THE STABLE HOMOTOPY OF $K(Z, n)$

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1. Introduction

There is the map $\Sigma^k K(Z, n) \to K(Z, n + k)$ which pulls the basic class back to the k-fold suspension of the basic class; let $F_{n,k}$ be the fiber of this map. We then get the commutative diagram

$$(1.1)\qquad \begin{array}{ccccc} \Sigma^{k-1}F_{n,1} & & & & \\ \downarrow & \searrow & & & \\ F_{n,k} & \to & \Sigma^k K(Z, n) & \to & K(Z, n+k) \\ \Big| & & \searrow & \nearrow & \\ \Big| & & & \Sigma^{k-1}K(Z, n+1) & \\ \downarrow & & \nearrow & & \\ F_{n+1,\,k-1} & & & & \end{array}$$

in which each of the straight sequences is a fiber space, at least in the stable range. The downward diagonal one is so because it is the suspension of a fiber triple. That the vertical sequence exists and is a fiber triple is shown by an elementary argument.

Note that $\pi_i(\Sigma^k K(Z, n)) = 0, Z, 0$ for, respectively, $i < n + k$, $i = n + k$, and $n + k < i < 2n + k$, and note also that $\pi_{k+i}(\Sigma^k K(Z, n)) = \pi_{k+i+1}(\Sigma^{k+1} K(Z, n))$, for k large relative to i and n. Thus let

$$\pi_i(n) = \pi_{2n+k+i}(\Sigma^k K(Z, n)),$$

k large. These $\pi_i(n)$ are the stable homotopy groups of $K(Z, n)$ of the title. The 2-primary part of $\pi_i(n)$ is computed here for $i = 0, \cdots, 7$.

The fiber space $\Sigma^{k-1}F_{n+1} \to F_{n,k} \to F_{n+1,k-1}$ yields the exact sequence $\cdots \to \pi_q(\Sigma^{k-1}F_{n+1}) \to \pi_q(F_{n,k}) \to \pi_q(F_{n+1,k-1}) \to \cdots$ which, for $q = 2n + k + i$, becomes

$$(1.2)\qquad \cdots \to \pi_i(n + 1) \xrightarrow{\partial} \pi_{2n+k+i}(\Sigma^{k-1}F_{n,1}) \to \pi_i(n) \to \pi_{i-1}(n + 1) \xrightarrow{\partial} \cdots$$

so that, using the known groups of $F_{n,1}$ [1], $\pi_i(n)$ is determined up to extension by $\pi_{i-1}(n + 1)$ and the two boundary maps. The boundary maps are determined by the following.

Lemma 1.3. *A homotopy class $[g] \in \pi_{i-1}(n + 1)$ has $\partial[g] \neq 0$ if and only if $[g]$ corresponds to a spherical cohomology class (that is, $g^*: H^*(F_{n,k}) \to H^*(S^{2n+k+i})$ is non-trivial). (Here $i \geq 1$.)*

Proof. Referring back to the diagram (1.1), we may as well assume that $g: S^{2n+k+i} \to \Sigma^{k-1}K(Z, n + 1)$; since we are in the stable range, $\partial[g]$ can be interpreted as $[hg]$, where $h: \Sigma^{k-1}K(Z, n + 1) \to \Sigma^k F_{n,1}$. Now if $\partial[g] \neq 0$, then

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$(hg)^* \neq 0$, as all of the homotopy of $F_{n,1}$ corresponds to spherical cohomology [1]. Hence $g^* \neq 0$.

Now assume that g^* is not the trivial homorphism. We proceed indirectly and thus assume that $\partial[g] = 0$, i.e. that $g = ih$, where $h: S^{2n+k+i} \to \Sigma^k K(Z, n)$ and $i: \Sigma^k K(Z, n) \to \Sigma^{k-1} K(z, n + 1)$:

$$\begin{array}{ccc} H^*(\Sigma^{k-1}K(Z, n+1) & \xrightarrow{i^*} & H^*(\Sigma^k K(Z, n)) \\ & {}_{g^*}\searrow \quad \swarrow_{h^*} & \\ & H^*(S). & \end{array}$$

Thus $h^* \neq 0$; say $h^*(i^*u) \neq 0$, where $u \in H^{2n+k+i}(\Sigma^{k-1}K(Z, n + 1))$. But then $u = \Sigma^{k-1}(\mathrm{Sq}^I \alpha_{n+1})$ where Sq^I is of degree $n + i$ and $i^*u = S^k \mathrm{Sq}^I \alpha_n$. But then either $\mathrm{Sq}^I \alpha_n$ is decomposable or $\mathrm{Sq}^I \alpha_n = \mathrm{Sq}^{n+i} \alpha_n = 0$, as $i \geq 1$. But if i^*u is decomposable, then $h^*i^*u = 0$, which is a contradiction.

