HOMOTOPY TYPES OF DIFFERENTIABLE MANIFOLDS

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1. Introduction

M. Kervaire has discovered a 10-dimensional combinatorial manifold which is not of the homotopy type of any differentiable manifold. The question arises as to what conditions are sufficient for a combinatorial manifold, or even a polyhedron, to be of the same homotopy type as a differentiable manifold. In [1] Professor W. Browder has succeeded in giving some necessary and sufficient conditions for polyhedra of dimensions $2k + 1$ and $4k$. In this note we shall prove that his conditions are also sufficient for a $2k$-dimensional polyhedron to be of the same homotopy type as a combinatorial $2k$-manifold which admits an almost differentiable structure. We shall also consider problems of homotopy type of almost parallelizable manifolds.

2. Preliminaries

Manifolds considered here are to be oriented, connected, compact and differentiable of class C^r, $r \geqq 1$, unless we specify the contrary. Imbeddings are differentiable imbeddings. To obtain a differentiable manifold we have on many occasions to round up the corners. For the sake of brevity we may sometimes omit mentioning this fact. The references for this procedure are [3] or [8]. We shall use the notations D^n for the n-disk and ∂ for the geometric boundary.

3. Known and new results

We state here results of [1] and new results which we are going to prove.

THEOREM 3.1 (Browder, [1]). *Let $n = 2m + 1 \geqq 5$, and let X be a connected finite polyhedron, with $\pi_1(X) = 0$. Suppose that the following two conditions are satisfied:*

i) X satisfies the Poincaré duality; that is, $H_n(X; Z) \cong Z$ and, if g is a generator, then the cap product $\cap\, g: H^i(X) \to H_{n-i}(X)$ is an isomorphism for all i.

ii) There exists an oriented k-vector bundle ζ over X such that $\Phi(g) \in H_{n+k}(T(\zeta))$ is spherical. Here Φ is the Thom isomorphism $\Phi: H_n(X) \to H_{n+k}(T(\zeta))$ and $T(\zeta)$ is the Thom complex.

Then X is of the same homotopy type as an n-dimensional closed differentiable manifold.

THEOREM 3.2 (Browder, [1]). *Let $n = 4m > 5$ and X be a connected finite*

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polyhedron with $\pi_1(X) = 0$. *Suppose that conditions i) and ii) are the same as in the previous theorem, with* $n = 4m$, *and that*

iii) the Hirzebruch index condition applies; that is, if $B: H^{2m}(X; R) \otimes H^{2m}(X; R) \to R$ *(R is the field of real numbers) is the cup product pairing, then the index of this pairing satisfies the following equality:*

$$Index = \langle L_m(\bar{p}_1, \bar{p}_2, \cdots, \bar{p}_m), g \rangle,$$

L_m *is the Hirzebruch polynomial, and* $\bar{p}_1, \cdots, \bar{p}_m$ *are the dual classes of the Pontrjagines classes of* ζ.

Then X is of the same homotopy type as a closed differentiable $4m$*-manifold.*

Remark (Browder, [1]). In case ζ is a trivial bundle in the theorems, condition ii) is equivalent to $\sum^k (g) \in H_{n+k}(\sum^k X)$ being spherical, where $\sum$ denotes suspension. The conclusions of theorems assert that X is of a homotopy type of a π-manifold.

THEOREM 3.3. *Let* $n =$ [illegible] *and X be a connected finite polyhedron with* $\pi_1(X) = 0$. *Suppose that conditions i) and ii) of Theorem* 3.1 *are satisfied. Then X is of the homotopy type of a combinatorial closed* $2m$*-manifold which admits a differential structure except possibly on one n-cell. Moreover, if* $n = 6$ *or* $n = 14$, *the manifold is in fact differentiable.*

THEOREM 3.4. *Let* $n = 4m > 5$ *and X be a connected finite polyhedron with* $\pi_1(X) = 0$. *Suppose that the following conditions are satisfied:*

i) X satisfies the Poincaré duality (see Theorem 3.1).

