A SAMELSON PRODUCT IN $SO(2n)$*

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Consider the sequence

$$\pi_j(O(2n)) \to \pi_j(O(2n+1)) \xrightarrow{p_*} \pi_j(S^{2n}) \xrightarrow{\partial_*} \pi_{j-1}(O(2n)).$$

The main result of [1] can be thought of as the following: the homomorphism ∂_* is just $i_*: \pi_j(S^{2n}) \to \pi_j(V_{4n,2n})$ followed by a monomorphism onto a direct summand $\partial_{2*}: \pi_j(V_{4n,2n}) \to \pi_{j-1}(S^{2n})$ for all $j < 4n - 1$. The main result of this note is concerned with what happens if $j = 4n - 1$.

Let $P(\iota_{2n})$ be the Whitehead product of ι_{2n} with itself, where ι_k represents a generator of $\pi_k(S^k)$. Then $\partial_* P(\iota_{2n})$ is called the Samelson product, and we are interested in the order of this element. In particular, we will prove

THEOREM A. *If $n \neq 1$, 2, or 4, then the order of $\partial_* P(\iota_{2n})$ is $a_n(2n-1)!/8$, where $a_n = 1$ if n is even and 2 if n is odd.*

For comparison with the result of [1] we have

THEOREM B. *If $i: S^{2n} \to V_{4n+1,2n+1}$ is a generator of $\pi_{2n}(V_{4n+1,2n+1})$ and if $n \neq 1$, 2, or 4, then $i_* P(\iota_{2n})$ is infinite cyclic and generates a direct summand.*

Consider the sequence

$$\begin{array}{ccccc} & & & & S^{2n} \\ & & & & \downarrow i \\ SO(2n) & \xrightarrow{i} & SO(4n+1) & \xrightarrow{p} & V_{4n+1,2n+1}. \end{array}$$

THEOREM C. *Let α_n generate $\pi_{4n-1}(SO(4n+1))$, $n \neq 1, 2$, or 4. Then*

$$p_*(\alpha_n) = (a_n/4)(2n-1)!\, i_* P(\iota_{2n}).$$

Lundell announced results [6] related to these (in particular to Theorem A), but these sharpen his.

The proof of Theorem A uses the strong form of James' result [5] which gives the commutative diagram, exact on the two components for all j,

$$\begin{array}{ccccccc} \to \pi_j(SO(2n)) & \to & \pi_j(SO(2n+1)) & \to & \pi_j(S^{2n}) & \longrightarrow & \\ \downarrow J & & \downarrow J & & \downarrow \Sigma & & \\ \to \pi_{j+2n}(S^{2n}) & \longrightarrow & \pi_{j+1+2n}(S^{2n+1}) & \longrightarrow & \pi_{j+1}(S^{4n+1}) & \to . & \end{array}$$

In addition we need the sharpened form of the Barratt-Mahowald result [1] (due to Barratt [2]), which asserts that, for $n \neq 1$, 2, or 4, if α_n generates

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$\pi_{4n-1}(SO(4n+1))$, then there is an element α_n' in $\pi_{6n-1}(S^{2n})$ such that $\Sigma^{2n+1}\alpha_n' = J\alpha_n$.

The above result of Barratt implies, using the diagram

$$\begin{array}{ccc} & & \pi_{4n}(BSO(4n+1)) \\ & & \uparrow i_* \\ \pi_{4n}(BSO(2n)) & \longrightarrow & \pi_{4n}(BSO(2n+1)) \\ \downarrow J & & \downarrow J \\ \pi_{6n-1}(S^{2n}) & \underset{\Sigma}{\longrightarrow} & \pi_{6n-1}(S^{2n+1}), \end{array}$$

that there is an element $\alpha \in \pi_{4n}(BSO(2n+1))$ such that $i_*\alpha$ generates $\pi_{4n}(BSO(4n+1))$ and $J(\alpha) \in im\Sigma$. Consider the bundle $S^{2n} \to Y \to S^{4n}$ induced over S^{4n} by α. In the homotopy sequence of this bundle, $\partial_*\iota_{4n}$ is the attaching map by which, in Y, the cell in dimension $4n$ is attached to the cell in dimension $2n$. Using James' result, we have $\Sigma\partial_*\iota_{4n} = 0$. Hence $\partial_*\iota_{4n} = \lambda P(\iota_{2n})$, where λ is some integer. Clearly the order of the Samelson product is just λ.