Next, note that, for $0 \leq i \leq 7$, the groups $\pi_i(n)$ are periodic of period 8 in n. This is because the Adem relations expressing $\mathrm{Sq}^i \mathrm{Sq}^n$ in terms of admissible elements are, for $i \leq 7$, periodic of period 8 in n. Thus we construct (section 4) a modified Posnikov tower for $F_{n,k}$, k large and $n \equiv 0(8)$ valid up to dimension $2n + k + 6$. This is made possible by the secondary operation φ computed in [2] and the tertiary operation λ, computed below. This yields $\pi_i(n)$, $n \equiv 0(8)$; using this and 1.2 and 1.3, we get $\pi_i(n)$, $n \equiv 7(8)$ up to extensions. This much information makes it easy to construct the modified Posnikov tower for $F_{n,k}$, $n \equiv 7(8)$, completing the inductive step.

After eight steps we get $\pi_i(n)$, $n \equiv 0(8)$ again, which is a good check. Note that the groups $\pi_i(n)$, $0 \leq i \leq 6$, yield those for $i = 7$, as $\pi_{2n+7}(F_{n,1}) = 0$. As a further check, the towers were actually computed for $0 \leq i \leq 7$.

2. Statement of results

THEOREM. *The 2-primary part of the first eight non-trivial stable homotopy groups of $K(Z, n)$ is given by the following table*:

$$\pi_i(n) = \pi_{2n+k+i}(\Sigma^k K(Z, n))$$

i \ n, mod 8	0	7	6	5	4	3	2	1
0	Z	Z_2	Z	Z_2	Z	Z_2	Z	Z_2
1	Z_2	0	Z_2	0	Z_2	0	Z_2	0
2	Z_2	$Z_2^s + Z_2$	0	Z_4	Z_2	$Z_2^s + Z_2$	0	Z_4
3	Z_4	Z_2	Z_2	0	Z_4	Z_2	Z_2	0
4	0	$Z_2^s + Z_4$	Z_2^s	Z_2^s	0	Z_8	$Z_2^s + Z_2$	Z_2^s
5	0	0	Z_4	0	0	0	Z_8	Z_2
6	$Z_2^s + Z_4$	Z_2^s	Z_2^s	$Z_2^s + Z_4$	Z_2^s	Z_2^s	Z_2^s	Z_{16}
7	Z_{16}	Z_4	0	0	Z_4	0	0	0

The mod p, $p > 2$ case was handled by Barcus [3]. In the table, Z_2^s indicates that the generator corresponds to a spherical cohomology class.

3. A tertiary cohomology operation

Let φ_h and φ_h' be the operations introduced in [1]; let Φ^n denote the vector operation $(\varphi_n, \varphi_{n+1}')$; and let Φ_i^n be the ith coordinate of Φ^n, $i = 1, 2$.

LEMMA 3.1. *For $n \equiv 0(4)$, there is a choice of φ_{n+1} and φ_{n+2}' such that* $\mathrm{Sq}^2 \Phi_1^{n+1} + \mathrm{Sq}^1 \Phi_2^{n+1} = \mathrm{Sq}^{n+3} \mathrm{Sq}^2$ *holds stably and with zero indeterminacy.*