ii) There is an oriented k-vector bundle ζ *over X such that it is an induced bundle from a vector bundle over the* $4m$*-sphere* S^{4m}; *that is,*

$$\begin{array}{ccc} E(\zeta) & \rightarrow & E(\eta) \\ \downarrow & \zeta = f^*\eta & \downarrow \\ X & \xrightarrow{\quad} & S, \end{array}$$

where f is of degree one, and $\Phi(g) \in H_{n+k}(T(\zeta))$ *is spherical,* Φ *being the Thom isomorphism.*

iii) Let the bundle η *over* S^{4m} *be determined by an element*

$$\lambda \in \pi_{4m-1}(SO(k)) \cong Z.$$

Then the index I of the cohomology pairing of X is $-s_m a_m \lambda (2m - 1)!$, *where* s_m *is* $2^{2m}(2^{2m-1} - 1)/(2m)!$ *times the mth Bernoulli number, and* a_m *is two if m is odd and one if m is even.*

Then X is of the same homotopy type of an almost parallelizable manifold.

Remark. Kervaire and Milnor have shown that if $\lambda \in \pi_{n-1}(SO(k))$ and $J(\lambda) = 0$, where J is the Hopf-Whitehead homomorphism, then to such a λ there corresponds an almost parallelizable manifold and vice versa [5]. Therefore λ in our theorem must be annihilated by J. To see that it is really so, we recall that our hypotheses imply the existence of a map of degree one, from a $(n + k)$-sphere to the Thom complex $T(\eta)$. The Thom complex $T(\eta)$ is an $S^k \cup (n+k)$-cell, where the cell is attached to the sphere by a map determined by $J(\lambda) \in \pi_{n+k-1}(S^k)$. The attaching is trivial because there is a map $T(\eta) \to S^{n+k}$ such that the composition is of degree one.

4. Proof of Theorem 3.3

Many stages of Professor Browder's proof are used here. Since $\Phi(g)$ is spherical, there is a map f such that

$$\begin{array}{ccc} S^{n+k} & \xrightarrow{f} & T(\zeta) \\ \cup & & \cup \\ N & & X \end{array} .$$

We can replace f by a differentiable map (still using the same label) within the same homotopy class. Furthermore, this map can be assumed to be transversally regular on X. Then $N = f^{-1}(X)$ is a closed submanifold in S^{n+k}, and its normal bundle is the induced bundle. All these assertions follow from theorems of Thom. We have now a map f from a closed manifold N into X. Since X is connected, we can make N connected by attaching tubes. We shall show that f is of degree one. N can easily be made simply connected by surgeries. We continue to use surgery to change N until f induces isomorphic homology. A theorem of J. H. C. Whitehead implies that N and X are homotopic. We proceed now to give proofs of these steps.

LEMMA 1. *If $N = f^{-1}(X)$, then $f:N \to X$ is of degree one.*

Proof. We simply examine the following two commutative diagrams and note that the correct generator goes to the generator:

$$\begin{array}{ccc} & f^*E(X) & \rightarrow E(X) \\ & \downarrow & \downarrow \\ S^{n+k} \supset & N & \longrightarrow X \end{array} ,$$

$$\begin{array}{ccc} H_n(N) & \xrightarrow{\Phi f^*\zeta} & H_{n+k}(f^*E, f^*E_0) \\ \downarrow & & \downarrow \\ H_n(X) & \xrightarrow{\Phi\zeta} & H_{n+k}(E, E_0) \end{array} .$$

f^*E_0 and E_0 are respectively the collections of non-zero vectors of the bundle f^*E and E. Q.E.D.

LEMMA 2. *If $N \to X$ as before, then the kernel of $f_*:H_*(N) \to H_*(X)$, is a direct summand and f_* is onto.*

Proof. Consider the following diagram:

$$\begin{array}{ccc} H^{n-j}(N) & \xleftarrow{f^*} & H^{n-j}(X) \\ P_N \downarrow \cong & & \cong \downarrow P_X \\ H_j(N) & \xrightarrow{f^*} & H_j(X) \end{array} .$$

If s is in $H_j(X)$, then there is $P_X^{-1}(s)$ in $H^{n-j}(X)$ such that $g \cap P_X^{-1}(s) = s$. It is easy to see that

$$f_* P_N f^* P^{-1}(s) = s;$$

that is,

$$\begin{aligned} f_* P_N f^* P^{-1}(s) &= f_*(i \cap f^*(P_X^{-1}(s))) \\ &= f_* i \cap P_X^{-1}(s) \\ &= g \cap P_X^{-1}(s) \\ &= s. \end{aligned}$$

Hence $f_* P_N f^* P_X^{-1}$ is the identity map of $H_*(X)$, and $H_*(N) = (P_N f^* P_X^{-1}) \cdot (H_*(X)) + \operatorname{Ker} f_*$.

