EMBEDDING AND IMMERSION OF PROJECTIVE SPACES

By J. Adem, S. Gitler, and M. Mahowald*

Introduction

Considerable progress has been made recently in the problem of embedding and immersing of manifolds. The families of projective spaces RP^n, CP^n and QP^n—the real, complex, and quaternionic, respectively— form an interesting collection of manifolds, since they seem to have embedding and immersion difficulties which are close to the worse possible for each dimension. For example, if $n = 2^j$ or $2^j + 1$, the exact results for embedding all manifolds are known and RP^n in each case proves that the result is best possible. In this note we prove the following.

Theorem 1. *If $n = 2^r + 5$ and $n > 13$, then RP^n embeds in R^{2n-3} but not in R^{2n-4}.*

We also obtain

Theorem 2. *If $n = 4(2^r + 2^s) + 3$ and $r > s$, then RP^n cannot be embedded* in R^{2n-8}.

This second result is not as sharp as Theorem 1, in the sense that it is not known whether this result is best possible. In particular we have that RP^{15} does not embed in R^{22}, but that it does embed in R^{24} (by [8], (1.4)).

From [12], it is known that, if $n = 2^r + 2^s$ and $r > s$, QP^n embeds in R^{8n-4}. We prove that this result is best possible, namely,

Theorem 3. *If $n = 2^r + 2^s$ and $r > s$, then QP^n embeds in R^{8n-4} but not in R^{8n-5}.*

This result answers in the negative a problem raised by Sanderson (Problem 4′ of [6]) to the effect that the best possible embeddings and immersions of QP^n coincide.

We also include the following immersion results:

Theorem 4. *If $n = 2^r + 2^s + 2^t$ and $r > s > t$, then QP^n immerses in R^{8n-6} but not in R^{8n-7}.*

Theorem 5. *If $n = 2^r + 2^s + 2^t$ and $r > s > t$, then RP^{4n+3} immerses in R^{8n-3}.*

Theorem 5 is an immediate consequence of Theorem 4 and (5.3) of [12]. The result implies that RP^{31} immerses in R^{53}. Together with the results of James [5], this result is best possible.

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1. Proof of the non-embedding theorems

1.1 *Proof of Theorem* 1. That RP^n immerses in R^{2n-3} is given by (7.2.2) of [7]; from (9.3) of [3] we have that RP^n immerses in R^{2n-4} but not in R^{2n-5}. It follows from Hirsch ([4], (6.1)) that, for any immersion of RP^n in R^{2n-4}, the normal bundle ν does not have a non-zero section. We now apply the methods of [7] and assume familiarity with the notation. Let $f:RP^n \to BSO(n-4)$ be the classifying map for ν. Then f does not admit a lifting to $RP^n \to BSO(n-5)$. The obstructions to the lifting are given in 4.1 of [7], and they are

$$k_1^1 = \chi, \qquad k_1^2:(\mathrm{Sq}^2 + W_2)\chi = 0, \qquad k_3^2:(\mathrm{Sq}^4 + W_4)\chi = 0,$$

$$(1.2) \qquad k_1^3:(\mathrm{Sq}^2 + W_2)k_1^2 = 0, \qquad k_3^3:\mathrm{Sq}^1 k_3^2 + (\mathrm{Sq}^2\,\mathrm{Sq}^1 + W_3)k_1^2 = 0,$$

$$\text{and} \quad k_1^4:\mathrm{Sq}^1 k_3^3 + (\mathrm{Sq}^2 + W_2)k_1^3 = 0.$$

Now $\chi(\nu) \in H^{n-4}(RP^n; Z) = 0$ and $W_2(\nu) \neq 0$. Therefore, $k_1^2(\nu) = 0$, because of the indeterminacy. Now $k_2^3(\nu)$ has zero indeterminacy, and, if $k_3^2(\nu) = 0$, then all the remaining successive k-invariants would vanish and we would obtain a lifting. Therefore $k_3^2(\nu) \neq 0$. Let $E_0(\nu)$ be the total space of the associated sphere bundle of ν and $\alpha \in H^{n-5}(E_0(\nu); Z_2)$, any class with $\delta\alpha = U$, where U is the Thom class of ν. Then, from (5.3.3) of [7], we obtain

$$(1.3) \qquad \mathrm{Sq}^4\,\alpha = k_3^2 \neq 0.$$

It follows that ν cannot be the normal bundle to an embedding of RP^n in R^{2n-4}, since (1.3) contradicts the existence of the Massey subalgebra [10], and and so Theorem 1 is proved.

1.3. *Proof of Theorem* 2. We first observe that if $n = 2^r + 7$, $r \geqq 3$, then Theorem 2 follows from Theorem 1.