Proof. Consider the fiber space with base $K_1(Z, n+1)$, fiber $K_2(Z_2, 2n+3) \times K_3(Z_2, 2n+5)$, k-invariants $k_2 = \mathrm{Sq}^{n+2}$ and $k_3 = \mathrm{Sq}^{n+4}$, and total space $E = K_1 \times K_2 \times K_3$. Then the relations $\mathrm{Sq}^2 \mathrm{Sq}^{n+2} = 0$ and $\mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^{n+2} + \mathrm{Sq}^1 \mathrm{Sq}^{n+4} = 0$ determine φ_{n+1} and φ_{n+2}', respectively. In [2] it is shown that φ_{n+1} can be chosen so that, for α_1, the basic class of $K_1 = K(Z, n+1)$, $\varphi_{n+1}(\alpha_1) \ni \alpha_1 \cup \mathrm{Sq}^2 \alpha_1$ and $\varphi_{n+2}(\alpha_1) \ni 0$. Thus, as the universal example for φ_{n+1}, we may take (E, α_1, u), where u is any combination of $\alpha_1 \cup \mathrm{Sq}^2 \alpha_1 + \epsilon \mathrm{Sq}^2 \alpha_2$, $\epsilon = 0, 1$. We choose $u = \alpha_1 \cup \mathrm{Sq}^2 \alpha_1 + \mathrm{Sq}^2 \alpha_2$. Similarly, we may take as universal example for φ_{n+2}', (E, α_1, v) where $v = \mathrm{Sq}^2 \mathrm{Sq}^1 \alpha_2 + \mathrm{Sq}^1 \alpha_3$. We then have

$$\mathrm{Sq}^2 \varphi_{n+1}\alpha_1 + \mathrm{Sq}^1 \varphi_{n+2}^{\ 1}\alpha_1 = \mathrm{Sq}^2 (\alpha_1 \cup \mathrm{Sq}^2 \alpha_1) + \mathrm{Sq}^2 \mathrm{Sq}^2 \alpha_2 + \mathrm{Sq}^1 \mathrm{Sq}^2 \mathrm{Sq}^1 \alpha_2 + \mathrm{Sq}^1 \mathrm{Sq}^1 \alpha_3 = \mathrm{Sq}^{n+3} \mathrm{Sq}^2 \alpha_1 .$$

To show that the indeterminacy is zero, let $g:X \to K(Z, n)$ be such that $g^* \alpha_1 = u \in H^*(X)$, and let $g_0, g_1:X \to E'$ be two liftings of g. Then

$$g_1'^*u = g_0^*u + \mathrm{Sq}^2 a_1$$
$$g_1'^*v = g_0^*v + \mathrm{Sq}^2 \mathrm{Sq}^1 a_1 + \mathrm{Sq}^1 a_2 ,$$

for some $a_1, a_2 \in H^*(X)$. Hence $\mathrm{Sq}^2 g_1^*u + \mathrm{Sq}^1 g_1^*v = \mathrm{Sq}^2 g_0^*u + \mathrm{Sq}^1 g_0^*v$; that is, the indeterminacy of the relation is zero.

LEMMA 3.2. *Let $n \equiv 0(8)$. Then the relation of Lemma 3.1 defines a tertiary operation λ_n, which may be so chosen that*

$$\alpha \cup \mathrm{Sq}^4 \alpha \in \lambda_n \alpha$$

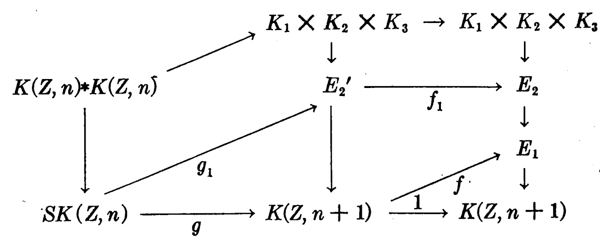
where α is the fundamental class of $K(Z, n+1)$.

Proof. Let $E_k \to K(Z, n+k)$ be the fiber space with fiber $K(Z_2, 2n+k+1) + K(Z_2, 2n+k+3)$, fundamental classes β_1 and β_2, and k invariants $\mathrm{Sq}^{n+2} \alpha$ and $\mathrm{Sq}^{n+4} \alpha$. For $k = 1$, this is trivial; let $f:K(Z, n) \to E_1$ be a cross section.

Now as $\mathrm{Sq}^2 \mathrm{Sq}^{n+2} = 0$ and $\mathrm{Sq}^2 \mathrm{Sq}^1 \mathrm{Sq}^{n+2} + \mathrm{Sq}^1 \mathrm{Sq}^{n+4} = 0$, there is the pair $(u, v) \in H^{2n+1+3}(E_1) \times H^{2n+1+2}(E_1)$ such that $i^*u = \beta_1$ and $i^*v = \beta_2$. Then $(E_1, (u, v), \alpha)$ is the universal example for Φ^n, where α is also used for the image of $\alpha \in H^*(K(Z, n+1))$ in E_1.

