# HOMOTOPY-SMOOTH SPHERE FIBRINGS

By W. A. Sutherland*

## 1. Introduction

This note contains miscellaneous remarks relating to the work of W. Browder and of M. Spivak described below. The title refers to the objects whose study is our general theme. In particular, the following seem to be of interest: an alternative proof, and slight strengthening, of one of Spivak's results (Proposition 2.1); a variant of J. Levine's embedding theorem for simply-connected manifolds (in §4); and a question raised about differentiable bundles by partial results (in §5).

Throughout, a "$q$-sphere bundle" will mean a fibre bundle with fibre the $q$-sphere $S^q$ and group an orthogonal group, and a "$q$-sphere fibring" will mean a Hurewicz fibring with fibres homotopy equivalent to $S^q$. A "closed" manifold is one which is compact and without boundary.

Let $M$ be a 1-connected smooth closed manifold, and let $B$ be the total space of a $q$-sphere fibring $\xi$ over $M$, with $q > 1$. We shall abbreviate "$B$ has the homotopy type of a smooth closed manifold" to "$B$ is homotopy-smooth." Let $X$ be any space having the homotopy type of a 1-connected $CW$-complex with finitely generated homology groups and satisfying the cap product form of Poincaré duality. It is not hard to show that $B$ satisfies these conditions. According to [1], certain necessary conditions for such an $X$ to be homotopy-smooth are often sufficient. One necessary condition is the existence of a reducible sphere bundle over $X$, where a "reducible" sphere fibring is one whose Thom space $T$ is reducible; i.e., the integral homology of $T$ is zero above some dimension where it is infinite cyclic with a spherical generator. In [8] Spivak proved that there is a unique stable fibre homotopy equivalence class of reducible sphere fibrings over $X$ and described that class for $B$. Proposition (2.1) gives an alternative proof of that description which permits an "unstable" conclusion. This is used in §4 to obtain metastable embeddings.

For $B$ to be homotopy-smooth, it is sufficient that $B$ be homotopy equivalent to the total space of a sphere bundle over $M$: in particular, that $\xi$ be fibre homotopy equivalent to a sphere bundle; in fact it is often sufficient that $\xi$ be stably fibre homotopy equivalent to a sphere bundle (Proposition (3.1)). However, examples in §3 show that no one of these conditions separately is necessary.

In §5 partial results are obtained which suggest the apparently difficult question, "When is a differentiable bundle, with sphere fibre, fibre homotopy equivalent to a sphere bundle?"

I should like to thank Dr. M. Spivak for showing me [8] in typescript and Dr. J. D. Stasheff for helpful comments.

* The author gratefully acknowledges the support of the N.S.F. under Contract No. GP-4362 during the preparation of this note.

## 2. A reducible sphere fibring over $B$

Let $SH(n)$ denote the space of oriented homotopy equivalences of $S^{n-1}$; let $BSH(n)$ denote the corresponding classifying space; and let $BSH$ denote the direct limit (for details see e.g. §2 of [10]). Given spaces $P$, $Q$ with basepoints, let $[P, Q]$ denote the set of homotopy classes of maps from $P$ to $Q$, where maps and homotopies preserve basepoints. Let $K$ be a finite $CW$-complex. By the oriented version of [9], $[K, BSH(n)]$ classifies the oriented fibre homotopy equivalence classes of oriented $(n-1)$-sphere fibrings over $K$; and, by for example (1.9) of [8], the Whitney join gives $[K, BSH]$ an abelian group structure. In particular, any oriented sphere fibring $\xi$ over $K$ has a "negative," i.e., a sphere fibring $\eta$ over $K$ such that the Whitney join $\xi * \eta$ is fibre homotopy trivial.

The integer $n$ will sometimes denote a trivial $(n-1)$-sphere fibring. The stable form of the following proposition is similar to (5.4) of [8].

PROPOSITION (2.1). *Let $M$ be a finite 1-connected $CW$-complex over which there is a reducible sphere fibring $\nu$. Let $\pi : B \to M$ be the projection of a $q$-sphere fibring $\xi$ $(q > 1)$ and let $\eta$ be a negative for $\xi$. Then the sphere fibring $\pi^*(\eta * \nu * 1)$ over $B$ is reducible.*

*Note.* The following key, in which $E(\ )$ denotes the total space of a fibring and $T(\ )$, its Thom space, may facilitate reading the proof: $A = E(\eta * \nu)$, $B = E(\xi)$, $P = E(\pi^*(\eta * \nu))$, $R = E(\xi * \eta * \nu)$, $T = T(\pi^*(\eta * \nu))$, $T' = T(\xi * \eta * \nu)$.