To compute λ we consider the diagram

$$\begin{array}{ccccc} & & BSO(2n) & \xrightarrow{p_1} & BSO(2n+1) \\ & {\scriptstyle i_1}\nearrow & \uparrow \bar{\alpha} & & \uparrow \alpha \\ S^{2n} & & & & \\ & {\scriptstyle i_2}\searrow & & & \\ & & Y & \underset{p_2}{\longrightarrow} & S^{4n} \end{array} .$$

It is clear that i_1 is the classifying map for the tangent bundle of S^{2n}, and hence $i_1{*}\chi = 2\kappa_{2n}$, where κ_{2n} generates $H^{2n}(S^{2n}; Z)$, and χ is the Euler class. Let y_i generate $H^i(Y; Z)$ for $i = 2n, 4n$, or $6n$. Let $p_n \in H^{4n}(BSO(2n+i))$, $i = 0$ or 1, be the nth Pontryagin class. Then $\alpha^* p_n = a_n(2n-1)!\,\kappa_{4n}$ where κ_{4n} generates $H^{4n}(S^{4n}; Z)$ and $a_n = 1$ if n is even and 2 if n is odd ([7], p. 131). Putting these facts together, we have $\bar{\alpha}^*\chi = 2y_{2n}$ and $\bar{\alpha}^*p_n = a_n(2n-1)!\,y_{4n}$. But, in $H*(BSO(2n))$, $\chi^2 = p_n$ ([7], p. 84). Hence $4y_{2n}^{\;2} = a_n(2n-1)!\,y_{4n}$. Finally, in any complex $S^{2n} \cup_{\lambda P(\iota_{2n})} e^{4n}$, the cup product square of the cell in dimension $2n$ is 2λ times the cell in dimension $4n$. Hence $\lambda = a_n(2n-1)!/8$.

To prove Theorem B we first recall some facts about the cohomology of Stiefel manifolds. According to Borel [3], $H*(V_{k+m,m}; Z_2)$ is an algebra in primative generators $h_i \in H^i(V_{k+m,m}; Z_2)$, $k \leq i < k+m$. The Steenrod algebra acts according to $\mathrm{Sq}^j h_i = \binom{i}{j} h_{i+j}$. If $k \equiv 0 \bmod 2$, then $H^k(V_{k+m,m}; Z) = Z$, while $H^{2i}(V_{k+m,m}; Z) = Z_2$ if $k \leq 2i < k+m-1$.

Let Y' be the space in the Postnikov tower of $V_{4k+1,2k+1}$ for which all the homotopy groups up through dimension $4k-2$ have been added. That is, there is a map $f: V_{4k+1,2k+1} \to Y'$ such that $f_*: \pi_j(V_{4k+1,2k+1}) \to \pi_j(Y')$ is an isomorphism for $j \leqq 4k-2$ and $\pi_j(Y') = 0$ for $j \geqq 4k-1$. Then f^* is an isomorphism through dimension $4j-2$. Define classes y_i' in $H*(Y')$ by $f^* y_i' = h_i$.

LEMMA 1. *Suppose* $2^{j-1} < 2k < 2^j$ *and* $2k = 2^i(2l-1)$. *Then in* $H^*(Y'; Z_2)$, $\mathrm{Sq}^{2k} y_{2k}' = \mathrm{Sq}^1 \mathrm{Sq}^{4k-2^j} y_{2^j-1}'$.

Proof. Using the Adem relation and Borel's formula, we see $\mathrm{Sq}^{2k} y_{2k}' = \mathrm{Sq}^{2^i} \mathrm{Sq}^{2^{i+1}l} y_{2k}'$ and $\mathrm{Sq}^{2^{i+1}l} y_{2k}' = \mathrm{Sq}^{2^{i+1}l+2k-2^j+1} y_{2^j-1}'$. Finally

$$\mathrm{Sq}^{2^i} \mathrm{Sq}^{2^{i+1}l+2k-2^j+1} y_{2^j-1}' = \mathrm{Sq}^{4k-2^j+1} y_{2^j-1}' = \mathrm{Sq}^1 \mathrm{Sq}^{4k-2^j} y_{2^j-1}'.$$

