EXISTENCE, UNIQUENESS, CONTINUITY, AND CONTINUATION OF SOLUTIONS FOR RETARDED DIFFERENTIAL EQUATIONS

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1. Introduction

In the last few years retarded differential equations have attracted the attention of many mathematicians, but although theorems related to the topics indicated in the title of this paper have been proved for different special cases (see references [2] through [11]), it is difficult to find in the current literature theorems that are general enough to cover most of the interesting applications of retarded differential equations

If one translates retarded differential equations to the language of functional differential equations, as for example is done in [5] and [8], it is not difficult to prove general existence and uniqueness theorems as extensions of the corresponding ones for ordinary differential equations. This is what we endeavour to do here even for the case of infinite retardations. The theorems to follow are applicable, of course, to ordinary differential equations; to so-called differential-difference equations, i.e., equations of the form

$$\dot{x}(t) = f[t, x(t), x(t - \tau_1), \cdots, x(t - \tau_n)], \qquad t \geq 0, \tag{1}$$

where $0 \leq \tau_1 \leq \tau_2 \leq \cdots \leq \tau_n$ and $\tau_j = \tau_j(t)$ are continuous functions of t; and also to more general past dependence equations, as for example

$$\dot{x}(t) = -\alpha \sum_{n=0}^{\infty} \frac{x(t-n)}{2^n}, \qquad t \geq 0 \tag{2}$$

or

$$\dot{x}(t) = f(t, x \mid [t-1, t], x \mid [0, t], x \mid (-\infty, t]), \qquad t \geq 0, \tag{3}$$

where $x \mid [a, b]$ means the function x restricted to the interval $[a, b]$.

2. Notation and definitions

As usual R^n will denote n-dimensional euclidean space with some norm $|\ \ |$. We shall be concerned with linear spaces of continuous functions defined on closed intervals $[a, b]$ or half open lines $(-\infty, b]$ with values in R^n. In the case of a closed interval $[a, b]$, the topology of the corresponding space of functions will be the usual uniform one; that is, if x is a continuous function on $[a, b]$, then the topology is defined by the norm $\| x \|$ given as

$$\| x \| = \sup \{ | x(t) |, a \leq t \leq b \},$$

and the resulting space is a Banach space. In the case $(-\infty, b]$, the topology will be given by a metric ρ defined as follows: if x and y are continuous functions on

$(-\infty, b]$, then

$$\rho(x, y) = \sup \{\min (2^{-k}, \| x - y \|_k); k = 0, 1, 2, \cdots\},$$

where $\| x - y \|_k = \sup \{| x(t) - y(t)|; b - k - 1 \leq t \leq b - k\}$. Such a metric is complete, and convergence is equivalent to uniform convergence on all compact subsets of $(-\infty, b]$. In this topology, a function x will be said to be bounded by H if $\| x \|_k < H$ for $k = 0, 1, 2, \cdots$. Notice that this concept of boundedness is stronger than the usual one for this space.

In the sequel we consider a given number r, $0 \leq r \leq \infty$, which we call the total lag. In specific cases this number is the maximum retardation needed at any time in the differential equation; for instance, in ordinary equations $r = 0$; in example (1), $r = \sup \{\tau_n(t), t \geq 0\}$; in (2) and (3), $r = \infty$.

Given two nonnegative numbers a and b, we denote by $C[r, a, b]$ the space of continuous functions from $[a - r, a + b]$ into R^n or from $(-\infty, a + b]$ into R^n, if $r = \infty$; by $C[r, a, b, H]$, we denote the subset of $C[r, a, b]$ bounded by H. A space that will play a special role is that for $a = b = 0$, that is, of continuous functions defined on $[-r, 0]$ or $(-\infty, 0]$; this space we shall simply denote by $C[r]$ or $C[r, H]$ for a bounded subset. The space $C[r]$ will be our phase space exactly in the same way as R^n is for ordinary differential equations. Actually, for $r = 0$, $C[r] = R^n$. For every t, $a \leq t \leq a + b$, we define a transformation from $C[r, a, b]$ into $C[r]$ which we call the t-restriction, and we denote its image by x_t, given by the following rule: if $x \in C[r, a, b]$, then x_t is the function of θ defined by $x(t + \theta)$ for $-r \leq \theta \leq 0$, clearly $x_t \in C[r]$. It is a routine matter to check that x_t is continuous in t, in the sense that if $t_n \to t$, $n \to \infty$, then $x_{t_n} \to x_t$ in the topology of $C[r]$. By $f(t, \psi)$ we shall designate a function with domain $[0, T) \times \Omega$, Ω open in $C[r, H]$, and range in R^n, where $0 < T \leq \infty$ and $0 < H \leq \infty$.