*Proof.* Let $\psi : A \to M$ be the projection of $\eta * \nu$. Let $M_\psi$, $M_\pi$ be the mapping cylinders of $\psi$, $\pi$; thus, to obtain $M_\psi$ from the disjoint union of $M$ and $A \times I$, where $I$ is the closed interval $[0, 1]$, one identifies $\psi(a)$ with $(a, 1)$ for each $a$ in $A$. Embed $A$ in $M_\psi$ by identifying $a$ with the image of $(a, 0)$ in $M_\psi$; embed $B$ in $M_\pi$ similarly. Let $\psi' : M_\psi \to M$ be the map induced by sending $(a, t)$ to $\psi(a)$ and $m$ to $m$ for any $a$ in $A$, $t$ in $I$, $m$ in $M$, and define $\pi'$ similarly. Let $P \subset Q \subset R$ be the total spaces of the fibrings induced by the diagonal map $\Delta$ of $M$ from the restrictions of $\pi' \times \psi'$ to the subspaces $B \times A$, $B \times M_\psi$, $B \times M_\psi \cup M_\pi \times A$ of $M_\pi \times M_\psi$. The Thom space $T$ of $\pi^*(\eta * \nu)$ is naturally homeomorphic with $Q \cup CP$, where, if $(p, t)$ in $P \times I$ represents a point of the cone $CP$, then all $(p, 1)$ are identified to a point and $(p, 0)$ is identified with $p$ in $Q$. By definition, $R$ is the total space of $\xi * (\eta * \nu)$. Let $h$ be the natural inclusion of $M$ in $M_\pi$. Since points of $R$ are specified by triples $(m, (x, y))$ where $m \in M$, $(x, y) \in B \times M_\psi \cup M_\pi \times A$, and $\Delta(m) = (\pi'(x), \psi'(y))$, it follows that a map $f : A \to R$ is defined by

$$f(a) = (\psi(a), (h\psi(a), a)).$$

I claim that $T$ is homotopy equivalent to $R \cup_f CA$, the result of identifying $(a, 0)$ in $CA$ with $f(a)$ in $R$ for all $a$ in $A$. (An explicit homotopy equivalence $\theta$ from $Q \cup CP$ to $R \cup_f CA$ is defined as follows: $\theta$ restricted to $Q$ is the inclusion in $R$; while if $a$ in $A$ and $b$ in $B$ are such that $\pi(b) = \psi(a) = m$, so that $((m, (b, a)), t)$ represents a point of $CP$, define

$$\theta((m, (b, a)), t) = \begin{cases} (m, ((b, 2t), a)) & \text{in } R, \text{ if } 0 \leqq 2t \leqq 1 \\ (a, 2t - 1) & \text{in } CA, \text{ if } 1 \leqq 2t \leqq 2. \end{cases}$$

It is easy to construct an explicit homotopy inverse for $\theta$.)

Since $\xi * \eta$ is fibre homotopy trivial and the Whitney join is associative, the Thom space $T'$ of $\xi * (\eta * \nu)$ has the homotopy type of an iterated suspension of the Thom space of $\nu$. Hence, since $\nu$ is reducible, so is $T'$. Now there exist maps $\alpha$ of $T'$ to the suspension $\Sigma R$ of $R$, and $\beta$ of $R$ to $R \cup_f CA$, and hence $\gamma$ of $T'$ to $\Sigma T$, which induce epimorphisms (in fact isomorphisms) of the highest-dimensional non-zero integral homology groups. Namely, let $\beta$ be the inclusion, and to define $\alpha$ note that $T'$ is $M \cup_\phi CR$ where $\phi$ is the projection of $\xi * (\eta * \nu)$; let $\alpha$ be the inclusion of $T'$ in the result of attaching $CM$ to $T'$. Since $T'$ is reducible and by the existence of such a $\gamma$, it follows that $\Sigma T$ is reducible, which completes the proof.