Let $\lambda: S^{4n} \to BSO(4n+1)$ be a generator of $\pi_{4n}(BSO(4n+1)) = Z$. We then have this diagram:

$$\begin{array}{ccccc} & & Y & \xrightarrow{p_1} & S^{4n} \\ & {}^{i_1}\nearrow & \Big\downarrow \lambda_1 & & \Big\downarrow \lambda \\ V_{4n+1,2n+1} & & & & \\ & {}_{i_2}\searrow & & & \\ & & BSO(2n) & \xrightarrow[p_2]{} & BSO(4n+1). \end{array}$$

Let κ_{4k} generate $H^{4k}(S^{4k}; Z)$. If $j < 4k$, then i_1* is an isomorphism and we define classes $y_i \in H^i(Y)$ such that $i_1 * y_i = h_i$.

LEMMA 2. *If* $n \neq 1, 2$, *or* 4, *then* $H^{4n}(Y; Z) = Z + Z_2$ *and the sequence* $H^{4n}(S^{4n}) \to H^{4n}(Y) \to H^{4n}(V_{4n+1,2n+1})$ *splits.*

Proof. Let p_n be the nth Pontryagin class. Now $\lambda^* p_n = a_n(2n-1)!\,\kappa_{4n}$. But, in $H*(BSO(2n); Z)$, $\chi^2 = p_n$, where χ is the Euler class. Also $\lambda_1 * \chi = 2y_{2n}$. Combining these results, we see that $\chi_1 * p_n = a_n(2n-1)!\, p_1 * \kappa_{4n}$ and that it also equals $4y_{2n}^2$. Since $i_1 * y_{2n}^2 \neq 0$ (even mod 2), y_{2n}^2 cannot be divided by 2; yet $4y_{2n}^2$ must be divided by $a_n(2n-1)!$, which is $8d$ for some d if $n \neq 1, 2$. This can only happen if $H^{4n}(Y; Z) = Z + Z_2$.

Assume now that $n \geqq 3$. Let $a = p_1 * \kappa_{4n}$, and let b generate the finite part. Let $S^{2n'}$ be the stage of the Postnikov tower of S^{2n} in which all the homotopy up through dimension $4n-2$ has been added. Note also that Y' is the corresponding stage for Y, as it is for $V_{4k+1,2k+1}$. Since $H^{4n}(Y; Z) = Z + Z_2$, $H^{4n}(Y'; Z) = Z + Z_2$ also. Let a' and b' be the generators of the infinite and the finite parts, respectively (a' is not unique).

LEMMA 3. *If* $n \neq 1, 2$, *or* 4 *then, in* $H*(Y'; Z)$, $y_{2n}'^2 = 2a' + b'$.

Proof. If $n \neq 2^j$, then Lemma 1 asserts that $(y_{2n}'^2) \bmod 2$ is in $\mathrm{Sq}^1 H^{4n-1}(Y'; Z_2)$. Hence $y_{2n}'^2 = 2da' + b'$, where d is some integer. Consider the map $f: S^{2n'} \to Y'$

which induces an isomorphism in homotopy in dimension $2n$. Then $f*(y_{2n}'^2)$ is twice a generator. This implies that $d = 1$.

Now suppose that $n = 2^j$. According to Brown and Peterson [4], the unstable secondary cohomology operation φ_{2n} based on $\mathrm{Sq}^2\,\mathrm{Sq}^{2n-1} = 0$, which is a relation on integer classes of dimension $2n$, is non-zero in the two-cell complex $S^{2n} \cup_{P(\iota_{2n})} e^{4n}$. Since $\mathrm{Sq}^{2n-1} = \mathrm{Sq}^1\,\mathrm{Sq}^{2n-1}$ and $H^{4n-1}(Y'; Z) = 0$, the operation is defined on y_{2n}'. Mod 2 we have $f*\varphi_{2n}(y_{2n}') \neq 0$, with zero indeterminacy. Hence, mod 2, $f*$ is non-trivial. But if $y_{2n}'^2 = a' + b'$, then $f*(a') \equiv 0 \bmod 2$ (since with integers for coefficients it is twice a generator). This completes the proof of the lemma.

Proof of Theorem B. Clearly the k-invariant for the infinite part of $\pi_{4k-1}(V_{4k+1,2k+1})$ is a' and $f*a'$ is the k-invariant for $[\iota_{2n}, \iota_{2n}]$.

Theorem C now is just a restatement of a portion of what was proved to get Theorems A and B.

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