LEMMA 3. *N can be made simply connected.*

Proof. Let $C_1, C_2, \cdots, C_r$ be simple closed curves in N, representing generators of the fundamental group of N. Since N is orientable, if we take a small tubular neighborhood of C_i, for each i, this neighborhood has a product structure, that is, $C_i \times D^{n-1}$. We remove the interior of each such neighborhood and fill it in with $D^2 \times S^{n-2}$. Since X is simply connected, the element represented by C_i is in the kernel of f_*. Q.E.D.

We may proceed now with this situation: there exists a map $f: N \to X$, where N is a closed manifold and N, X are simply connected. Also $f_*: H_*(N) \to H_*(X)$ is onto, and $\operatorname{Ker} f_*$ is a direct summand.

LEMMA 4. *$\pi_2(N) \to \pi_2(X)$ is onto.*

Proof. N, X are simply connected. We have the following commutative diagram:

$$\begin{array}{ccc} \pi_2(N) & \to & H_2(N) \\ \downarrow & & \downarrow \text{ onto.} \\ \pi_2(X) & \to & H_2(X) \end{array} \qquad \text{Q.E.D.}$$

In order to have isomorphic homology induced by f_*, we have to use surgery to kill the kernel of f_*, knowing that f_* is onto. However, surgery is applied only to kill homotopy groups [5]. Therefore we have to know that elements in the kernel f_* are spherical.

LEMMA 5. *If $f_*: H_j(N) \to H_j(X)$ is isomorphic, $2 \leqq j \leqq i$, then the kernel $f_*: H_{i+1}(N) \to H_{i+1}(X)$ is spherical, for fixed i.*

Proof. Consider the mapping cylinder Z_f of the map f. Both N and X can be naturally imbedded in Z_f. The map f is decomposed into two maps; that is,

$$N \to Z_f \to X.$$

The first map is an inclusion and the second map is a strong deformation retraction. We replace X by Z_f homotopically. Let us consider the following well-known exact sequences:

$$\begin{array}{ccccccccc} & & & & \pi_j(X) & & & & \\ & & & & \wr\wr & & & & \\ \cdots \to & \pi_{j+1}(Z_f, N) & \to & \pi_j(N) & \to \pi_j(Z_f) & \to & \pi_j(Z_f, N) & \to & \cdots \\ & \downarrow & & \downarrow & \downarrow & & \downarrow & & \\ \cdots \to & H_{j+1}(Z_f, N) & \to & H_j(N) & \to H_j(Z_f) & \to & H_j(Z_f, N) & \to & \cdots \\ & & & & \wr\wr & & & & \\ & & & & H_j(X) & & & & \end{array}$$

Since $f_*:H_j(N) \to H_j(X)$ is isomorphic for all $j \leqq i$, $H_j(Z_f, N) = 0$ and $\pi_{i+2}(Z_f, N) \to H_{i+2}(Z_f, N)$ is onto, by the relative Hurewicz theorem. The assertion follows from examining the sequences. Q.E.D.

Since N is simply connected, every element of $H_2(N)$ is spherical. We can perform surgery to kill the $\operatorname{Ker} f_*$, $f_*:H_2(N) \to H_2(X)$, to obtain isomorphic homology in this level. We then apply the above lemma to obtain isomorphic homology on the higher level.