We should remark at this point that function spaces other than the ones mentioned may be of interest in the applications and also other topologies may be useful. However, the class we consider seems to be the most important, certainly is the simplest and brings out most of the essential elements necessary for generalizations.

Let $f(t, \psi)$ be a function as indicated above, and let $\dot{x}(t)$ denote the right hand derivative of a function $x(t) \in R^n$ at the point t, then a retarded differential equation (functional differential equation) is a functional relation of the form

$$\dot{x}(t) = f(t, x_t). \tag{4}$$

Given any point (a, φ) in the domain of $f(t, \psi)$, we say that (4) has a solution with initial condition (a, φ) if there exists a number $b > 0$, $a + b \leq T$, and a function $x = x(a, \varphi) \in C[r, a, b]$, such that

$$x_a = \varphi, x_t \in \Omega, \quad a \leq t < a + b, \quad \text{and}$$
$$\dot{x}(t) = f(t, x_t), \quad a \leq t < a + b.$$

3. Existence

In this section we shall extend a classical existence theorem of ordinary differential equations to equation (4).

THEOREM 1. *If $f(t, \psi)$ is continuous on $[0, T) \times \Omega$, and (a, φ) is any point in the domain of $f(t, \psi)$, then there exists a solution of (4) with initial condition (a, φ). Furthermore, the solution is continuously differentiable for $t > a$.*

Proof. In order to give the proof, we have to consider separately the cases $r < \infty$ and $r = \infty$, because of the different topologies. We shall give the details for the case $r = \infty$, the other being a simplified version of this one. The proof will make use of Schauder's fixed point theorem on an operator W acting on a subset of $C[-\infty, a, b]$, $a + b < T$, and defined as follows:

$$W[x](t) = \tilde{x}(t) + \int_{-\infty}^{t} f^*(s, x_s)\, ds, \tag{5}$$

with $-\infty < t \leq a + b$, where

$$\tilde{x}(t) = \begin{cases} \varphi(t - a) = \varphi(\theta);\ -\infty < \theta \leq 0[-\infty < t \leq a] \\ \varphi(0);\ a < t \leq a + b \end{cases} \tag{6}$$

and

$$f^*(t, \psi) = \begin{cases} 0;\ (-\infty, a) \times \Omega \\ f(t, \psi);\ [a, a + b] \times \Omega. \end{cases} \tag{7}$$

Clearly, if $W[x]$ is such that $W_t[x] \in \Omega$, $a \leq t < a + b$, and had a fixed point, the theorem would be proved.

For any positive numbers b and M, denote by $D[b, M]$ the subset of $C[-\infty, a, b]$ determined by the conditions

$$x \in D[b, M] \Leftrightarrow \begin{cases} x(t) = \tilde{x}(t);\ -\infty < t \leq a \\ |x(t) - \varphi(0)| = |x(t) - \tilde{x}(a)| < M;\ a \leq t \leq a + b. \end{cases} \tag{8}$$

First we want to prove that b and M can be chosen in such a way that $f(t, x_t)$ is bounded for all $a \leq t \leq a + b$ and $x \in D[b, M]$. Now, since $f(t, \psi)$ is continuous at (a, φ), there exists a neighborhood $V_a \times V_\varphi$ in $[0, T] \times \Omega$ such that $(t, \psi) \in V_a \times V_\varphi$ implies $|f(t, \psi)| < A$ for some constant A. Let the neighborhood V_φ of φ be described as the set of $\psi \in \Omega$ such that $\rho(\varphi, \psi) < 2^{-N} < \rho_0/2$, $\rho_0 = \text{dist}(\varphi, \partial\Omega)$, for some convenient integer $N > 0$. It is immediate from the definition of the metric ρ that, if ψ satisfies the condition

$$\sup\{|\varphi(\theta) - \psi(\theta)|;\ -N - 1 \leq \theta \leq 0\} < 2^{-N-1},$$

then $\psi \in V_\varphi$. Let b^* be chosen such that $|\varphi(\theta) - \varphi(\theta')| < 2^{-N-2}$, for $|\theta - \theta| < b^*$ and $\theta, \theta' \in [-N - 1, 0]$, and let $M = 2^{-N-2}$. Then, for all $x \in D[b^*, M]$, we have

