THE FINITE PLANES OF OSTROM

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1. Introduction

In 1961, T. G. Ostrom communicated a discovery on the construction of finite planes to Daniel R. Hughes. That construction is very interesting since it provides new information about known planes as well as new planes. However, it was not clear then what the algebraic significance of the construction was, and this paper is the result of an attempt by the author to provide a clear exposition of the construction as well as the algebraic background for further applications of it. It was written in October 1961 without any intention of later publication, but has already been referred to in [1] and [2] by Ostrom as important background material for those two papers. Thus its publication now seems desirable.

2. Affine planes

Our study is simplified by a presentation in the realm of affine rather than projective planes and so we shall begin with a review of Bruck's exposition in [3]. The review will also serve to introduce the notations we shall use.

An *affine plane* is a set $\mathfrak{M}$ of elements $P, Q, \cdots$ (called *points*) and subsets $\mathfrak{L}$ (called *lines*), subject to the following axioms of incidence.

I. If P and Q are distinct points of $\mathfrak{M}$, there is one and only one $\mathfrak{L} = [P:Q]$ containing both P and Q.

II. If $\mathfrak{L}_1$ and $\mathfrak{L}_2$ are distinct lines of $\mathfrak{M}$, there is at most one point P of $\mathfrak{M}$ on both $\mathfrak{L}_1$ and $\mathfrak{L}_2$.

III. If a point P of $\mathfrak{M}$ is not on a line $\mathfrak{L}$ of $\mathfrak{M}$, there is exactly one line $\mathfrak{L}'$ of $\mathfrak{M}$ containing P and not containing any point of $\mathfrak{L}$. We say that $\mathfrak{L}'$ is *parallel* to $\mathfrak{L}$.

IV. There are at least four distinct points of $\mathfrak{M}$ no three of which are on a line of $\mathfrak{M}$.

By III, every line of $\mathfrak{M}$ containing P, except the line $\mathfrak{L}'$ parallel to $\mathfrak{L}$, must intersect $\mathfrak{L}$. There is then an induced one-to-one correspondence between the lines $\mathfrak{L}_i$ through P not parallel to $\mathfrak{L}$ and the points P_i of $\mathfrak{L}$, where $\mathfrak{L}_i = [P:P_i]$. It is then easy to show that, if one line of $\mathfrak{M}$ has n points, every line of $\mathfrak{M}$ has n points, that there are n^2 points in $\mathfrak{M}$, and that every point of $\mathfrak{M}$ is on $n + 1$ lines. We call n the *order* of $\mathfrak{M}$.

The relation of parallelism can be shown to be an equivalence relation. The *parallel class* of a line $\mathfrak{L}$, in a plane $\mathfrak{M}$ of order n, has exactly n lines in it.

3. Coordinates

Coordinates can be introduced in an affine plane $\mathfrak{M}$ as follows. We select an arbitrary point E of $\mathfrak{M}$ (called the *origin*) and three distinct lines through E called the *x-axis,* the *y-axis,* and the *unit line*), as in fig. 1. Select any point U

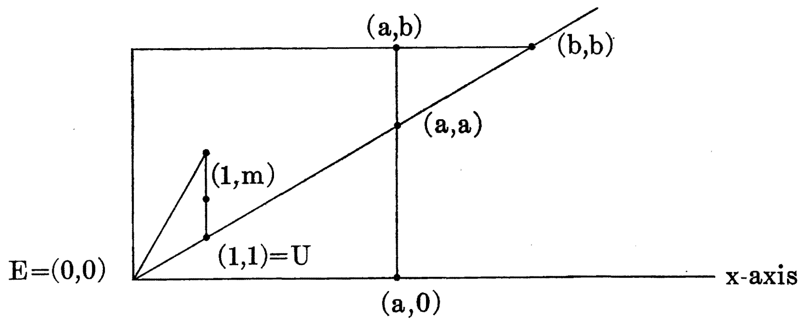

FIG. 1

(the *unit point*) on the unit line distinct from E. We now select a set $\mathfrak{R}$ of elements subject to the following two restrictions:

(a) there are two distinct elements among the elements of $\mathfrak{R}$ (we call them *zero* and *one* and designate them by 0 and 1);

(b) there is a one-to-one correspondence between the points P of the unit line and the elements $a = a_P$ of $\mathfrak{R}$ such that $a_E = 0$, and $a_U = 1$.

We introduce the labels $E = (0, 0)$, $U = (1, 1)$, and $P = (a_P, a_P)$ for the points P of the unit line. Suppose now that Q is any point of $\mathfrak{M}$. The line through Q in the parallel class of the y-axis meets $[E: U]$ in a unique point (a, a), and the line through Q in the parallel class of the x-axis meets EU in a unique point (b, b). We shall then write

$$(1) \qquad Q = (a, b),$$

and have assigned the unique coordinate pair (a, b) to Q where a and b are in $\mathfrak{R}$.

The points of the x-axis have coordinates $(x, 0)$, and we say that the equation of the x-axis is $y = 0$. The points of the y-axis have coordinates $(0, y)$, and we say that the equation ot the y-axis is $x = 0$. The line through (a, b) in the parallel class of the y-axis consists of all points (a, y), and we say that the equation of this line is $x = a$. Similarly, we say that the equation of the line through (a, b) in the parallel class of the x-axis is $y = b$. We shall also describe the unit line as the line whose equation is $y = x$.

We now consider any line $\mathfrak{L}$. There is a unique line $\mathfrak{L}'$ in the parallel class of $\mathfrak{L}$ which passes through the origin E. If $\mathfrak{L}'$ is the y-axis, we shall assign no *slope* to $\mathfrak{L}$. Its equation is then $x = a$. Otherwise $\mathfrak{L}'$ meets the *slope line* $x = 1$ in a unique point $(1, m)$, where m is the *slope of every line in the parallel class of* $\mathfrak{L}$. In particular, the lines $y = b$ all have zero slope, and every line in the parallel class of the unit line has slope 1.