There are some points of our procedure to be noticed. First, we have to show that there is a product structure for each imbedding; that is, there is an imbedding $\varphi:S^p \times D^{n-p} \to N$ for each element in the kernel f_*. Second, our theorem is concerned with even dimensions. As we shall see, we can kill the kernel f_* of dimensions less than m. For killing the mth dimensional elements we have to use an unusual kind of surgery—a removal of a handle body. The difficulty of the usual surgery at this dimension is that the Arf invariant appears. The invariant is defined when the $(m - 1)$-dimensional homology kernel is killed. Only when the Arf invariant vanishes can surgery work in dimension m.

LEMMA 6. *Let $\varphi:S^p \times D^{n-p} \to N$ be an imbedding and let $\varphi \mid S^p \times 0 = \varphi'$. Suppose that $f\cdot\varphi'$ is homotopic to a constant map. Then there is a map $f'':N' \to X$, where $N' = \chi(N, \varphi)$, the modified manifold of N* [5], *and f'' is of degree one.*

Proof. If $f\cdot\varphi' \sim *$, then $f\cdot\varphi \sim *$. By homotopy extension theorem, $f \sim f'$ and $f'(\varphi(S^p \times D^{n-p})) = *$. Then we set $f'' = f'$ outside $D^{p+1} \times S^{n-p-1}$ and $f''(D^{p+1} \times S^{n-p-1}) = *$. It is clear that f'' is of degree one if f was of degree one. Q.E.D.

LEMMA 7. *Let $S^p \to N^n \to X$, with $f\cdot\varphi \sim *$, $p < (1/2)n$. Then the normal bundle ν of S^p in N^n is trivial.*

Proof. Let τ denote the tangent bundle. Since $\varphi^*(\tau(N)) = \tau(S^p) \oplus \nu$, and $\tau(N) \oplus \eta$ is trivial, where $f^*(\zeta) = \eta$, (the normal bundle of N in S^{n+k}), hence $\varphi^*(\tau(N) \oplus \eta) = \tau(S^p) \oplus \nu \oplus \varphi^*(\eta)$ is trivial. Since $\eta = f^*(\zeta), \varphi^*\eta = \varphi^*f^*\zeta = (f\varphi)^*\zeta$ is trivial (recall that $f\cdot\varphi \sim *$). But $\tau(S^p) \oplus \epsilon$ is trivial if ϵ is trivial. Hence $\tau(S^p) \oplus \varphi^*(\eta) \oplus \nu = \tau(S^p) \oplus \varphi^*\eta \oplus \nu = \epsilon \oplus \nu$ is trivial, where $\epsilon = \tau(S^p) \oplus \varphi^*\eta$ is trivial. ν is a $(n - p)$-dimensional bundle over S^p, and, since $n - p > p$, this implies that ν is trivial (see [6], Lemma 4).

hypothesis iii), it is easy to see that $\text{Ker} f_* \otimes \text{Ker} f_*$ has zero signature. Since the normal bundle is almost trivial, the new manifold is almost parallelizable. For the case $n = 6$ or $n = 14$, we may use the argument of removal of a handle body as in the proof of Theorem 3.3. Therefore Theorem 3.4 is true for $n = 6$ or $n = 14$ without hypothesis iii). Q.E.D.

COROLLARY. *The last theorem is also true for* $n = 6$ *or* $n = 14$ *without the index hypothesis.*

Added in the proof. Recent results of E. H. Brown and F. P. Peterson (Bull. Amer. Math. Soc., **71**[1965] 190–93) have settled the problem on homotopy types of $(8k + 2)$-dimensional π-manifolds. More precisely, we have the following.

THEOREM. *Let X be a finite connected polyhedron with $\pi_1(X) = 0$. The sufficient conditions for X to be the same homotopy type of an $(8k + 2)$-dimensional π-manifold are the following $(2n = 8k + 2 > 5)$:*

1) *X satisfies the Poincaré duality;*

2) *$\Sigma_*^m(g) \in H_{2n+m}(\Sigma^m X;Z)$ is spherical for some integer m, where Σ^m denotes m-time suspensions and Σ_*^m, the corresponding homomorphism;*

3) *the Steenrod operation $\text{Sq}^2 : H^{2n-2}(X;Z_2) \to H^{2n}(X;Z_2)$ is trivial;*

4) *the Kervaire invariant $\Psi(X)$ is zero.*

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