$$\sup\{|\varphi(\theta) - x_t(\theta)|;\ a \leq t \leq a + b^*,\ -N - 1 \leq \theta \leq 0\} < 2^{-N-1},$$

and therefore $x_t \in V_\varphi$. So, with the added provisions that $a + b^* < T$ and $[a, a + b^*] \subset V_a$, we have

$$|f(t, x_t)| < A;\ a \leq t \leq a + b^*,\ x \in D[b^*, M]. \tag{9}$$

Finally choose $b \leq b^*$ in such a way that

$$bA \leq M, \tag{10}$$

and let $D[b, M, A]$ be the subset of $D[b, M]$ of functions that satisfy

$$|x(t_1) - x(t_2)| < A\,|t_1 - t_2|; \qquad t_1, t_2 \geq a. \tag{11}$$

It is a routine matter to check that $D[b, M, A]$ is a convex compact subset of $C[\infty, a, b]$ with the induced topology. Making use of (6)–(11), one sees immediately that, for all $x \in D[b, M, A]$, we have $W[x] \in D[b, M, A]$. On the other hand,

$$|W[x](t) - W[y](t)| \leq \begin{cases} 0; \; -\infty < t \leq a \\ \int_a^t |f(s, x_s) - f(s, y_s)|\, ds; \; a \leq t \leq a + b, \end{cases} \tag{12}$$

for all $x, y \in D[b, M, A]$. But when t ranges over $[a, a + b]$ and x, over $D[b, M, A]$, then (t, x_t) ranges on a compact subset of $[a, a + b] \times C[\infty]$ on which f is then uniformly continuous. This fact and (12) imply that W is continuous on $D[b, M, A]$. Now, as stated at the beginning, we use Schauder's fixed point theorem to complete the proof of Theorem 1.

4. Continuity and uniqueness

Consider the functional differential equations

$$\dot{x}(t) = f(t, x_t) \tag{13}$$

and

$$\dot{y}(t) = g(t, y_t), \tag{14}$$

where f and g are continuous locally bounded functions into R^n with domain $[0, T) \times \Omega$ and f is locally lipschitzian with respect to the second variable with Lipschitz function $k(t)$. According to Theorem 1, for any pair of points (a, φ) and (a', φ') in the common domain of f and g, there exist numbers b and b' and functions $x = x(a, \varphi)$ and $y = y(a', \varphi')$ defined on $a - r \leq t \leq a + b$ and $a' - r \leq t \leq a' + b'$, respectively, and such that $x(a, \varphi)$ satisfies (13) and $y(a', \varphi')$ satisfies (14), with the corresponding initial conditions. Let $k(t)$ be the Lipschitz function of f associated with a region in Ω which contains all the elements x_t, $a \leq t \leq a + b$, y_t, $a' \leq t \leq a' + b'$. Let α, β be positive numbers such that $|a - a'| = \alpha$, $\|\varphi - \varphi'\| = \beta$ ($\rho(\varphi, \varphi') = \beta$ in the case $r = \infty$), B a local common bound for f and g and γ a local bound for $|f(t, \psi) - g(t, \psi)|$. In order to be more specific let us suppose that $a \leq a' < a + b \leq a' + b'$. We can now prove

THEOREM 2. *With the former conditions and notations, the solutions x and y of* (13) *and* (14) *respectively satisfy the relation*

$$[\epsilon(\alpha)\beta + B\alpha + (b + a - a')\gamma]e^{\int_a^t k(s)ds} \geq \begin{cases} \rho(x_t, y_t); & \text{if } r = \infty \\ \|x_t - y_t\|; & \text{if } r < \infty \end{cases} \tag{15}$$

for $a' \leq t \leq a + b$; where $\epsilon(\alpha)$ is a continuous function of α, $\epsilon(\alpha) \to 1$ as $\alpha \to 0$.