Assume finally that $\mathfrak{L}$ is a line intersecting the y-axis in a point $(0, b)$. We call b the *y-intercept* of $\mathfrak{L}$ and $(0, b)$ the *intercept point* of $\mathfrak{L}$. If $\mathfrak{L}$ has slope m, its points are all points (x, y) where

$$(2) \qquad y = F(x, m, b)$$

is uniquely determined for every x, m, b of $\mathfrak{R}$. On the line $[E: U]$ we have $y = x$;

that is,

$$F(x, 1, 0) = x. \tag{3}$$

Indeed, we have defined a system

$$(\mathcal{R}, F) \tag{4}$$

consisting of the set $\mathcal{R}$ and the ternary operation $F(x, m, b)$ having a set of properties which are indeed equivalent to the axioms I, II, and III. We shall now state these properties and shall introduce two binary operations determined by $F(x, m, b)$.

Observe first that a quasigroup is a system $(\mathcal{G}, \phi)$ consisting of a set $\mathcal{G}$ of elements $a, b, c, \cdots$ and a function $\phi(a, b)$ with arguments a, b and values $\phi(a, b)$ in $\mathcal{G}$ such that if any *two* of the elements $a, b, c = \phi(a, b)$ are given the third element is a uniquely existing element of $\mathcal{G}$. A quasigroup is a *loop* if $\mathcal{G}$ contains an element e called its *identity* element such that $\phi(e, a) = \phi(a, e) = a$ for every a of $\mathcal{G}$.

We now define a binary operation called *addition* by

$$a + b = F(a, 1, b) \tag{5}$$

and a binary operation called *multiplication* by

$$ab = F(a, b, 0). \tag{6}$$

Then the axioms I, II, and III may be seen to be equivalent to the following properties.

V. The system $(\mathcal{R}, +)$ is a *loop* with 0 as identity element.

VI. Let $\mathcal{R}^*$ be the set obtained by omitting 0 from $\mathcal{R}$. Then the system consisting of $\mathcal{R}$ and the product operation is a loop $(\mathcal{R}, \circ)$ with 1 as identity element.

VII. The pair of equations

$$y_1 = F(x_1, m, b), \qquad y_2 = F(x_2, m, b) \tag{7}$$

has a unique solution m, b for any two ordered pairs (x_1, y_1) and (x_2, y_2) with $x_1 \neq x_2$.

VIII. The equation

$$F(x, m_1, b_1) = F(x, m_2, b_2) \tag{8}$$

has a unique solution x if $m_1 \neq m_2$.

Every affine plane can now be coordinatized by a system $(\mathcal{R}, F)$, where the ternary function $F = F(x, m, b)$ is subject to the axioms V, VI, VII, and VIII above. Conversely, every system $(\mathcal{R}, F)$ satisfying our axioms determines an affine plane whose points are the pairs (x, y) with x and y in $\mathcal{R}$ and whose lines are the lines $x = c$ and $y = F(x, m, b)$. It will be convenient to use the notation

$$\{m, b\} \tag{9}$$

for the line consisting of the points (x, y) with $y = F(x, m, b)$. We shall call the system $(\mathfrak{R}, F)$ the *ternary ring* of the plane $\mathfrak{M}$ and the system $(\mathfrak{R}, +, \circ)$ the *binary ring* of $\mathfrak{M}$.

4. The Ostrom planes

We shall restrict our attention to finite affine planes of order q^2. Then we shall say that $\mathfrak{M}$ is *derivable* if there exists a planar ternary ring $(\mathfrak{R}, F)$ of $\mathfrak{M}$ such that the following are true.

IX. The loop $(\mathfrak{R}, +)$ is an abelian group.

X. The system $(\mathfrak{R}, F)$ contains a subsystem $(\mathfrak{K}, F)$ of order q.

XI. The function $F(x, \alpha, y) = x\alpha + y$ for every x and y of $\mathfrak{R}$ and α of $\mathfrak{K}$.

XII. The properties $(x\alpha)\beta = x(\alpha\beta)$, $(x + y)\alpha = x\alpha + y\alpha$, $x(\alpha + \beta) = x\alpha + x\beta$ hold for every x and y of $\mathfrak{R}$ and α and β of $\mathfrak{K}$.

Our properties imply that the system $(\mathfrak{K}, +, \circ)$ is an associative division ring. Since $\mathfrak{R}$ is finite, so is $\mathfrak{K}$; and hence $\mathfrak{K}$ is a field of q elements. Evidently $(\mathfrak{R}, +)$ is a *vector space of dimension two over* $\mathfrak{K}$ *relative to the product* $x\alpha$ *as scalar product.* Let t be *any element* of $\mathfrak{R}$ not in $\mathfrak{K}$. Then every element of $\mathfrak{R}$ is uniquely expressible in the form

$$a = \alpha + t\beta, \tag{10}$$

for α and β in $\mathfrak{K}$.

We are now ready to define the *planes of Ostrom.* Define $a\mathfrak{K}$, for every nonzero a of $\mathfrak{R}$, to be the set of all $a\lambda$ for λ in $\mathfrak{K}$. Then the set $a\mathfrak{K} + b$ consists of all $a\lambda + b$, for λ in $\mathfrak{K}$. We shall now define what will be called a *derived* plane $\mathfrak{M}'$ of $\mathfrak{M}$. It depends on the coordinate system for $\mathfrak{M}$ (that is, on the ternary ring $(\mathfrak{R}, F)$ and the subring $(\mathfrak{K}, F)$) and may not be unique. Its points will be the points (x, y) of $\mathfrak{M}$. The lines

$$\{m, c\} \quad (m \text{ not in } \mathfrak{K}) \tag{11}$$

of $\mathfrak{M}$ are also to be lines of $\mathfrak{M}'$. Its remaining lines are defined to be the sets

$$\{a\mathfrak{K} + b, a\mathfrak{K} + c\} \tag{12}$$

defined for every $a \neq 0$, b, and c of $\mathfrak{R}$. Each such line consists of all points

$$(a\lambda + b, a\mu + c), \tag{13}$$

where $a \neq 0$, b, and c are fixed, and λ and μ range over all elements of $\mathfrak{K}$. There remains the necessity of seeing that the incidence axioms are satisfied. Our proof stems from considerations due to Hughes.