Let E_2 be the fiber space over E_1 with fiber $K_1(Z, 2n+1) \times K_2(Z_2, 2n+3) \times K_3(Z_2, 2n+4)$, and k-invariants $\mathrm{Sq}^{n+1} \alpha$, u, and v. Let α_1, α_2, and α_3 be the fundamental classes of the fiber. Let $K_1 \times K_2 \times K_3 \to E_2' \to K(Z, n+1)$ be the fiber space induced by $E_2 \to E_1$ by $f:K(Z, n+1) \to E_1$.

Now the map $g: SK(Z, n) \to K(Z, n+1)$, which brings the basic class α back to the suspension $s\alpha' \in H^*(SK(Z, n))$ of the basic class α' of $K(Z, n)$, lifts to a map $SK(Z, n) \to E_2$, as the appropriate k-invariants vanish on $SK(Z, n)$, for dimensional reasons. Note that this last map factors as $f_1 g_1$ through E_2', as E_2' is induced from $E_2 \to E_1$. Thus we have the commutative diagram



in which the top row are fibers to the maps g, $E_2' \to K(Z, n+1)$ and $E_2 \to E_1$.

Now the class $\mathrm{Sq}^4\, \alpha_1 + \mathrm{Sq}^2\, \alpha_2 + \mathrm{Sq}^1\, \alpha_1$ transgresses to zero in $H^*(E_1)$, by the previous lemma; we choose $w \in H^*(E_2)$ so that it restricts to $\mathrm{Sq}^4\, \alpha_1 + \mathrm{Sq}^2\, \alpha_2 + \mathrm{Sq}^1\, \alpha_1$ in the fiber $K_1 \times K_2 \times K_3$.

Then (E_2, w) is the universal example of λ_n. Consider the commutative diagram

$$\begin{array}{ccccc}
H^*(E_2) & \xrightarrow[i_1^*]{} & H^*(K_1 \times K_2 \times K_3) & \xrightarrow{\bar{i}} & H^*(E_1) \\
\downarrow f_1 & & \downarrow 1 & & \downarrow f \\
H^*(E_2') & \xrightarrow[i_2^*]{} & H^*(K_1 \times K_2 \times K_3) & \longrightarrow & H^*(K(Z, n+1)) \\
\downarrow & & \downarrow g_1 & & \downarrow 1 \\
H^*(SK(Z, n) & \xrightarrow[i_3^*]{} & H^*(K(Z, n)^*K(Z, n)) & \longrightarrow & H^*(K(Z, n+1)),
\end{array}$$

in which the horizontal sequences are exact in the range which concerns us. Let $u' = f_1^* u$. Now it is known [2] that $g_1^* \alpha_1 = \alpha * \alpha$ and $g_1^* \alpha_2 = \alpha * \mathrm{Sq}^2\, \alpha$, where α is the fundamental class of $K(Z, n)$. Thus

$$\begin{aligned} g_1^*(\mathrm{Sq}^4\, \alpha_1 + \mathrm{Sq}^2\, \alpha_2) &= \mathrm{Sq}^4\, \beta * \beta + \mathrm{Sq}^2\, \beta * \mathrm{Sq}^2\, \beta + \beta * \mathrm{Sq}^4\, \beta + \mathrm{Sq}^2\, \beta * \mathrm{Sq}^2\, \beta \\ &= \mathrm{Sq}^4\, \beta * \beta + \beta * \mathrm{Sq}^4\, \beta. \end{aligned}$$

But $i_3^*(s(\mathrm{Sq}^4\, \beta \cup \beta)) = \mathrm{Sq}^4\, \beta * \beta + \beta * \mathrm{Sq}^4\, \beta$ also. Thus the lifting g_1 can be chosen so that $g_1^* w = s(\mathrm{Sq}^4\, \beta \cup \beta)$. That is, $\alpha \cup \mathrm{Sq}^4\, \alpha \in \lambda_n \alpha$.