*Remark.* I am unable to decide in general whether $\pi^*(\eta * \nu)$ is reducible. However, this *is* true when either (a) $\nu \sim \mu * 1$ ("is fibre homotopy equivalent to") for some sphere fibring $\mu$; or (b) $\nu * \eta \sim \zeta * 1$ for some $\zeta$, and the fibre dimension of $\nu * \eta * \xi$ is greater than the dimension of $M$. For suppose (a) holds, and let $T''$ denote the Thom space of $\xi * \eta * \mu$. Then $T''$ is reducible, and it is easy to see that $T''$ has the homotopy type of $R \cup CM$ where the cone $CM$ is attached by a map of $M$ into the subspace $f(A)$ of $R$ (namely by a cross section of $\nu * \eta$). Hence there is a map of $R \cup CM$ into $R \cup_f CA$ extending the identity map of $R$; so $R \cup CA$ is reducible since $R \cup CM$ is reducible. The argument for (b) is similar.

## 3. Examples

Let $\pi : B \to M$ be the projection of a $q$-sphere fibring $\xi$ over a 1-connected smooth closed $m$-manifold, with $q > 1$. This section contains remarks on when $B$ is homotopy-smooth.

Let $\xi_s$ denote the stable class of $\xi$ in $[M, BSH]$. Let $BSO$ be a classifying space for the stable rotation group, and let $J : [M, BSO] \to [M, BSH]$ be induced by the natural inclusions of $SO(n)$ in $SH(n)$. Let $\eta$ be a negative for $\xi$, and let $\nu$ be the normal bundle of some embedding of $M$ in Euclidean space. By Theorem A of [8] and (2.1), there is a reducible sphere bundle over $B$ if and only if $\pi^*(\eta * \nu)_s$ is in $J[B, BSO]$, which is true if and only if $\pi^*(\xi_s)$ is in $J[B, BSO]$. Thus, for example, when the dimension $b$ of $B$ is odd, $B$ is homotopy-smooth if and only if $\pi^*(\xi_s)$ is in $J[B, BSO]$. An attempt to describe this condition in terms of $\xi$ itself, by comparing the statements "$B$ is homotopy-smooth" and "$\xi_s$ is in $J[M, BSO]$," leads to the following.

PROPOSITION (3.1). *With the above notation, suppose that $\xi_s$ is in $J[M, BSO]$ and that either $b \not\equiv 2 \bmod 4$ or $3m \leqq 2b - 1$. Then $B$ is homotopy-smooth.*

*Proof.* By (2.1), $\pi^*(\zeta * \nu)_s$ is reducible where $\zeta$ is a sphere bundle such that $J\zeta_s = -\xi_s$. The result for $b \not\equiv 2$ mod 4 now follows by applying [1] (noting, when $b$ is divisible by 4, that the index of $B$ is zero, by the multiplicative property of the index for fibrings, and the Hirzebruch $L$-genus in the dual Pontrjagin classes of $\pi^*(\zeta * \nu)_s$ is zero, for dimensional reasons).

If $3m \leqq 2b - 1$, then, by (2.2) and (3.2) of [4] and (3.2) of [10], $\xi$ is fibre homotopy equivalent to a sphere bundle (cf. proof of (4.1) below); so $B$ is homotopy-smooth.

If $\xi$ admits a cross section and $B$ is homotopy-smooth, then $\xi_s$ is in $J[M, BSO]$, since $\pi^*(\xi_s)$ is in $J[B, BSO]$.

EXAMPLE (3.3). *There is a sphere fibring $\xi$ whose total space $B$ is homotopy-smooth, and $\xi_s$ is not in the image of $J$.*

In fact, in the example, $B$ has the homotopy type of a $\pi$-manifold. To construct $\xi$, consider the composite isomorphism