We have already selected an element t in $\mathfrak{R}$ and not in $\mathfrak{K}$. Then every point of $\mathfrak{M}$ may be regarded as being a pair

$$(x, y) = (\xi_1 + t\xi_2, \eta_1 + t\eta_2), \tag{14}$$

where ξ_1, ξ_2, η_1, and η_2 are unique elements of $\mathfrak{K}$. We define a transformation σ of our set $\mathfrak{M}$ of points (x, y) by

$$(x, y)^\sigma = (\xi_1 + t\eta_1, \xi_2 + t\eta_2). \tag{15}$$

Then σ is nonsingular, and

(16) $$\sigma^2 = \epsilon$$

is the identity transformation.

Assume first that

(17) $$a = t\alpha, \qquad b = \beta + t\beta_0, \qquad c = \gamma + t\gamma_0 \quad (\alpha \neq 0, \beta, \beta_0, \gamma, \gamma_0 \text{ in } \mathcal{K}).$$

Then $a\mathcal{K} + b = (t\alpha)\mathcal{K} + b = t\mathcal{K} + \beta + t\beta_0 = t\mathcal{K} + \beta$, and so every

(18) $$\{t\alpha\mathcal{K} + b, t\alpha\mathcal{K} + c\} = \{t\mathcal{K} + \beta, t\mathcal{K} + \gamma\}.$$

Each such set consists of all points $(\beta + t\lambda, \gamma + t\mu)$ and

(19) $$(\beta + t\lambda, \gamma + t\mu)^\sigma = (\beta + t\gamma, \lambda + t\mu).$$

We have proved the following result.

LEMMA 1. *The transformation σ maps each set $\{t\mathcal{K} + \beta, t\mathcal{K} + \gamma\}$ onto the line $x = \beta + t\gamma$ of* $\mathfrak{M}$.

Assume next that $a = \alpha_1 + t\alpha_2$, where $\alpha_1 \neq 0$ and α_2 are in $\mathcal{K}$. Then we can define a unique element

(20) $$\alpha = \alpha_2\alpha_1^{-1}$$

of $\mathcal{K}$, and we know that $a\mathcal{K} = a_\alpha\mathcal{K}$, where

(21) $$a_\alpha = 1 + t\alpha,$$

for every α of $\mathcal{K}$. But then $\{a\mathcal{K} + b, a\mathcal{K} + c\} = \{a_\alpha\mathcal{K} + b, a_\alpha\mathcal{K} + c\}$, $(1 + t\alpha)\lambda + \beta + t\beta_0 = \lambda + \beta + t(\alpha\lambda + \beta_0)$, and so $(a_\alpha\lambda + \beta, a_\alpha\mu + c)^\sigma = (\lambda + t\mu + \beta + t\lambda, \lambda\alpha + t\mu\alpha + \beta_0 + t\gamma_0)$, from which

(22) $$(a_\alpha\lambda + b, a_\alpha\mu + c)^\sigma = (x, x\alpha + d),$$

where x and d are given by

(23) $$x = (\lambda + \beta) + t(\mu + \gamma), \quad d = (\beta_0 - \beta_\alpha) + t(\gamma_0 - \gamma_\alpha).$$

We have now shown that σ *maps every set* $\{a\mathcal{K} + b, a\mathcal{K} + c\}$ *onto either a line* $x = \beta + t\gamma$ *or a line* $\{\alpha, d\}$ *of* $\mathfrak{M}$. Moreover, every such line of $\mathfrak{M}$ is the image $\mathcal{L}^\sigma$ of a set $\mathcal{L}' = \{a\mathcal{K} + b, a\mathcal{K} + c\}$. Since these are precisely the lines of $\mathfrak{M}$ omitted from our set of lines of $\mathfrak{M}'$, there are exactly as many lines in $\mathfrak{M}'$ as in $\mathfrak{M}$. If $\mathcal{L}_1' = \{a_1\mathcal{K} + b_1, a_1\mathcal{K} + c_1\}$ and $\mathcal{L}_2' = \{a_2\mathcal{K} + b_2, a_2\mathcal{K} + c_2\}$ are two distinct lines of $\mathfrak{M}'$, they have a point of intersection if and only if $(\mathcal{L}_1')^\sigma$ and $(\mathcal{L}_2')^\sigma$ have a point of intersection. Since two lines $\{m, b\}$ and $\{m_1, b_1\}$ of $\mathfrak{M}$ intersect as lines of $\mathfrak{M}'$ if and only if they intersect as lines of $\mathfrak{M}'$, it suffices to show that every line $\{m, d\}$, for m not in $\mathcal{K}$, intersects with every line $\{a\mathcal{K} + b, a\mathcal{K} + c\}$ in a unique point. We consider the parallel system of all $\{m, k\}$. This is a set of q^2 lines, each having q^2 points and no two distinct lines of the system having a point in common. Thus every point in $\mathfrak{M}'$ is on one and only one line of the parallel system. If the set $\{a\mathcal{K} + b, a\mathcal{K} + c\}$ contained two distinct points on a single line $\{m, k\}$ of this parallel system, the line joining these two points would be that line $y = F(x, m, k)$ of the system. Let the two points be $(a\alpha + b, a\beta + c)$ and $(a\gamma + b, a\delta + c)$. If $\alpha = \gamma$, the line in question is the line $x = a\alpha + b$ and is *not* the line $y = F(x, m, k)$. If $\alpha \neq \gamma$, the equations $a\beta + c = (a\alpha + b)\lambda + h$ and $a\delta + c = (a\gamma + b)\lambda + h$ are satisfied if and only if $a\beta - (a\alpha + b)\lambda = a\delta - (a\gamma + b)\lambda$, $a(\beta - \delta) = [a(\alpha - \gamma)]\lambda$, $\beta - \delta = (\alpha - \gamma)\lambda$, $\lambda = (\alpha - \gamma)^{-1} \cdot (\beta - \delta)$ is in $\mathcal{K}$. Thus there is *no* line $\{m, k\}$ with *two* of its points in $\{a\mathcal{K} + b, a\mathcal{K} + c\}$. Since there are q^2 such lines and q^2 points in $\{a\mathcal{K} + b, a\mathcal{K} + c\}$, there is *exactly one point* of $\{a\mathcal{K} + b, a\mathcal{K} + c\}$ on each line of the parallel system of all $\{m, k\}$. Thus there is exactly one point of $\{a\mathcal{K} + b, a\mathcal{K} + c\}$ on our given line $\{m, d\}$.

