$, since M is compact and invariant. Suppose now $y \notin M$. By Lemma 8, if $\alpha > 0$, then there is a $t < 0$ such that $\pi(y, t) \in B(M, \alpha)$. There is thus a sequence $\{t_n\}$ of negative reals such that $\pi(y, t_n) \to x \in M$. If this sequence is bounded below, then there is a convergent subsequence $\{t_n'\}$ of $\{t_n\}$, $t_n' \to \tau \leqq 0$. But then $\pi(y, t_n') \to \pi(y, \tau) = x$, and so $x \in \gamma(y)$. Since M is invariant, this implies $\gamma(y) \subset M$ and, in particular also, $y \in M$—which is a contradiction. Thus the sequence $\{t_n\}$ cannot have a bounded subsequence; i.e., $t_n \to -\infty$. But then $x \in \Lambda^-(x)$; and since $x \in M$, we have proved that $\Lambda^-(y) \cap M \neq \emptyset$. This proves the theorem.

We recall that Carlos Perello's example (given in [1]) of an attractor on a torus indicated that, even for an attractor, $\Lambda^-(y) \subset M$ does not hold for each $y \in D^+(M)$ in general. The following theorem gives the characterization of the set $D^+(M)$ in the case of a weak attractor M.

THEOREM 4. *If a compact invariant set M is a weak attractor, then*

$$D^+(M) = \{y \in X : \Lambda^-(y) \cap M \neq \emptyset\}.$$

Proof. If the set on the right is denoted by P, then $D^+(M) \subset P$ follows from Theorem 3. But $P \subset D^+(M)$ always holds (even when M is not a weak attractor); for if $z \in \Lambda^-(y) \cap M$, then there is a sequence $\{t_n\}$ in R^-, with $t_n \to -\infty$ and $\pi(y, t_n) \to z$. This implies that $y \in \Lambda_D{}^+(z) \subset \Lambda_D{}^+(M) \subset D^+(M)$, and the theorem is proved.

The examples of Mendelson and Digel in the plane show that a weak attractor need not be a weak negative attractor. On the other hand the two examples on the torus indicate that the possibility that a weak attractor is also a negative weak attractor is not always ruled out. The following theorem throws light in this direction.

THEOREM 5. *A positive weak attractor M is also a negative weak attractor if and only if*

$$D^+(M) = A(M).$$

Proof. The set $A(M)$ is a neighborhood of M; therefore, if $D^+(M) = A(M)$, we have, by Theorem 3, $\Lambda^-(y) \cap M \neq \emptyset$ for each $y \in A(M)$. Thus M is a negative weak attractor. If $D^+(M) \neq A(M)$, then $D^+(M)$ cannot be a neighborhood of M. Thus every neighborhood U of M meets $A(M) \setminus D^+(M)$. If however, $y \in U \cap (A(M) \setminus D^+(M))$, then $\Lambda^-(y) \cap M = \emptyset$; for otherwise $y \in D^+(M)$, by Theorem 4. Hence M is not a negative weak attractor. This proves the theorem.

5. Dynamical systems defined in a planar region

To start with, we note the following consequence of Theorem 5 for general dynamical systems defined in the euclidean plane.

THEOREM 6. *If the set M is a weak attractor, then it cannot be a negative weak attractor.*

Proof. M can be a negative weak attractor only if $D^+(M) = A(M)$. This is impossible in the plane, as $D^+(M)$ is compact and hence closed and bounded. But the only closed sets which are also open are the empty set and the whole plane. Since $A(M)$ is open, we cannot have $D^+(M) = A(M)$. This proves the theorem.

For the remaining part we assume that our dynamical system is defined in the euclidean plane by a system of differential equations

$$\dot{x}_1 = f_1(x_1, x_2), \quad \dot{x}_2 = f_2(x_1, x_2), \quad \left(\cdot = \frac{d}{dt}\right).$$

From now on M will stand for a compact and invariant set.

We shall let x stand for the pair (x_1, x_2) and shall continue using the same notation as in the previous sections. Our first result is a stronger version of Theorem 3 in [1].