$$\pi_{13}(SH(14)) \to \pi_{13}(F(14)) \to \pi_{27}(S^{14}),$$

where $F(14)$ is the subspace of basepoint-preserving maps in $SH(15)$, the latter isomorphism is as in (2.10) of [12], and the former is induced by unreduced suspension (see (2.2) of [4]; $SH(n)$ is $G_{n-1}$ of that reference). Let $[\iota, \iota]$ denote the Whitehead square of a generator of $\pi_{14}(S^{14})$, and let $\alpha_1$, $\beta_1$ be the homotopy classes thus named in [11] (p. 180). Let $\xi'$ in $\pi_{13}(SH(14))$ correspond under the above isomorphism to $[\iota, \iota] + 2\alpha_1 \circ \beta_1$ in $\pi_{27}(S^{14})$. As in [5] (p. 128), $\xi'$ defines a quasi-fibration; replace this by a Hurewicz fibring $\xi$ (see e.g. p. 241 of [9]). Then the suspension $\Sigma B$ has the homotopy type of $e^{28} \cup_{2\alpha_1 \circ \beta_1} S^{14} \cup_{2\iota} e^{15}$ (where subscripts specify the maps attaching the cells $e^{15}$ and $e^{28}$ to $S^{14}$) and hence of $(S^{14} \cup_{2\iota} e^{15}) \cup_* e^{28}$, where $*$ denotes the constant map. Hence any trivial $q$-sphere bundle over $B$ with $q$ large enough is reducible, and by [1] $B$ has the homotopy type of a $\pi$-manifold. Also, $\xi_s$ is non-zero while $\pi_{14}(BSO)$ is zero, so $\xi_s$ is not in the image of $J$.

However, Theorem 7.5 of [5], extended as mentioned (p. 128 of [5]), shows that $B$ is homotopy equivalent to the total space of the tangent sphere bundle of $S^{14}$.

EXAMPLE (3.4). *There is a 3-sphere fibring $\eta$ over $S^8$, with total space $B$ which is homotopy-smooth but not homotopy equivalent to the total space of any 3-sphere bundle.*

To construct $\eta$, consider the homomorphisms

$$\pi_7(F(3)) \to \pi_7(SH(4)) \to \pi_7(F(4))$$

induced by natural inclusions. The composition is an isomorphism of groups of order 15. Let $\eta$ correspond, as in (3.3), to the image $\eta'$ in $\pi_7(SH(4))$ of a non-zero element of $\pi_7(F(3))$. Then $\eta'$ is of odd order. Also, $\eta$ admits a cross section. (Incidentally, for $\eta'$ of order 5, $\eta \sim \zeta * 1$ is false for all $\zeta$.) Since $J\pi_8(BSO) = \pi_8(BSH)$, $B$ is homotopy-smooth by (3.1). Now $\pi_7(SO(4))$ is finite and has no elements of odd order; hence its image in $\pi_7(SH(4))$ does not contain $\eta'$. Hence, by (1.6) and (1.10) of [6], extended as in [5] (p. 128), $B$ is not homotopy equivalent to the total space of a 3-sphere bundle.

## 4. Metastable embeddings

Suppose that $B$ of §3 is homotopy-smooth. This section gives metastable embeddings of any smooth closed manifold homotopy equivalent to $B$ analogous to some of the results of [3]. We use the following variant of Theorem 1 of [7].

PROPOSITION (4.1). *Let $B$ be a 1-connected smooth closed $b$-manifold with $b > 4$, and let $\xi$ be a reducible $(r - 1)$-sphere fibring over $B$, with $2r > b + 2$. Then there is an embedding of $B$ in $S^{b+r}$ whose normal sphere bundle is fibre homotopy equivalent to $\xi$*

(a) *over $B$, if $b = 6$, 14 or $b \not\equiv 2 \bmod 4$,*

(b) *over $B - U$, if $b \equiv 2 \bmod 4$, where $U$ is an open $b$-disk in $B$.*

*Proof.* Since $\xi$ is reducible, $\xi$ is stably fibre homotopy equivalent to the stable normal bundle $\nu$ of $B$ by Theorem A of [8]. By (2.2) and (3.2) of [4], the homomorphism from $\pi_i(BSO, BSO(r))$ to $\pi_i(BSH, BSH(r))$ induced by the natural maps is bijective for $i < 2r - 2$ and surjective for $i = 2r - 2$. Since $2r - 2 \geqq b + 1$, it follows from (3.2) of [10] that there is an $r$-sphere bundle over $B$ which is stably equivalent to $\nu$ and fibre homotopy equivalent to $\xi$ and, hence, reducible. The result now follows by Theorem 1 of [7].

The following two sample propositions, which follow easily from (2.1) and (4.1), are analogous to (1.1) and (1.4) of [3].