We have now shown that if P and Q are distinct points they are either on a line $\{m, b\}$ for m not in $\mathcal{K}$, on a line $x = h$, or on a line $y = x\alpha + b$. In the first case the line $\{m, b\}$ of $\mathfrak{M}$ is also a line of $\mathfrak{M}'$. In the second case the points are (b, y) and (b, y_2), where $y_1 \neq y_2$, and these are points on the unique line $\{(y_2 - y_1)\mathcal{K} + b, (y_2 - y_1)\mathcal{K} + y_1\}$ of $\mathfrak{M}'$. In the final case the points are $P = (x_1, y)$ and $Q = (x_2, y_2)$, where $x_1 \neq x_2$, $y_1 = x_1\alpha + d$ and $y_2 = x_2\alpha + d$. These are points on the line $\{a\mathcal{K} + b, a\mathcal{K} + c\}$, where $b = x_1$, $c = y_1$, $x_2 - x_1 = a$, $y_2 - y_1 = (x_2 - x_1)\alpha = a\alpha$, $P = (b, c)$ and $Q = (a + b, a\alpha + c)$. This completes our proof of the existence of a line $\mathcal{L}' = [P:Q]$ in $\mathfrak{M}'$ for every pair of distinct points P and Q. We have already shown that Axiom II holds in $\mathfrak{M}'$, and so $\mathcal{L}'$ is unique.

Axiom IV holds trivially. The parallel Axiom III is also trivial for $\mathfrak{M}'$, since it evidently holds for lines of slope m not in $\mathcal{K}$ and holds for the remaining lines of $\mathfrak{M}'$ (since it holds for their images under the transformation σ). This completes our proof that $\mathfrak{M}'$ is an affine plane.

5. Coordinates for $\mathfrak{M}'$

We are now ready to introduce a set of coordinates in $\mathfrak{M}'$ and to indicate some properties of the corresponding ternary ring $(\mathfrak{R}, \Phi)$. Note that the coordinate ring will depend upon our mapping σ and therefore on our selection of an element t such that $\mathfrak{R} = \mathcal{K} + t\mathcal{K}$.

We shall take as our x-axis the line

$$\mathcal{L}_x = (\mathcal{K}, \mathcal{K}) \tag{24}$$

whose image $\mathcal{L}_x^\sigma$ is the x-axis of $\mathfrak{M}$ relative to $(\mathfrak{R}, F)$. We select the line

$$\mathcal{L}_y = (t\mathcal{K}, t\mathcal{K}) \tag{25}$$

as the y-axis of $\mathfrak{M}'$ and have $\mathcal{L}_y^\sigma$ as the y-axis of $\mathfrak{M}$. The *origin* of $\mathfrak{M}'$ will be taken to be the origin

$$E = (0, 0) = E^\sigma \tag{26}$$

of $\mathfrak{M}$. Select the line

$$\mathcal{L}_u = (a\mathcal{K}, a\mathcal{K}), \quad a = 1 + t, \tag{27}$$

as the *unit line* of $\mathfrak{M}'$, so that L^{σ}_u is the unit line $y = x$ of $\mathfrak{M}$. Finally, let

$$(28) \qquad U' = (1 + t, 0), \qquad (U')^{\sigma} = (1, 1),$$

so that the image under σ of the *unit point* U' of $\mathfrak{M}'$ is the unit point U of $\mathfrak{M}$.

We now consider an arbitrary point

$$(29) \qquad P = (\xi_1 + t\xi_2, \eta_1 + t\eta_2)$$

of $\mathfrak{M}$ relative to $(\mathfrak{R}, F)$. The line of $\mathfrak{M}'$ through P parallel to $\mathfrak{L}_y$ is a line $(t\mathfrak{K} + \beta, t\mathfrak{K} + \gamma)$, where, necessarily,

$$(30) \qquad \beta = \xi_1, \quad \gamma = \eta_1 .$$

The intersection of the line $\{t\mathfrak{K} + \xi_1, t\mathfrak{K} + \eta_1\}$ with the unit line $\mathfrak{L}_u = \{(1 + t)\mathfrak{K}, (1 + t)\mathfrak{K}\}$ is a point with $(1 + t)\lambda = t\rho + \eta_1$ and $(1 + t)\mu = t\tau + \xi_1$ and is a point P_x, where

$$(31) \qquad P_x = (a\xi_1, a\eta_1), \quad P_x^{\sigma} = (\xi_1 + t\eta_1, \xi_1 + t\eta_1).$$

We similarly consider a line through P parallel to $\mathfrak{L}_x$. This has been seen to be a line $\{\mathfrak{K} + t\beta, \mathfrak{K} + t\delta\}$; and so $\beta = \xi_2, \gamma = \eta_2$, and our line is $\{\mathfrak{K} + t\xi_2, \mathfrak{K} + t\eta_2\}$. Its intersection with the unit line $\{a\mathfrak{K}, a\mathfrak{K}\}$ is a point $P_y = (\xi_2 + t\xi_2, \eta_2 + t\eta_2)$, and $P_y^{\sigma} = (\xi_2 + t\eta_2, \xi_2 + t\eta_2)$. It follows that the coordinate pair for our point $P = (\xi_1 + t\xi_2, \eta_1 + t\eta_2)$, in the coordinate system proposed for $\mathfrak{M}'$, is $(\xi_1 + t\eta_1, \xi_2 + t\eta_2) = P^{\sigma}$. We state this result as follows.