4. The towers

We wish to construct modified Posnikov towers for $\sum^k K(Z, n)$, beginning with the map $\Sigma^k K(Z, n) \xrightarrow[f]{} K(Z, n+k)$. Note that the kernel of f^* includes all $\mathrm{Sq}^I\, \alpha$, α the basic class of $K(Z, n+k)$, where I is of excess $>n$ (and less than k, but here k is large); thus we must kill these classes by adding homotopy groups in the fibers of our towers. Next, the cokernel consists of the k-fold suspension of various cup-product classes, $\mathrm{Sq}^2\,\alpha \cup \alpha$, $\mathrm{Sq}^3\,\alpha \cup \alpha, \cdots$. (It turns out that $\mathrm{Sq}^2\,\alpha \cup \alpha$, $\mathrm{Sq}^4\,\alpha \cup \alpha$, and some class in dimension $2n + 6$ are crucial for our purposes.)

Hence these (cup-product) classes must be added to $K(Z, n + k)$, unless they occur as secondary or higher order operations based on relations between classes introduced in the fibers of towers. Thus the tower for $n \equiv 0(8)$ has two Z_2 groups (α_1, α_2) added at $i = 1$, and $i = 3$ at the first stage, E_1. Their k-invariants are $\mathrm{Sq}^{n+2}\,\alpha$ and $\mathrm{Sq}^{n+4}\,\alpha$. Then $\mathrm{Sq}^{n+1}\,\alpha$ pulls up to E_1 and is killed at the second stage. E_2 is formed by adding $3 - Z_2$ groups $(\beta_1, \beta_2, \beta_3)$ at $i = 0, 2, 3$, and k-invariants $\mathrm{Sq}^{n+1}\,\alpha$, $\mathrm{Sq}^2\,\alpha_1$, and $\mathrm{Sq}^1\,\alpha_2 + \mathrm{Sq}^{2,1}\,\alpha_1$, respectively. We then find $\mathrm{Sq}^2\,\beta_1 = 0$, yielding the operation φ, where $\varphi(S^k\alpha_n) = S^k(\mathrm{Sq}^2\,\alpha_n \cup \alpha_n)$, and next, $\mathrm{Sq}^4\,\beta_1 + \mathrm{Sq}^2\,\beta_2 + \mathrm{Sq}^1\,\beta_3 = 0$, yielding the operation λ, where $\lambda(S^k\alpha_n) = S^k(\mathrm{Sq}^4\,\alpha_n \cup \alpha_n)$, (section 3). One computes and finds that this takes care of all cup-product classes except for one in dimension, $i = 6$. Thus one more class, s_1, must be added for $i = 6$, and we know it corresponds to a spherical cohomology class. The balance of the tower for $i = 0$ is a straightforward computation. One reads off the homotopy groups in the usual way: e.g. $\pi_3(n)$ has order 4, because both α_2 and β_3 occur in this dimension. Then as the k-invariant of β_3 involves $\mathrm{Sq}^1\,\alpha_2$, $\pi_3(n) = Z_4$.

This is the general pattern, with two exceptions. First, Z_4-groups occur for $n \equiv 7$ and $n \equiv 6(8)$, without being built up. That these classes are of order 4 was known from the previous cases; as a check, one can easily compute the corresponding MP-towers with integral coefficients. The second exception is that the class labeled β_3 in the tower for $n \equiv 2(8)$ corresponds to a spherical cohomology class. One can discover this by the process of elimination; in fact we know no better way to discover it. A little thought convinces one that this would not be unlikely, however. Thus, the quaternary operation φ_2 in dimension $i = 6$ detects a cup-product class; $\mathrm{Sq}^1\,\varphi_2$ detects another cup-product, and it is to this class that the relation $\mathrm{Sq}^7\,\alpha_1 + \mathrm{Sq}^6\,\alpha_2$ would otherwise have led. Thus β_3 is added to kill a class which needs killing only through a fluke. This is the usual situation where there is a differential in the Adams spectral sequence.

The towers

$n = 0(8)$

i	π_i								
0	Z				β_1				
1	Z_2		α_1	$\mathrm{Sq}^{n+1}\alpha$					
2	Z_2	$\mathrm{Sq}^{n+2}\alpha$			β_2	$\mathrm{Sq}^2\beta_1 : \varphi$			
3	Z_4		α_2	$\mathrm{Sq}^2\alpha_1$	β_3				
4	0	$\mathrm{Sq}^{n+4}\alpha$		$\mathrm{Sq}^1\alpha_2 + \mathrm{Sq}^{2,1}\alpha_1$		$\mathrm{Sq}^4\beta_1 + \mathrm{Sq}^2\beta_2 + \mathrm{Sq}^1\beta_3 : \lambda$			
5	0								
6	$Z_2^{*} + Z_4$	s_1					γ_1		γ_2
						$\mathrm{Sq}^{3,1}\beta_3$		$\mathrm{Sq}^1\gamma_1$	

* An element of order 4.