Now assume $n = 2^r + 2^s + 3$, $r > s \geqq 3$, and consider the secondary operation from one to three variables Ω_{8k+4} associated with the relations

$$(1.4) \qquad \begin{aligned} \mathrm{Sq}^1\,\mathrm{Sq}^{8k+4} + \mathrm{Sq}^2\,\mathrm{Sq}^{8k+3} + \mathrm{Sq}^{8k+4}\,\mathrm{Sq}^1 &= 0,\\ \mathrm{Sq}^4\,\mathrm{Sq}^{8k+3} + \mathrm{Sq}^{8k+5}\,\mathrm{Sq}^2 &= 0,\\ \mathrm{Sq}^4\,\mathrm{Sq}^{8k+4} + \mathrm{Sq}^{8k+6}\,\mathrm{Sq}^2 + \mathrm{Sq}^{8k+7}\,\mathrm{Sq}^1 &= 0. \end{aligned}$$

Ω_{8k+4} is defined in classes $x \in H^q(X)$ such that $\mathrm{Sq}^1 x = 0$, $\mathrm{Sq}^2 x = 0$, $\mathrm{Sq}^{8k+3} x = 0$, $\mathrm{Sq}^{8k+4} x = 0$ and its value $\Omega_{8k+4}(x)$ lies in $H^{q+8k+4}(X) \oplus H^{q+8k+6}(X) \oplus H^{q+8k+7}(X)$ modulo the subgroup $Q(X) = (\mathrm{Sq}^{8k+4}, 0, \mathrm{Sq}^{8k+7})\Delta H^q(X) + (0, \mathrm{Sq}^{8k+5}, \mathrm{Sq}^{8k+6})\Delta H^{q+1}(X) + (\mathrm{Sq}^2, \mathrm{Sq}^4, 0)\Delta H^{q+8k+2}(X) + (\mathrm{Sq}^1, 0, \mathrm{Sq}^4)\Delta H^{q+8k+3}(X)$, where $\Delta: H^p(X) \to H^p(X) \oplus H^p(X) \oplus H^p(X)$ is the diagonal homomorphism given by $\Delta(x) = (x, x, x)$, for all $x \in H^p(X)$.

The following lemma gives a sufficient condition for Ω_{8k+4} to vanish. Let $A_j(u)$ be the total image of all stable primary operations which raise dimension by j acting on u.

LEMMA 1.5. *Let $u \in H^q(X)$ be such that $\Omega_{8k+4}(u)$ is defined. If $q = 8k + 2$ and $A_{8k+a}(u) = 0$ and $u \smile \mathrm{Sq}^a u = 0$ for $a = 4, 5$, then $\Omega_{8k+4}(u) = 0$.*

The proof is entirely analogous to that of [2] (p. 63) and therefore will be omitted.

LEMMA 1.6. *If ν is the normal bundle to an embedding of RP^n and U is the Thom class of ν, then $\Omega_{n-7}(U)$ is defined and, with diagonal indeterminacy, is non-zero. Moreover $A_{n-\epsilon}(U) = 0$ for $\epsilon = 4, 5, 7$.*

Proof. Since the operation Ω_{n-7} is stable and so are the conclusions of the lemma, we may as well work with the stable normal bundle. The Thom space of the stable normal bundles is RP^{2^N-1}/RP^{2^N-n-2}, where $N = \varphi(n)$ is the Hurwitz-Radon number. Consider the diagram of fibrations

$$\begin{array}{ccc} RP^{2^N-1} & \xrightarrow{\pi_1} & CP^{2^{N-1}-1} \\ & {\scriptstyle\pi_3}\searrow \quad \swarrow{\scriptstyle\pi_2} & \\ & QP^{2^{N-2}-1} & \end{array}$$

Let $x \in H^1(RP^\infty)$, $w \in H^2(CP^\infty)$, and $y \in H^4(QP^\infty)$ be the generators. Then $\pi_1{}^*w = x^2$, $\pi_2{}^*y = w^2$, and $\pi_3{}^*y = x^4$. If $4q = 2^N - n - 1$, then it is easy to see that $\Omega_{n-7}(y^q)$ is defined and has zero indeterminacy. In fact $\Omega_{n-7}(y^q) = (\varphi_{n-7}{}'(y^q), 0, 0)$, where $\varphi_{n-7}{}'$ is an operation associated with the first relation of (1.4). Moreover, if φ_{n-7} is an operation associated with the relation $\mathrm{Sq}^1\,\mathrm{Sq}^{n-7} + (\mathrm{Sq}^2\,\mathrm{Sq}^1)\mathrm{Sq}^{n-9} + \mathrm{Sq}^{n-7}\,\mathrm{Sq}^1 = 0$, then $\varphi_{n-7}{}'(y^q) = \varphi_{n-7}(y^q)$, with zero indeterminacy. Now $\pi_2{}^*\varphi_{n-7}(y^q) = \varphi_{n-7}(w^{2q})$, and $\varphi_{n-7}(w^{2q}) \not\ni 0$ with zero indeterminacy by [2] (p. 68). It follows that $\varphi_{n-7}(y^q)$, and thus $\Omega_{n-7}(y^q)$, is non-zero. Finally, it is easy to verify that $\Omega_{n-7}(x^{4q})$ has diagonal indeterminacy; so, by naturality with respect to $\pi_3{}^*$, we conclude that $\Omega_{n-7}(x^{4q}) \not\ni 0$. The remaining conclusions follow by simple calculation.