THEOREM 1. *Let $\mathfrak{M}'$ be a derived plane of a derivable plane $\mathfrak{M}$ coordinatized by $(\mathfrak{R}, F)$, where the binary ring $(\mathfrak{R}, +, \circ)$ has the properties IX, X, XI, XII relative to a kernel $\mathfrak{K}$ such that $\mathfrak{R} = \mathfrak{K} + t\mathfrak{K}$. Select E as origin, $U' = (1 + t, 0)$ as unit point, $\{\mathfrak{K}, \mathfrak{K}\}$ as x-axis, $\{t\mathfrak{K}, t\mathfrak{K}\}$ as y-axis,* and $\{(1 + t)\mathfrak{K}, (1 + t)\mathfrak{K}\}$ *as unit line, and map the element $\xi + t\eta$ of $\mathfrak{R}$ onto the point $(\xi + t\xi, \eta + t\eta)$ of the derived plane $\mathfrak{M}'$. Then the points $P = (\xi_1 + t\xi_2, \eta_1 + t\eta_2)$ of the set $\mathfrak{M}$ have as coordinate pairs the image pairs $P^{\sigma} = (\xi_1 + t\eta_1, \xi_2 + t\eta_2)$ in this coordinate system of $\mathfrak{M}'$.*

We already know that the lines $\{t\mathfrak{K} + \beta, t\mathfrak{K} + \gamma\}$ of $\mathfrak{M}'$ are the lines $x = \beta + t\gamma$ in our new coordinate system. The coordinates of the points on the line

$$(32) \qquad \{(1 + t)\mathfrak{K} + \beta + t\beta_0, (1 + t)\mathfrak{K} + \gamma + t\gamma_0\}$$

satisfy the equation

$$(33) \qquad y = x + k,$$

where

$$(34) \qquad \begin{aligned} x &= x + t\mu + \beta + t\gamma, \\ k &= \beta_0 - \beta + t(\gamma_0 - \gamma), \end{aligned}$$

and the indicated operations are those of the binary ring $(\mathfrak{R}, +, \circ)$ of $\mathfrak{M}$. This line intersects the slope line $x = 1$ of $\mathfrak{M}'$ in a point where $x = 1$ and $y = 1 + k$.

Thus the line has slope 1. It follows that the ternary operation $\Phi(x, m, k)$ has the property

$$\Phi(x, 1, k) = x + k; \tag{35}$$

that is, *addition is the same for* Φ *as for* F. We also see that the equation of the line

$$\{(1 + t\alpha)\mathcal{K}, (1 + t\alpha)\mathcal{K}\} \tag{36}$$

in $\mathfrak{M}'$ is $y = x\alpha$, where the indicated product is that of $(\mathfrak{R}, +, \circ)$. This is the line whose intersection with the x-axis of $\mathfrak{M}'$ is the origin $(0, 0)$. Thus

$$\Phi(x, \alpha, 0) = x\alpha. \tag{37}$$

But the line

$$\{(1 + t\alpha)\mathcal{K} + \beta + t\beta_0, (1 + t\alpha)\mathcal{K} + \gamma + t\gamma_0\} \tag{38}$$

satisfies the condition that $y = x\alpha + k$ in the coordinate system for $\mathfrak{M}'$, where the indicated operations are those of the binary ring of $\mathfrak{M}$ and

$$x = \lambda + \beta + t(\mu + \gamma), \quad k = (\beta_0 - \beta_\alpha) + t(\gamma_0 - \gamma_\alpha). \tag{39}$$

Let us designate the binary ring of $\mathfrak{M}'$ as

$$\mathfrak{R}' = (\mathfrak{R}, +, \circ), \tag{40}$$

where

$$x \circ y = \Phi(x, y, 0). \tag{41}$$

Then $\mathcal{K}$ is a binary subring of $\mathfrak{R}'$, $\mathfrak{R}'$ is a vector space over $\mathcal{K}$, $\mathfrak{R}'$ must have dimension 2 over $\mathcal{K}$, and $x \circ \alpha = x\alpha$ for every x in $\mathfrak{R}$ and α of $\mathcal{K}$. Also

$$\Phi(x, \alpha, k) = x\alpha + k = F(x, \alpha, k). \tag{42}$$

It is now clear that the axioms IX, X, XI, XII hold for $\mathfrak{M}'$ relative to $\Phi(x, m, k)$.

Let us now examine the facts more closely. We have begun with a plane $\mathfrak{M}$ with a coordinate system depending on a choice of coordinatizing elements leading to a ternary ring defined by a function $F(x, m, k)$. We are now able to select a field $\mathcal{K}$ such that the derivability axioms are satisfied. We construct the plane $\mathfrak{M}'$ and select a basis of the binary ring $(\mathfrak{R}, +, \circ)$ over $\mathcal{K}$ which consists of 1 and t, where t is in $\mathfrak{R}$ and is not in $\mathcal{K}$. This yields the mapping σ of points of $\mathfrak{M}$ onto points of $\mathfrak{M}$, and we have the property that if $P = (x, y)$ is the coordinate representation of a point of $\mathfrak{M}$ relative to $(\mathfrak{R}, +, \circ)$ then $P^\sigma = (x, y)^\sigma$ is the coordinate representation of the points of $\mathfrak{M}'$ relative to its binary ring $(\mathfrak{R}, +, \circ)$. We may thus regard σ as being a nonsingular mapping of the points of $\mathfrak{M}$ regarded *as being coordinate pairs* (x, y) onto the points of $\mathfrak{M}'$ regarded as being these same pairs. This mapping has period two. It does not map all of the lines of $\mathfrak{M}$ onto the lines of $\mathfrak{M}'$. It does map the lines $\{m, k\}$ of $\mathfrak{M}$ onto lines $\{m', k'\} =$

$\{m, k\}^\sigma$ of $\mathfrak{M}'$ in a $(1 - 1)$ fashion defined by the fact that if (x, y) is on a line $\{m, k\}$ of $\mathfrak{M}$, then (x, y) is on the line $\{m, k\}^\sigma = \{m', k'\}$ of $\mathfrak{M}'$.