Proof. Again we consider only the case $r = \infty$, the other being simpler. Using (5) for both solutions x and y and subtracting them, we obtain

$$\begin{aligned}|x(t, a, \varphi) - y(t, a', \varphi')| &\\ &\leq |\tilde{x}(t) - \tilde{y}(t)| + |\textstyle\int_{-\infty}^{t} f^*(s, x_s)\, ds - \int_{-\infty}^{t} g^*(s, y_s)\, ds| \\ &\leq |\tilde{x}(t) - \tilde{y}(t)| + |\textstyle\int_{-\infty}^{t} [f^*(s, x_s) - f^*(s, y_s)]\, ds| \\ &\qquad + |\textstyle\int_{-\infty}^{t} [f^*(s, y_s) - g^*(s, y_s)]\, ds| \\ &\leq |\tilde{x}(t) - \tilde{y}(t)| + \textstyle\int_{-\infty}^{t} k^*(s)\rho(x_s, y_s)\, ds \\ &\qquad + \textstyle\int_{-\infty}^{t} |f^*(s, y_s) - g^*(s, y_s)|\, ds,\end{aligned}$$

on the interval $-\infty < t \leq a + b$ and where $\tilde{y}(t)$ and g^* are defined in the same way as $\tilde{x}(t)$ and f^* by (6) and (7), and $k^*(t) = 0$ for $t < a$, $k^*(t) = k(t)$ for $t > a$.

From this inequality we immediately get

$$\rho[x_t(a, \varphi), y_t(a', \varphi')] \leq \rho(x_t, y_t) + \textstyle\int_a^t k(s)\rho(x_s, y_s)\, ds + B\alpha + (b + a - a')\gamma,$$

for $a' \leq t \leq a + b$. On the other hand, clearly $\rho(x_t, y_t) \leq \epsilon(\alpha)\beta$, for some continuous function $\epsilon(\alpha) \to 1$ as $\alpha \to 0$. Substituting this in our last inequality, we get (15) by using Gronwall's Lemma, and Theorem 2 is proved.

COROLLARY 1. *Suppose that f is continuous on $[0, T) \times \Omega$ and is locally lipschitzian with respect to the second variable with Lipschitz function $k(t)$. If $x(a, \varphi)$ and $x(a, \varphi')$ are solutions of (15) with initial values (a, φ) and (a, φ'), respectively, then*

$$\| x_t(a, \varphi) - x_t(a, \varphi')\| \leq \| \varphi - \varphi' \| e^{\int_a^t k(s)ds}; \qquad r < \infty$$

and

$$\rho[x_t(a, \varphi), x_t(a, \varphi')] \leq \rho(\varphi, \varphi')e^{\int_a^t k(s)ds}; \qquad r = \infty$$

for all $t \geq a$ in the common domain of the definition of $x(a, \varphi)$ and $x(a, \varphi')$.

COROLLARY 2. *Suppose that $f(t, \psi, \lambda)$ is continuous for $|\lambda| < \lambda_0$ and $(t, \psi) \in [0, T) \times \Omega$ and is locally lipschitzian in ψ. Then the solution $x(\lambda, a, \varphi)$ of $\dot{x}(t) = f(t, x_t, \lambda)$, $|\lambda| < \lambda_0$, with initial value (a, φ), is continuous in λ.*

Certainly a uniqueness result under a Lipschitz condition can be obtained from (15) by considering the case $\alpha = \beta = \gamma = 0$, but some comment is in order. Since the phase space $C[r]$ is determined by the total lag r, it might well happen that when considering the solution to some functional equation at some starting time $t = a$ the initial function $\varphi \in C[r]$ carries superfluous information. For example, if one is interested in an equation of the form $\dot{x}(t) = f(x\,|[0, t])$ at the initial time $t = 0$, the phase space would be $C[\infty]$, and therefore the initial point would be a function $\varphi(\theta)$ defined on $-\infty < \theta \leq 0$. Nevertheless, it is clear that the solution of such an equation at $t = 0$ should depend only on $\varphi(0)$, all the other values of φ being of no consequence to the equation, and that therefore the uniqueness should be stated in terms of $\varphi(0)$. Therefore we shall prove the following.

THEOREM 3. *Let f be continuous and locally lipschitzian with respect to the second variable in* $[0, T) \times \Omega$. *If, for a given* $a \in [0, T]$, $f(t, x_t)$, $t \geq a$, *depends only on the values of* $x_t(\theta)$ for $-r_0 \leq \theta \leq 0$, $0 \leq r_0 \leq r$, *then the solution of* (4) *is uniquely determined by* $(a, \{\varphi\})$, *where* $\{\varphi\}$ *represents the class of functions* $\varphi \in \Omega$ *that coincide in* $-r_0 \leq \theta \leq 0$.