$n \equiv 7(8)$

i	π_i						
0	Z_2		α_1				
1	0	$\mathrm{Sq}^{n+1}\alpha$					
2	$Z_{2^s} + Z_2$	s_1			β_1		
3	Z_2		α_2	$\mathrm{Sq}^{2,1}\alpha_1$			
4	$Z_{2^s} + Z_4$	$s_2\mathrm{Sq}^2 s_1 + \mathrm{Sq}^{n+2,2}$	α				$\gamma_1{}^*$
5	0					$\mathrm{Sq}^3\beta_1$	
6	Z_{2^s}	s_3					

$n \equiv 6(8)$

i	π_i						
0	Z		α_1				
1	Z_2	$\mathrm{Sq}^{n+1}\alpha$	α_2				
2	0	$\mathrm{Sq}^{n+2}\alpha$		$\mathrm{Sq}^2\alpha_1 + \mathrm{Sq}^1\alpha_2$:	φ		
3	Z_2				β_1		
4	Z_{2^s}	s_1		$\mathrm{Sq}^{2,1}\alpha_1$			
5	Z_4						$\gamma_1{}^*$
6	Z_{2^s}	s_2				$\mathrm{Sq}^3\beta_1$	

$n \equiv 5(8)$

dim	π_i						
0	Z_2		α_1				
1	0	$\mathrm{Sq}^{n+1}\alpha$					
2	Z_4		α_2	$\mathrm{Sq}^2\alpha_2{:}\varphi$	β_1		
3	0	$\mathrm{Sq}^{n+3}\alpha$		$\mathrm{Sq}^{2,1}\alpha_1 + \mathrm{Sq}^1\alpha_2$			
4	Z_{2^s}	s_1					
5	0						
6	$Z_{2^s} + Z_4$	s_2			β_2		γ_1
				$\mathrm{Sq}^7\alpha_1 + \mathrm{Sq}^{4,2,1}\alpha_1$		$\mathrm{Sq}^{4,1}\beta_1 + \mathrm{Sq}^1\beta_2$	

$n \equiv 4(8)$

dim	π_i						
0	Z				β_1		
1	Z_2		α_1	$\mathrm{Sq}^{n+1}\alpha$			
2	Z_2	$\mathrm{Sq}^{n+2}\alpha$			β_2	$\mathrm{Sq}^2\beta_1{:}\varphi$	
3	Z_4		α_2	$\mathrm{Sq}^2\alpha_1$	β_3		
4	0	$\mathrm{Sq}^{n+4}\alpha$		$\mathrm{Sq}^{2,1}\alpha_1 + \mathrm{Sq}^1\alpha_2$		$\mathrm{Sq}^4\beta_1 + \mathrm{Sq}^2\beta_2 + \mathrm{Sq}^1\beta_3$:	λ
5	0						
6	Z_{2^s}	s_1					

$n \equiv 3(8)$

dim	π_i						
0	Z_2		α_1				
1	0	$\mathrm{Sq}^{n+1}\alpha$					
2	$Z_{2^s} + Z_2$	s_1			β_1		
3	Z_2		α_2	$\mathrm{Sq}^{2,1}\alpha_1$			
4	Z_8	$\mathrm{Sq}^{n+2,2}\alpha + \mathrm{Sq}^2 s_1$	α_3		β_2	$\mathrm{Sq}^2\beta_1{:}\varphi_x$	γ_1
5	0	$\mathrm{Sq}^{n+5}\alpha + \mathrm{Sq}^{n+3,2}\alpha$		$\mathrm{Sq}^{4,1}\alpha_1 + \mathrm{Sq}^1\alpha_3$		$\mathrm{Sq}^{2,1}\beta_1 + \mathrm{Sq}^1\beta_2$	
6	Z_{2^s}	s_2		$\mathrm{Sq}^6\alpha_1 + \mathrm{Sq}^3\alpha_2$:	φ_w		