We now apply our derivation process to $\mathfrak{M}'$ and call the new plane $\mathfrak{M}''$. The points of $\mathfrak{M}''$ are obtained from points (x', y') of $\mathfrak{M}'$ as the images $(x', y')^\sigma$ with exactly the same σ. Then, if $(x', y') = (x, y)^\sigma$, it follows that $(x', y')^\sigma = (x, y)^{\sigma\sigma} = (x, y)$. If $\{a\mathcal{K} + b, a\mathcal{K} + c\}$ is a line of $\mathfrak{M}''$, it consists of points $(x', y') = (x, y)^\sigma$, where $x' = a\lambda + b$ and $y' = a\mu + c$ and the corresponding pairs $(x', y')^\sigma$ are on the lines $\{a\mathcal{K} + b, a\mathcal{K} + c\}^\sigma$ of $\mathfrak{M}''$. But these are clearly the lines of $\mathfrak{M}$. Hence $\mathfrak{M}'' = \mathfrak{M}$. We state our results as follows.

Theorem 2. *Let $\mathfrak{M}$ be a finite affine plane with a coordinatizing ternary ring defined by a function $F(x, m, k)$, and let $\mathfrak{M}$ be derivable so that the corresponding binary ring $(\mathcal{R}, +, \circ)$ is a vector space of dimension two over a kernel $\mathcal{K}$. Let $\mathfrak{M}'$ be the derived plane determined by the coordinate system, and select a basis* 1, t *of $(\mathcal{R}, +, \circ)$ over $\mathcal{K}$ and a coordinatizing function $\Phi(x, m, k)$ for $\mathfrak{M}'$. Then $\mathfrak{M}'$ is derivable relative to the same field $\mathcal{K}$, and the derived plane of $\mathfrak{M}'$ relative to $\Phi(x, m, k)$ and $\mathcal{K}$ is $\mathfrak{M}$.*

6. Determination of $\phi(x, m, k)$

We shall now indicate a procedure for determining the ternary function $\Phi(x, m, k)$. We have already seen that $\Phi(x, \alpha, k) = x\alpha + k$ for every α of $\mathcal{K}$, so that the corresponding binary ring $\mathcal{R}' = (\mathcal{R}, +, \circ)$ satisfies the condition $x \circ \alpha = x \circ \alpha$ for every x of $\mathcal{R}$ and α of $\mathcal{K}$.

Let us first show how the operation $x \circ y$ may be determined. We introduce the notations

$$\begin{aligned} P &= (x', y') = (\xi_1 + t\xi_2, \eta_1 + t\eta_2), \\ P^\sigma &= (x, y) = (\xi_1 + t\eta_1, \xi_2 + t\eta_2), \\ m' &= \lambda_1 + t\lambda_2, \quad m = \mu_1 + t\mu_2, \\ k' &= \gamma_1 + t\gamma_2, \quad k = \delta_1 + t\delta_2. \end{aligned} \tag{43}$$

Then (x', y') is the coordinate pair representing any point P of $\mathfrak{M}$ relative to the coordinate system of $\mathfrak{M}'$, and (x, y) is the coordinate pair for the same point relative to the given coordinate system of $\mathfrak{M}$. Assume that P is the general point on the line $\{m', k'\}$ of $\mathfrak{M}'$ whose equation is $y' = \Phi(x' \ m', k')$. This is the line $y' = x' \circ m'$ if and only if $\Phi(0, m', k') = 0$, that is, $P = (0, 0)$. But then $P^\sigma = (0, 0)$, and so the corresponding line of $\mathfrak{M}$ must be a line whose equation is $y = x \circ m$. Thus we have the equations

$$\begin{aligned} \eta_1 + t\eta_2 &= (\xi_1 + t\xi_2) \circ (\lambda_1 + t\lambda_2), \\ \xi_2 + t\eta_2 &= (\xi_1 + t\eta_2) \circ (\mu_1 + t\mu_2). \end{aligned} \tag{44}$$

Take $\xi_1 = 1, \xi_2 = 0$, and obtain $\eta_1 = \lambda_1$, $\eta_2 = \lambda_2$. Then

$$(1 + t\lambda_1)(\mu_1 + t\mu_2) = t\lambda_2 \tag{45}$$

is the equation determining $m = \mu_1 + t\mu_2$ as a function of $m' = \lambda_1 + t\lambda_2$. We also have the relation $(\xi_1 + t\eta_1)m = \xi_2 + t\eta_2$ as determining $\Phi(x', m', 0) = \eta_1 + t\eta_2 = x' \circ m'$ as a function of $x' = \xi_1 + t\xi_2$ and $m' = \lambda_1 + t\lambda_2$.

Our result determines $\mathfrak{M}$ as a function of m' for the parallel class of lines of slope m'. It remains to determine k as a function of k' and then to determine Φ. We observe that the intercept point is $(0, k')$, and we see that $P = (0 + t0, \gamma_1 + t\gamma_2)$ and $P^\sigma = (0 + t\gamma_1, 0 + t\gamma_2)$. Thus

$$F(t\gamma_1, m, k) = t\gamma_2. \tag{46}$$

Since we will have already determined m, this determines k as a function of m' and k'. Our final equation is

$$\xi_2 + t\eta_2 = F(\xi_1 + t\eta_1, m, k), \tag{47}$$

and this determines $\Phi(x', m, k') = \eta_1 + t\eta_2$.

