AN ELEMENTARY PROOF OF AN IDENTITY OF GOULD'S

BY DAVID BLACKWELL* AND LESTER DUBINS†

This note offers an elementary proof of a combinatorial identity of Gould (Formula 4 in [4]).

José Adem called this identity and its literature to our attention and pointed out to us that it had not yet received a purely finitistic proof. We are indebted to him and to Samuel Gitler for several stimulating conversations.

For historical remarks see [5]; for further developments see [6]; and for an application to the "Adem Relations" see [2] and [1].

Our proof of Gould's identity uses the circular symmetry technique that Dvoretzky and Motzkin introduced in their elegant treatment of the "ballot-box problem" [3].

Throughout this note, δ, β, and n are positive integers.

Let ρ be a cyclic permutation onto itself of a set E, whose order is the number of elements of E; and let g be a nonnegative, integer-valued function defined on E. For any $e \in E$ such that, for all i, $1 \leq i \leq n$,

$$\sum_{j=0}^{i\beta-1} g(\rho^j e) < i, \tag{1}$$

e is *winning* for E, g, β, n, ρ.

THEOREM 1. *Suppose that E has $\delta + \beta n$ elements. Then, for each g such that*

$$\sum_{e \in E} g(e) = n, \tag{2}$$

there are precisely δ winning e.

Proof. As is not difficult to verify, there must exist an $e \in E$ and an integer $k \geq 1$ such that $g(\rho^j e) = 0$ for $1 \leq j \leq k\beta - 1$ and $g(\rho^{k\beta} e) = n$. Plainly, none of the $k\beta$ elements $\rho e, \cdots, \rho^{k\beta} e$ are winning for E, g, β, n, ρ. Consider E', g', β', n', ρ', where the following obtain: E' is E with the $k\beta$ elements $\rho e, \cdots, \rho^{k\beta} e$ deleted; g' is g restricted to E'; $\beta' = \beta$ and $n' = n - k$; $\rho' = \rho$ restricted to E', except that $\rho'(e) = \rho^{k\beta+1}(e)$. As is easily verified an element of E' is winning for E, g, β, n ρ if and only if it is winning for E', g', β', n', ρ'. The proof is completed by induction on n.

A set S is *special* for an $(n + 1)$-tuple of disjoint sets $\{B_0, B_1, \cdots, B_n\}$, say $\mathcal{B}$ for short, if these three conditions are satisfied:

(i) $S \subset B_0 \cup \cdots \cup B_n$;

(ii) S contains precisely n elements;

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(iii) for all i, $0 \leq i \leq n$, the intersection of S with the union of the B_j's for $0 \leq j \leq i$ contains at least $i + 1$ elements.

Let $N(\mathcal{B})$ be the number of special sets S.

COROLLARY 1. *If B_0 contains δ elements and, for all i, $1 \leq i \leq n$, B_i contains β elements, then*

$$N(\mathcal{B}) = \binom{\delta + n\beta}{n} \frac{\delta}{\delta + n\beta}. \tag{3}$$

For one may compute the number M of pairs (j, S), $0 \leq j \leq n\beta + \delta$, $S \subset E = B_0 \cup \cdots \cup B_n$, for which $\Phi(j, S) = \rho^j S$ is special, where ρ is a suitable cyclic permutation of E, in two ways:

$$M = \binom{\delta + n\beta}{n} \delta, \tag{4}$$

as flows from Theorem 1 by letting g range over the indicator (characteristic) functions of subsets of E with n elements; and

$$M = (\delta + n\beta) N(\mathcal{B}), \tag{5}$$

since Φ assumes each of its values (special or not) for exactly $(\delta + n\beta)$ pairs. Q.E.D.

Let $A_0, \cdots, A_n$ be a sequence of $n + 1$ sets. For each set S with fewer than $n + 1$ elements, define a triple $T(S) = (k, S', S'')$ thus: Denoting the intersection of S with the union of the $j + 1$ sets $A_0, \cdots, A_j$, by S_j, let k be the first j such that S_j contains fewer than $j + 1$ elements; let S' be S_k; and let S'' be $S - S'$.

LEMMA 1. *If $B_0, \cdots, B_n$ are pairwise disjoint, and $A_i \subset B_i$ for $0 \leq i \leq n$, then, as S ranges over those sets that are special for $(B_0, \cdots, B_n)$, T defines a one-one correspondence onto the set of all triples (k, S', S'') such that $0 \leq k \leq n$, S' is special for $A_0, \cdots, A_k$ and S'' is special for $C_0^k, \cdots, C_{n-k}^k$, where*

$$C_0^k = (B_0 \cup \cdots \cup B_k) - (A_0 \cup \cdots \cup A_k) \tag{6}$$

and

$$C_j^k = B_{j+k}, \qquad \textit{for} \qquad 1 \leq j \leq n - k. \tag{7}$$

GOULD'S IDENTITY. *Using Gould's notation,*

$$A_k(\alpha, \beta) = \binom{\alpha + \beta k}{k} \frac{\alpha}{\alpha + \beta k}, \tag{8}$$

his result is this. For all real numbers α, β, γ, and for all positive integers n,

$$\textstyle\sum_{k=0}^{n} A_k(\alpha, \beta) A_{n-k}(\gamma, \beta) = A_n(\alpha + \gamma, \beta). \tag{9}$$

Proof of Gould's Identity. Suppose first that α, β, and γ are positive integers. Let $B_0, \cdots, B_n$ be $n + 1$ disjoint sets where B_0 contains $\alpha + \gamma$ elements and each B_i for $i \geq 1$ contains β elements. Then, according to Corollary 1, the right side of (9) is the number of sets that are special for $(B_0, \cdots, B_n)$. On the other hand, if A_0 is a subset of B_0 with α elements and $A_i = B_i$, for $i \geq 1$, then Lemma 1, together with Corollary 1, implies that the left side of (9) also represents the number of sets special for $(B_0, \cdots, B_n)$. So (9) holds whenever α, β and γ are positive integers. Since the left side of (9) is a polynomial in α, β and γ, as is the right side, (9) must hold for all real α, β, and γ.

University of California, Berkeley

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