THEOREM 7. *Let the set M be a weak attractor, and let M be connected. Then, for each $x \in D^+(M)$, we have*

$$(\Lambda^+(x) \cup \Lambda^-(x)) \subset M.$$

Proof. Let $x \in D^+(M)$. Then, by Theorem 3, $\Lambda^-(x) \cap M \neq \emptyset$, and as $D^+(M) \subset A(M)$, we also have $\Lambda^+(x) \cap M \neq \emptyset$. If $x \in M$, then $\Lambda^+(x) \cup \Lambda^-(x) \subset M$ (for M is compact and invariant). Let now $x \in D^+(M) \setminus M$, and suppose, if possible, that $\Lambda^+(x) \not\subset M$ ($\Lambda^-(x) \not\subset M$). Choose $y \in (\Lambda^+(x) \setminus M)$ ($y \in (\Lambda^-(x) \setminus M)$). Certainly y is not a critical point; for otherwise $\gamma(y) = \Lambda^+(y) = \Lambda^-(y) = \{y\}$, and, since $y \in D^+(M) \subset A(M)$, we will have $y \in M$, as $\Lambda^+(y) \cap M \neq \emptyset$. Hence y is a regular point. Again $y \notin \gamma(x)$. For if $y \in \gamma(x)$, then $\gamma(x)$ will be a periodic orbit, so that $\gamma(x) = \Lambda^+(x) = \Lambda^-(x)$. Since $\Lambda^+(x) \cap M \neq \emptyset$ and M is invariant, we will then have $\gamma(x) \subset M$ and, in particular also, $y \in M$—which contradicts the assumption that $y \notin M$. Since y is a regular point and $y \notin \gamma(x)$, we can draw a transversal l through y with the property that $\gamma^+(x)$ ($\gamma^-(x)$) intersects l in a monotone sequence of points $\{P_n\}$, $P_n \to y$. The portion of the semi-orbit $\gamma^+(x)$ ($\gamma^-(x)$) between any two successive points, say P_1 and P_2, of this sequence and the part of the transversal between them form a Jordan curve J. This curve J divides the plane into two connected sets A_1 and A_2 which are disjoint. Further, one of them (say A_1) is positively invariant, and the other, A_2, is negatively invariant. Consequently, $\Lambda^+(x) \subset A_1$ and $\Lambda^-(x) \subset A_2$. But, as M is connected and $M \cap J = \emptyset$, we must have $M \subset A_1$, since $M \cap \Lambda^+(x) \neq \emptyset$. This shows that $M \cap \Lambda^-(x) = \emptyset$, contradicting Theorem 3. This proves the theorem.

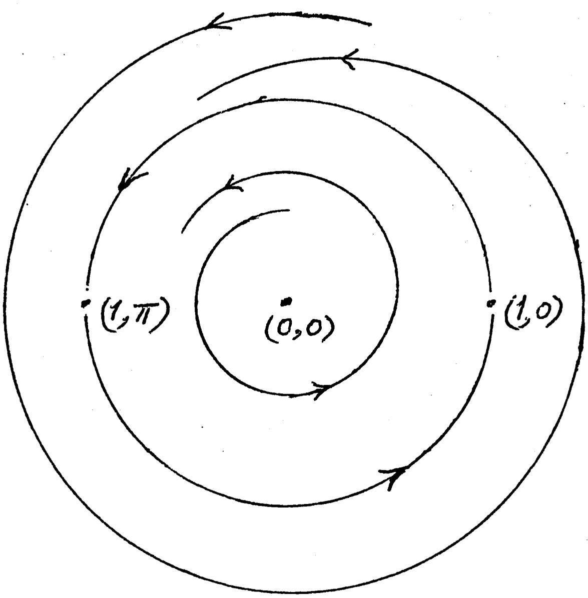
Indeed, if M is not connected, the theorem does not hold.

Example 1. Consider the dynamical system in the plane, given in polar coordinates by the equations

$$\dot{r} = r(1 - r),$$

$$\dot{\theta} = \begin{cases} \sin^2\theta + \dfrac{1}{\log 3}, & 0 < r \leqq 3/4, \\ \sin^2\theta + \dfrac{1}{\log(r/1 - r)}, & 3/4 < r < 1, \\ \sin^2\theta, & r = 1, \\ \sin^2\theta + \dfrac{1}{\log(r/r - 1)}, & r > 1. \end{cases}$$

The phase portrait is given in Figure 1. The set M consisting of the three critical points $(0, 0)$, $(1, 0)$, and $(1, \pi)$ is a weak attractor. This set M is not connected.



$D^+(M)$ is the set of all points with $r \leqq 1$. Note that, for any point P with $0 < r < 1$, $\Lambda^+(P)$ is the set of all points with $r = 1$. Thus $\Lambda^+(P) \not\subset M$.