7. Translation planes and shears planes

Let $\mathfrak{M}$ be a finite affine plane. Then a mapping τ of $\mathfrak{M}$ onto is called a *collineation* of $\mathfrak{M}$ if τ is a nonsingular transformation on the points of $\mathfrak{M}$ and $\mathfrak{L}^\tau$ is a line of $\mathfrak{M}$ for every line $\mathfrak{L}$ of $\mathfrak{M}$. The mappings $\tau = \tau(h, k)$, defined for every pair of elements h and k of $\mathfrak{R}$ by

$$(x, y)^\tau = (x + h, y + k), \tag{48}$$

are trivially $(1 - 1)$-transformations of $\mathfrak{M}$. We call $\mathfrak{M}$ a *translation plane if all of these mappings are collineations of* $\mathfrak{M}$. In this case the set of all translations forms a subgroup of the collineation group of $\mathfrak{M}$ called its *translation group*. It is easy to derive the following known results.

THEOREM 3. *A plane $\mathfrak{M}$ is a translation plane if and only if the following conditions hold.*

XIII. The binary ring $(\mathfrak{R}, +, \circ)$ is a Veblen-Wedderburn system; that is, addition is associative and $(x + y)z = xz + yz$ for every x, y, z of $\mathfrak{R}$.

XIV. The ternary function $F(x, m, k) = xm + k$.

It is shown on page 91 of [5] that XIII implies that addition is also commutative and that the translation group is abelian. Conversely, if XIII and XIV hold, the plane coordinatized by the ternary ring $(\mathfrak{R}, F)$ is a translation plane. It is also easy to see that all translations are slope preserving. We now derive the following result.

THEOREM 4. *Let $\mathfrak{M}$ be a derivable plane and $\mathfrak{M}'$ a corresponding derived plane of $\mathfrak{M}$. Then, if $\mathfrak{M}$ is a translation plane, so is $\mathfrak{M}'$.*

For, if $\tau = \tau(h, k)$, where $h = \lambda_1 + t\lambda_2$ and $k = \mu_1 + t\mu_2$, we have $P = (\xi_1 + t\xi_2, \eta_1 + t\eta_2)$, $P^\tau = (\xi_1 + t\xi_2 + \lambda_1 + t\lambda_2, \eta_1 + t\eta_2 + \mu_1 + t\mu_2)$, $P^\sigma = (\xi_1 + t\eta_1, \xi_2 + t\eta_2)$, and $P^{\tau\sigma} = (\xi_1 + \lambda_1 + t\eta_1 + t\mu_1, \xi_2 + \lambda_2 + t\eta_2 + t\mu_2) = P^{\sigma\rho}$, where

$$\rho = \tau(\lambda_1 + t\mu_1, \lambda_2 + t\mu_2) = \sigma\tau\sigma. \tag{49}$$

Thus $\tau(\lambda_1 + t\lambda_2, \mu_1 + t\mu_2)$ induces a translation $\tau(\lambda_1 + t\mu_1, \lambda_2 + t\mu_2)$ for every $h = \lambda_1 + t\lambda_2$, and $k = \mu_1 + t\mu_2$ in $\mathcal{R}$. Since σ leaves the lines $\{m, c\}$ of $\mathfrak{M}$ with m not in $\mathcal{K}$ fixed and τ maps lines $\{m, c\}$ onto lines $\{m, c_0\}$, it follows that $\sigma\tau\sigma$ carries every line $\{m, c\}$, with m not in $\mathcal{K}$, onto a line $\{m, d\}$ of $\mathfrak{M}$ with the same slope m (as a line of $\mathfrak{M}$). We also observe that a line $\{a\mathcal{K} + b, a\mathcal{K} + c\}$, with $a \neq 0$, b and c in $\mathcal{K}$, consists of points $(a\lambda + b, a\mu + c)$ and $(a\lambda + b, a\mu + c)^\tau = (a\lambda + b + h, a\mu + c + k)$. Thus $\{a\mathcal{K} + b, a\mathcal{K} + c\}^\tau = \{a\mathcal{K} + b + h, a\mathcal{K} + c + k\}$, and so τ maps every line of $\mathfrak{M}'$ onto a line of $\mathfrak{M}''$. This completes our proof.

Shears planes occur in the theory of projective planes as the *duals* of translation planes. They can be defined for affine planes as follows: Write

(50) $$F(x, m, c) = xm + c,$$

where addition is an abelian group. Assume that

(51) $$x(m + z) = xm + xz.$$

Then $(\mathcal{R}, +, \circ)$ is a (left) Veblen-Wedderburn system, and the ternary system $(\mathcal{R}, F)$ coordinatizes a plane $\mathfrak{M}$, called a shears plane. The translations $\tau = \tau(0, k)$, defined by

(52) $$(x, y)^\tau = (x, y + k),$$

are collineations of $\mathfrak{M}$ for every k of $\mathcal{R}$. For τ maps every line $x = a$ onto itself and, if $y = xm + c$, then $y + k = xm + (c + k)$, so that $\{m, c\}^\tau = \{m, c + k\}$. The mappings $\rho = \rho(d)$, defined by

(53) $$(x, y)^\rho = (x, y + xd),$$

are nonsingular transformations of $\mathfrak{M}$ which map all lines $x = a$ onto themselves. They map the points $(x, y) = (x, xm + c)$ onto $(x, y)^\rho = (x, xm + c + xd) = (x, xm' + c)$ for $m' = m + d$, since $x(m + d) = xm + xd$. We call these collineations the *shears* of $\mathfrak{M}$. The group generated by all translations $\tau(0, k)$ and all shears $\rho(d)$ is an abelian group. We are now ready to consider derived planes.