PROPOSITION (4.2). *Let $\pi: B \to M$ be the projection of a $(p - 1)$-sphere fibring $\xi$ over a smooth 1-connected closed $m$-manifold $M$, and let $\eta$ be a $(q - 1)$-sphere negative for $\xi$. Suppose that $M$ embeds in $S^t$ with normal sphere bundle $\nu$. Let $m + p > 5$, with $2(t + q) > 3m + p - 1$. Let $f: B' \to B$ be a homotopy equivalence, where $B'$ is a smooth closed manifold. Then $B'$ embeds in $S^{t+p+q}$ with normal sphere bundle fibre homotopy equivalent to $f^*\pi^*(\nu * \eta * 1)$*

(a) *over $B'$, if $m + p - 1 = 6$, 14 or $m + p - 1 \not\equiv 2 \bmod 4$*

(b) *over $B' - U$, if $m + p - 1 \equiv 2 \bmod 4$, where $U$ is an open disk in $B'$.*

Apart from being restricted to metastable range, this loses one dimension on [3] in general; as remarked after (1.2), that dimension may be regained if, for example, $\nu$ admits a cross section.

PROPOSITION (4.3). *Let $B$ be a smooth closed manifold homotopy equivalent to the total space of a $(p - 1)$-sphere fibring $\xi$ over $S^n$, where $\xi$ admits a cross section; and let $\eta$ be a $(q - 1)$-sphere negative for $\xi$, with $\eta * r$ fibre homotopy non-trivial. Suppose $n, p > 1$ and $2(q + r) > n + p - 1$. Then $B$ embeds in $S^{n+p+q+r}$ with fibre homotopy non-trivial normal sphere bundle.*

## 5. Differentiable bundles

A differentiable bundle is one whose total space, base, fibre, and projection are smooth. Let $\pi: B \to M$ be the projection of a differentiable bundle $\xi$, with fibre $S^q$ $(q > 1)$ and closed 1-connected base $M$. Let $\hat{\xi}$ denote the tangent sphere bundle along the fibres. If $\xi$ were a sphere bundle, $\hat{\xi}*1$ and $\pi^*(\xi)$ would be orthogonally equivalent. In the present circumstances, Theorem A of [8] and the known form of the stable normal bundle of $B$ show that $\hat{\xi}$ and $\pi^*(\xi)$ are stably fibre homotopy equivalent. In particular, if $\xi'$ denotes $\xi$ considered as a fibring, then, in the notation of §3, $\pi^*(\xi_s')$ is in $J[B, BSO]$. If $\xi'$ admits a cross section, this implies that $\xi_s'$ is in $J[M, BSO]$; and if, further, the dimension of $M$ is less than $2q - 2$, then, as in the proof of (4.1) above, $\xi$ is fibre homotopy equivalent to a sphere bundle. In particular, the set of fibre homotopy equivalence classes of bundles like $\xi$ stabilizes to $J[M, BSO]$ as $q$ increases. (It does not seem clear how the differentiable bundle equivalence classes of such $\xi$ might stabilize.)

These remarks give rise to the question at the end of §1. The following stronger result in a particular case follows from [2].

PROPOSITION (5.1). *Let $\xi$ be a differentiable bundle with fibre and base both spheres. Then $\xi$ is fibre homotopy equivalent to a sphere bundle.*

An equivalent statement follows: let $G$ be the group of diffeomorphisms of degree $+1$ of $S^q$, with the $C^\infty$-topology and basepoint the identity map, and let $J'\colon \pi_i(G) \to \pi_{i+q+1}(S^{q+1})$ be the analogue of $J\colon \pi_i(SO(q + 1)) \to \pi_{i+q+1}(S^{q+1})$. Then the images of $J$ and $J'$ coincide.

*Proof.* Let $G_T \subset G_1$ be the subspaces of $G$, of maps respectively holding a closed neighbourhood $T$ of a point $P$ in $S^q$ pointwise fixed and "1-tangent to the identity at $P$" (see p. 335 of [2]). Then the natural injections of $G_T$ in $G_1$ and of $SO(q + 1) \times G_1$ in $G$ induce isomorphisms of homotopy groups (see p. 335 and Prop. 11, p. 341, of [2]). Now the analogue of $J$ applied to $\pi_i(G_T)$ is zero ($T$ may be stretched over all of $S^q$ by a fixed homotopy equivalence), and (5.1) follows easily.

MASSACHUSETTS INSTITUTE OF TECHNOLOGY, CAMBRIDGE

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