Theorem 7 suggests that in the case when M is connected the distinction between weak attractors and attractors must lie in the behavior of trajectories not contained in $D^+(M)$.

THEOREM 8. *Let the set M be connected. Let M be a weak attractor. If M is not an attractor, then there are spiraling orbits in $A(M) \setminus D^+(M)$. If, further, $D^+(M)$ is simply connected, then every such orbit is a spiral.*

Proof. There exists $x \in A(M) \setminus D^+(M)$ such that $\Lambda^+(x) \not\subset M$. But $\Lambda^+(x) \subset \partial D^+(M)$. There is therefore a $y \in \Lambda^+(x) \cap \partial D^+(M)$, $y \notin M$. Since $y \in A(M) \setminus M$, y is a regular point. Further $y \notin \gamma(x)$. There is thus a transversal l having y as an inner point, and the semi-orbit $\gamma^+(x)$ intersects l in a monotone sequence $\{P_n\}$, with $P_n \to y$. Thus $\gamma^+(x)$ is a spiraling trajectory (see [4], pp. 43–44). Let now $D^+(M)$ be simply connected. In the above discussion, we let J_i denote the Jordan curve formed by the portion of the semi-orbit $\gamma^+(x)$ between P_i and P_{i+1} and the portion of the transversal l between P_i and P_{i+1}. Each J_i divides the plane into two parts, A_i and B_i, one of which is simply connected. We shall assume that A_i is simply connected. In general, there are two possibilities: (i) $A_i \supset A_{i+1}$ for each i, and (ii) $A_i \subset A_{i+1}$ for each i. Note that in case (ii) $M \subset B_i$ for each i, and, since $D^+(M)$ is connected and contains M, we have in fact $D^+(M) \subset B_i$. But in this case $\Lambda^+(x) = \partial(\bigcup_{i=1}^{\infty} A_i)$. As $\bigcup_{i=1}^{\infty} A_j$ is simply connected and $\Lambda^+(x) \subset D^+(M)$, we must have $D^+(M) \supset \bigcup_{i=1}^{\infty} A_i$, since $D^+(M)$ is simply connected. This is absurd because $D^+(M) \cap A_i = \emptyset$ for each i. This shows that the case (ii) is impossible. In the event of case (i), we have

$D^+(M) \subset A_i$ for each i. Note now that any trajectory which intersects the transversal l, say at a point Q_i between the points P_i and P_{i+1}, must intersect l in a monotone sequence of points $\{Q_j\}$, Q_j lying between P_j and P_{j+1}, $j \geqq i$. (This is so because such a trajectory must enter each of the Jordan curves J_j from the region B_j into A_j, and this can be done only between the points P_j and P_{j+i}.) Notice further that for any point $y \in A(M) \setminus D^+(M)$, $y \notin \gamma(x)$, we have $y \notin A_j$ for sufficiently large j. Hence each such trajectory must intersect the transversal l. This shows that each such trajectory is a spiral and the theorem is proved.

THEOREM 9. *Let the set M be simply connected. Let M be an unstable attractor. Then the set $\Lambda^+(x)$ for each orbit $\gamma(x)$ in $A(M) \setminus M$ contains critical points only.*

Proof. If, for some $x \in A(M) \setminus M$, $\Lambda^+(x)$ contains a regular point y, which necessarily lies in the boundary of M, then, following the method of proof of Theorem 8, it is seen that M will be stable. This contradicts the hypothesis that M is unstable. We leave the details to the reader.

COROLLARY. *Let M be simply connected; and let M be an unstable attractor. Assume further that ∂M contains regular points. Then no semi-orbit $\gamma^+(x)$, with $x \in A(M) \setminus M$, can approach M spirally.*

Remark. Theorem 9 and the above corollary leave still open the question of spiraling orbits in the set $A(M) \setminus M$ corresponding to an unstable attractor M, whose boundary ∂M consists only of critical points. The answer to this question may depend on an answer to the following problem.

Problem. Let M be an isolated critical point; let M be an attractor; and let there exist a spiraling semi-orbit $\gamma^+(x)$ approaching M. Can M be unstable?

Added in proof. Dr. Peter Seibert has informed the author that the answer to the above problem is in the affirmative. In fact Mendelson's example [3] can be transformed into an unstable critical point such that all orbits besides the critical point are spirals approaching it.

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