Proof. Given two functions $\varphi_1 \neq \varphi_2$ in the same class $\{\varphi\}$, we know that there exist unique solutions $x^{(1)} = x(a, \varphi_1)$ and $x^{(2)} = x(a, \varphi_2)$ of (4) such that ${x_a}^{(1)} = \varphi_1$ and ${x_a}^{(2)} = \varphi_2$ and satisfying the equation in some interval $a \leq t < a + b \leq T$. Suppose that $x^{(1)}(t) \neq x^{(2)}(t)$ for at least some $t \in [a, a + b)$. Construct the function $x \in C[r, a, b]$ defined as follows:

$$x(t) = \begin{cases} x^{(1)}(t); & a - r_0 \leq t < a + b \\ x^{(2)}(t); & a - r \leq t \leq a - r_0 . \end{cases}$$

Then, because of the hypothesis, x is a solution to (4) with initial condition (a, φ_2) and different from $x^{(2)}$. This is a contradiction and the theorem is proved.

It is not difficult to see that continuity theorems under a uniqueness condition (as for example Th. 7.4, Ch. I, in [1]), can be easily obtained and we shall not give any details.

5. Other Lipschitz conditions

For $r \leq \infty$, Theorem 2 gives a unified treatment of the relationship between the solutions of (13) and (14) with respect to the initial values when one assumes $f(t, \varphi)$ is locally lipschitzian in φ relative to the metric ρ. For the case when r is finite, Theorem 2 is sufficient for most applications. Unfortunately, when $r = \infty$, a local Lipschitz condition relative to the metric ρ is too severe. To illustrate this point, consider the linear system

$$\dot{x}(t) = f(x_t)$$

$$f(\varphi) = \int_{-\infty}^{0} L(\theta)\varphi(\theta)\, d\theta, \quad L(\theta) \geq 0, \quad \int_{-\infty}^{0} L(\theta)\, d\theta < \infty .$$

The function f is certainly defined on the set Ω of all bounded functions in $C[\infty]$. In Ω, is $f(\varphi)$ locally lipschitzian in φ with respect to the metric ρ? More precisely, for any finite $H > 0$, does there exist a constant K_H such that

$$|f(\varphi)| \leq K_H \rho(\varphi, 0), \quad \text{for} \quad \varphi \quad \text{in} \quad C[\infty, H]?$$

To show that this is not the case, it is sufficient to show that there is a kernel function $L(\theta)$ and no constant K such that

$$|f(\varphi)| \leq K\rho(\varphi, 0) \quad \text{for} \quad \varphi \quad \text{in} \quad C[\infty], \sup\{|\varphi(\theta)|, -\infty \leq \theta \leq 0\} = 1,$$

since $f(\varphi)$ is linear in φ. Suppose that such a K did exist. For any positive real numbers $\lambda < 1$, ϵ, τ, let

$$\varphi(\theta) = \begin{cases} \lambda \text{ when } -\tau \leq \theta \leq 0 \\ g(\theta) \text{ when } -\tau - \epsilon \leq \theta \leq -\tau \\ 1 \text{ when } \theta \leq -\tau - \epsilon \end{cases}$$

where $g(\theta)$ is a decreasing continuous function with $g(-\tau - \epsilon) = 1$, $g(-\tau) = \lambda$. For this φ,

$$f(\varphi) = \lambda \int_{-\tau}^{0} L(\theta)\, d\theta + \int_{-\infty}^{-\tau} L(\theta)\, d\theta + \mu(\epsilon, \lambda, \tau),$$

where $\mu(\epsilon, \lambda, \tau) \to 0$ as $\epsilon \to 0$ uniformly in λ, τ. If $\lambda = 2^{-k}$ and $\tau = k$, then $\rho(\varphi, 0) = 2^{-k}$ and, therefore, for every integer k, it would be necessary to have a constant K such that

$$2^{-k} \int_{-k}^{0} L(\theta)\, d\theta + \int_{-\infty}^{-k} L(\theta)\, d\theta + \mu(\epsilon, 2^{-k}, k) < K2^{-k}.$$

It is clear that there exist integrable kernels $L(\theta)$ for which this is not true.

For such a simple equation as the above, one would certainly expect an existence and uniqueness result in Ω, and this indicates the need for a less restrictive definition of lipschitzianity. At first glance one might also expect the same properties for an even larger class of functions than Ω. In fact, the domain of definition of f contains many unbounded functions in $C[\infty]$. However, if this larger class of functions is chosen, the function f will not be continuous. This remark simply serves as a word of caution for the case $r = \infty$.