We assume that $\mathfrak{M}$ is a derivable shears plane. Then the binary ring $(\mathcal{R}, +, \circ)$ is a vector space of dimension two over a field $\mathcal{K}$, and we construct $\mathfrak{M}'$. The mappings $\tau = \tau(0, k)$ map $\{m, c\}$ onto $\{m, c + k\}$. They also map $\{a\mathcal{K} + b, a\mathcal{K} + c\}$ onto $\{a\mathcal{K} + b, a\mathcal{K} + c + k\}$. Hence all translations $\tau(0, k)$ of $\mathfrak{M}$ are also translations of $\mathfrak{M}'$. Let us now consider the shear $\rho = \rho(d)$ of $\mathfrak{M}$. It maps $\{m, c\}$ onto $\{m + d, c\}$. If both m and $m + d$ are not in $\mathcal{K}$, the mapping $\rho = \rho(d)$ does map lines of $\mathfrak{M}'$ of slope m onto lines of slope $m + d$ not in $\mathcal{K}$. This does occur when $d = \delta$ in $\mathcal{K}$. Also $\rho(d)$ maps $(a\lambda + b, a\lambda + c)$ onto $[a\lambda + b, a\mu + c + (a\lambda + b)d]$; and, if $d = \delta$ is in $\mathcal{K}$, then $a\mu + c + (a\lambda + b)\delta = a(u + \lambda\delta) + c + b\delta$, so that $\{a\mathcal{K} + b, a\mathcal{K} + c\}^\rho = \{a\mathcal{K} + b, a\mathcal{K} + c + b\delta\}$. Thus $\rho(\delta)$ is a shear of $\mathfrak{M}'$ for every δ of $\mathcal{K}$. We have derived the following result.

THEOREM 5. *Let $\mathfrak{M}$ be a derivable shears plane and $\mathfrak{M}'$, the corresponding derived plane. Then the translations $\tau(0, k)$ and the shears $\rho(\delta)$ are collineations of $\mathfrak{M}'$ for every k of $\mathcal{R}$ and δ of $\mathcal{K}$.*

It should be clear that some shears $\rho(d)$ are not shears of $\mathfrak{M}'$. For example, let m be an element of $\mathfrak{R}$ not in $\mathfrak{K}$, and let $d = \delta - m$ where δ is in $\mathfrak{K}$. Then $\rho(d)$ maps m, c onto δ, c, and δ, c is *not* a line of $\mathfrak{M}'$. This does not imply that $\mathfrak{M}'$ is not a shears plane, since the shears of $\mathfrak{M}'$ are the mappings $(x, y) \rightarrow (x, y + x \circ d)$, where $x \circ d = \Phi(x, d0)$. It does not seem possible to study these mappings in the general shears case.

8. The derived plane of a Desarguesian plane

Let the binary ring $(\mathfrak{R}, +, \circ)$ of a plane $\mathfrak{M}$ be a quadratic extension field of a field $\mathfrak{K}$. This is a unique field in the finite case. If t is any element of $\mathfrak{R}$ not in $\mathfrak{K}$, we know that

$$1 = t(t\alpha + \beta), \tag{54}$$

where the polynomial $f(\omega) = \omega^2\alpha + \omega\beta - 1$ is irreducible in $\mathfrak{K}$. Since $\mathfrak{M}$ is a translation plane, the derived plane $\mathfrak{M}'$ is a translation plane also, and thus

$$\Phi(x', m', c') = (x' \circ m') + c'. \tag{55}$$

It remains to determine the binary operation $x' \circ m'$.

We introduce the notations of (43) and use the fact that $(\mathfrak{R}, +, \circ)$ is a field to see that $m(1 + t\lambda_1) = t\lambda_2$. Then $(\xi_2 + t\eta_2)(1 + t\lambda_1) = (\xi_1 + t\eta_1)m(1 + t\lambda_1)$, and so

$$(\xi_2 + t\eta_2)(1 + t\lambda_1) = (\xi_1 + t\eta_1)t\lambda_2 . \tag{56}$$

Then $\xi_2(\alpha t + \beta) + \eta_2 + \xi_2\lambda_1 + t\eta_2\lambda_1 = (\xi_1 + t\eta_1)\lambda_2$, and so

$$\eta_2 + t(\eta_2\lambda_1 - \eta_1\lambda_2) = \xi_1\lambda_2 - (\xi_2\lambda_1) - (t\alpha + \beta)\xi_2 , \tag{57}$$

so that

$$\begin{aligned} \eta_2 &= \xi_1\lambda_2 - \xi_2\lambda_1 - \beta\xi_2 , \\ \eta_1 &= \xi_1\lambda_1 - (\lambda_1^2 + \beta\lambda_1 - \alpha)\xi_2\lambda_2^{-1}. \end{aligned} \tag{58}$$

Evidently $\omega^2 f(-\omega^{-1}) = \omega^2(\alpha\omega^{-2} - \beta\omega^{-1} - 1) = \alpha - \beta\omega - \omega^2 = -(\omega^2 + \beta\omega - \alpha)$, and so $g(\omega) = \omega^2 + \beta\omega - \alpha$ is irreducible.

Every irreducible polynomial $g(\omega)$ defines a corresponding polynomial $f(\omega)$ with a root t defining the unique plane $\mathfrak{M}$. The Hall planes defined on page 364 of [4] by $g(\omega)$ are all isomorphic to the derived plane $\mathfrak{M}'$ of $\mathfrak{M}$. We state our results as follows.

THEOREM 6 *The derived plane of a Desarguesian plane coordinatized by the field of degree two over a field $\mathfrak{K}$ is a Hall plane. All Hall planes of the same order are isomorphic.*

Among the problems which arise as a consequence of this discussion are the following: (1) to determine the nature of $\mathfrak{M}'$ if $\mathfrak{M}$ is the shears plane coordinatized by a Hall $(V - W)$-system; (2) to determine the nature of $\mathfrak{M}'$ for planes coordinatized by appropriate ternary rings such as twisted fields, Andre systems, and semi-nuclear algebras; (3) and to study the uniqueness of $\mathfrak{M}'$ in the affine case. We shall not discuss these questions further.

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