Suppose now that RP^n embeds in R^{2n-8}. Then, from the existence of the Massey subalgebra [10], it follows that there is a class $\alpha \in H^{n-9}(E_0)$, E_0 being the sphere bundle of the embedding, with $\delta\alpha = U$. Then $\mathrm{Sq}^k\,\alpha = 0$ for $k = 1, 2, 5, 6, n - 9$. If $\mathrm{Sq}^4\,\alpha = \alpha\cdot W_4(\nu) + x^{n-5}$, then $\mathrm{Sq}^6\,\alpha \neq 0$, so that $\mathrm{Sq}^4\,\alpha = \alpha\cdot x^4$. Therefore, $\alpha \smile \mathrm{Sq}^4\,\alpha = \alpha^2 x^4 = 0$, since $\mathrm{Sq}^{n-9}\,\alpha = 0$. Thus $\alpha \smile \mathrm{Sq}^a\,\alpha = 0$ for $a = 4, 5$, and α satisfies the conditions of (1.5), which contradicts $\delta\Omega_{n-7}(\alpha) = \Omega_{n-7}(U) \not\ni 0$. Thus Theorem 2 is established.

1.7 *Proof of Theorem* 3. That QP^n embeds in R^{8n-4} and immerses in R^{8n-5} is given by Sanderson in [12]. In [13] it is proved that QP^n does not immerse in R^{8n-6}. Again, by (6.1) of Hirsch [4], for any immersion of QP^n in R^{8n-5}, the normal bundle ν does not have a section. From (1.2) we see that the only obstruction to a section is $k_1{}^2(\nu) \in H^{4n-4}(QP^n)$ with zero indeterminacy. Then, similarly to (1.3), we have $\mathrm{Sq}^2\,\alpha = k_1{}^2 \not\ni 0$, where $\alpha \in H^{4n-6}(E_0(\nu))$ satisfies $\delta\alpha = U$. This again contradicts the existence of a Massey subalgebra. Thus ν cannot be the normal bundle to an embedding, and Theorem 3 follows.

2. Proof of the immersion theorems

It suffices to prove Theorem 4.

2.1 *Proof of Theorem* 4. In (5.1) of [12] it is proved that QP^n immerses in R^{8n-5}. Take any such immersion, with normal bundle ν. We will show that ν has a non-zero section. From (1.2) there is only one obstruction $k_1^2(\nu) \in H^{4n-4}(QP^n)$. Now, by (3.5.1) of [9], we have that, if $U \in H^{4n-5}(E, E_0)$ is the Thom class of ν,

$$\Phi_{4n-4}'(U) = U \cup (k_1^2 + W_2 \cdot W_{4n-6}) = U \cup k_1^2,$$

since $W_2 = W_2(\nu) = 0$ and Φ_{4n-4}' is a secondary operation associated with the relation $\mathrm{Sq}^1 \mathrm{Sq}^{4n-4} + \mathrm{Sq}^2 \mathrm{Sq}^{4n-5} = 0$, valid for integral classes. It is easy to see in our case that $\Phi_{4n-4}'(U) = \Phi_{4n-4}(U)$, with zero indeterminacy, where Φ_{4n-4} is associated with the relation $\mathrm{Sq}^1 \mathrm{Sq}^{4n-4} + (\mathrm{Sq}^2 \mathrm{Sq}^1)\mathrm{Sq}^{4n-4} = 0$. Since ν is the normal bundle to an embedding of QP^n and Φ_{4n-4} is a stable operation, $\Phi_{4n-4}(U)$ is non-zero if and only if

$$c(\Phi_{4n-4}) : H^4(QP^n) \to H^{4n}(QP^n)$$

is non-zero, where $c(\Phi_{4n-4})$ is the dual operation to Φ_{4n-4} (see [2], (5.1)). This secondary operation is the one considered by Maunder in [11], and one can use his main theorem to compute it; or, using the methods of [2], one obtains that, for the given value of n, $c(\Phi_{4n-4})(y) = 0$, where $y \in H^4(QP^n)$ is the generator. Therefore $k_1^2 = 0$ and ν has a section. We apply Hirsch's theorem (6.1) of [4] to complete the proof of Theorem 4.

CENTRO DE INVESTIGACIÓN DEL I P N, MÉXICO, D.F.
INSTITUTE FOR ADVANCED STUDY, PRINCETON, NEW JERSEY
NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS

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