Consider again systems (13) and (14), but under the following hypothesis. Let Ω be the set of all bounded functions in $C[r, H]$, and suppose that f and g are continuous functions with domain $[0, T) \times \Omega$ and range in R^n such that f and g are bounded on $[0, \tau] \times (\Omega \cap C[r, H_1])$ for every $\tau < T$, $H_1 < H$. Also, suppose that for every $H_1 < H$, there is a function $k_{H_1}(t)$, continuous for $0 \leq t < T$, such that

$$(16) \quad |f(t, \varphi) - f(t, \psi)| \leq k_{H_1}(t) \sup \{|\varphi(\theta) - \psi(\theta)|, -r \leq \theta \leq 0\},$$

for $0 \leq t < T$ and φ, ψ in $\Omega \cap C[r, H_1]$. According to Theorem 1, for any pair of points (a, φ) and (a', φ') in the common domain of f and g, there exist numbers b and b' and functions $x = x(a, \varphi)$, and $y = y(a', \varphi')$, defined on $a - r \leq t \leq a + b$ and $a' - r \leq t \leq a' + b'$, respectively, such that x satisfies (13) and y satisfies (14), with the corresponding initial values (a, φ) and (a', φ'). For definiteness, suppose that $a \leq a' < a + b \leq a' + b'$. Let $k(t)$ be the Lipschitz function of f associated with a region S in Ω which contains all the elements x_t, $a \leq t \leq a + b$, y_t, $a' \leq t \leq a' + b'$. Suppose that B is a common bound for f and g in $[a, a' + b'] \times S$, $|f(t, \psi) - g(t, \psi)| < \gamma$ in $[a, a' + b'] \times S$, and $\alpha = a' - a$.

THEOREM 4. *With the former conditions and notations, the solutions x and y of* (13) *and* (14) *respectively satisfy*

$$\sup \{|x_t(\theta) - y_t(\theta)|, -r \leq \theta \leq 0\}$$

$$\leq [\epsilon(\alpha) \sup \{|\varphi(\theta) - \varphi'(\theta)|, -r \leq \theta \leq 0\} + B\alpha + (b + a - a')\gamma]e^{\int_a^t k(s)ds},$$

for $a' \leq t \leq a + b$, *where* $\epsilon(\alpha)$ *is a continuous function of* α, $\epsilon(\alpha) \to 1$ *as* $\alpha \to 0$.

The proof is the same as the proof of Theorem 2. For $\alpha = 0$ and $\gamma = 0$, we obtain the analog of Corollary 1 which yields continuity with respect to the initial function. The analog of Corollary 2 is also easily stated. Theorem 3 is also valid with the weaker Lipschitz condition (16).

In the case $r = \infty$ and with the Lipschitz condition of (16), we have been forced to consider only bounded functions in $C[\infty]$. In general, it is necessary to consider functions which may be unbounded. The above results can be improved in the following way. Suppose that $g(\theta)$, $-\infty < \theta \leq 0$ is any given function in $C[\infty]$, and let Ω be the set of functions φ in $C[\infty]$ such that φg is bounded. The metric is defined in $C[\infty]$ in exactly the same way, and the Lipschitz condition of (16) may be replaced by

$$|f(t, \varphi) - f(t, \psi)| \leq k_{H_1}(t) \sup \{|\varphi(\theta)g(\theta) - \psi(\theta)g(\theta)|, -\infty < \theta \leq 0\}.$$

The estimates in Theorem 4 will then involve an additional function of t and all supremums taken with the function g inside.

6. Continuation

Let us suppose that $f(t, \psi)$ is continuous and bounded in some domain $[0, T) \times \Omega$. Let $x(a, \varphi)$ be a solution of (4) with initial condition (a, φ) in the domain of f and defined in an interval $a \leq t < a + b \leq T$.

THEOREM 5. *Under the former conditions, the solution* $x(a, \varphi)$ *of* (4) *has a limit as* $t \to (a + b)^-$; *and, if* $x_{a+b} \in \Omega$, *then the solution can be extended to the right of* $a + b$.

Proof. The proof is exactly the same as that for Theorem 4.1, Chapter I, of [1].

Let Ω be the set of bounded functions in $C[r]$. If $f(t, \psi)$ is continuous on $[0, \infty) \times \Omega$, is linear in ψ, and if there is a function $k(t)$ continuous for $t \geq 0$ such that

$$|f(t, \psi)| \leq k(t) \sup \{|\psi(\theta)|, -r \leq \theta \leq 0\},$$

then Theorems 4 and 5 imply that the solutions of

$$\dot{x}(t) = f(t, x_t)$$

can be continued up to $+\infty$. Theorem 2 does not imply this property except for r finite.

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