FUNCTIONAL EQUATIONS AND GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS

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Introduction

Let R^n be a real n-dimensional Euclidean space where $|x|$ denotes the Euclidean norm of $x \in R^n$. Let X be the set of all functions $x(t)$ defined and continuous on the compact interval $[T_1, T_2]$ with values in R^n; i.e., $x: [T_1, T_2] \to R^n$. X will be considered as a Banach space with the norm $|x|_X = \sup_{t \in [T_1, T_2]} |x(t)|$. Let $0 < h < T_2 - T_1$, and denote by H the space of functions $y(t)$ defined and continuous in $[-h, 0]$, with values in R^n, and the norm defined by $y_H = \sup_{t \in [-h, 0]} |y(t)|$ for every $y \in H$.

If $x \in X$, define $x_t \in H$ for $t \in [T_1 + h, T_2]$ as follows: $x_t(\sigma) = x(t + \sigma)$, for $-h \leq \sigma \leq 0$. In other words x_t is the restriction of x to the interval $[t - h, t]$.

Let X_1 be a subset of X with the following property: if $x = x(t)$ for $t \in [T_1, T_2]$ is an element of X_1 and if $\bar{t} \in [T_1, T_2]$, then

$$\tilde{x} = \tilde{x}(t) = \begin{cases} x(t) & \text{for} \quad t \in [T_1, \bar{t}] \\ x(\bar{t}) & \text{for} \quad t \in [\bar{t}, T_2] \end{cases}$$

is also an element of X_1. Let H_1 be a subset of H such that $x \in X_1$ implies $x_t \in H_1$ for $t \in [T_1 + h, T_2]$. Consider the functional equation

$$dx/dt = f(x_t, t) \tag{1}$$

where $f: H_1 \times [T_1 + h, T_2] \to R^n$ and, for any fixed $x \in X_1$, $f(x_t, t)$ is Lebesgue integrable in $t \in [T_1 + h, T_2]$. Further, suppose that the primitive function of $f(x_t, t)$ has the following moduli of continuity for fixed $x^1, x^2 \in X_1$:

$$\left| \int_{\tau_1}^{\tau_2} f(x_\sigma^1, \sigma)\, d\sigma \right| \leq \omega_1(\tau_2 - \tau_1), \tag{2}$$

$$\left| \int_{\tau_1}^{\tau_2} [f(x_\sigma^1, \sigma) - f(x_\sigma^2, \sigma)]\, d\sigma \right| \leq \omega_2(\tau_2 - \tau_1) \sup_{\sigma \in [\tau_1, \tau_2]} \omega_3\{|(x^1 - x^2)_\sigma|_H\}, \tag{3}$$

for $T_1 + h \leq \tau_1 \leq \tau_2 \leq T_2$ and $\tau_2 - \tau_1 \leq \sigma^*$, where $\sigma^* > 0$ depends only on T_1, T_2, h. The functions ω_1, ω_2 are continuous and increasing on $[0, \sigma^*]$; ω_3 has the same property on $[0, \infty]$, $\omega_i(0) = 0$, $\sum_{j=1}^{\infty} 2^j \psi(\eta/2^j)$ uniformly convergent for $\eta \in [0, \sigma^*]$, $\psi(\eta) = \omega_3(\omega_1(\eta))\omega_2(\eta)$. Note that one can take $\omega_j(\eta) = \beta_j \eta^{\alpha_j}$, $\beta_j, \alpha_j > 0$, $j = 1, 2, 3$, $\alpha_1\alpha_3 + \alpha_2 > 1$. If especially $\alpha_j = 1$ for $j = 1, 2, 3$ and f is continuous in $H_1 \times [T_1 + h, T_2]$, then the usual conditions for existence and uniqueness of solutions of (1) are fulfilled [5].

It is the aim of this paper to show that equation (1) is a special case of a generalized ordinary differential equation in Banach spaces introduced by J. Kurzweil [1], [2]. The idea of associating a generalized equation to an equation of type (1), where $\omega_j(\eta)$ are linear, belongs to J. Kurzweil. This paper contains a generalization of this result to the case of moduli $\omega_j(\eta)$ stated above.

Some basic definitions and results of the theory of generalized differential equations will be stated in the following section.

1. Generalized differential equations in Banach spaces

The generalization of the concept of a differential equation is based on a more general definition of the integral of Riemann-Stieltjes.

Let $\tau_* < \tau^*$ and $\tilde{S} = \tilde{S}[\tau_*, \tau^*]$ be the system of sets $S \subset R^2$ with the following property: for every $\tau \in [\tau_*, \tau^*]$ there exists $\delta(\tau) > 0$ such that $(\tau, t) \in S$, if $\tau, t \in [\tau_*, \tau^*]$, $|\tau - t| \leq \delta(\tau)$. Let $U(\tau, t)$ be a function defined on some $S \in \tilde{S}[\tau_*, \tau^*]$ with values in a Banach space Y.

Let A be a finite sequence of numbers $\{\alpha_0, \tau_1, \alpha_1, \cdots, \tau_k, \sigma_k\}$, $\tau_* = \alpha_0 < \alpha_1 < \cdots < \alpha_k = \tau^*$, $\tau_i \in [\alpha_{i-1}, \alpha_i]$ for $i = 1, 2, \cdots, k$. A is called a partition of $[\tau_*, \tau^*]$ subordinate to $S \in \tilde{S}$ if $(\tau_i, t) \in S$ for $i = 1, 2, \cdots, k$, $t \in [\alpha_{i-1}, \alpha_i]$.

Denote by $\tilde{A}(S)$ the set of all A subordinate to S, and let $B(A) = \sum_{j=1}^{k} U(\tau_j, \alpha_j) - U(\tau_j, \alpha_{j-1})$ for $A \in \tilde{A}(S)$.

Definition 1. *$U(\tau, t)$ is K-integrable if, for every $\epsilon > 0$, there exists $S \in \tilde{S}[\tau_*, \tau^*]$ such that $|B(A_1) - B(A_2)|_Y < \epsilon$ whenever $A_1, A_2 \in \tilde{A}(S)$. It is easy to prove* [2] *that U is K-integrable if and only if there exists a unique vector $b \in Y$ such that, given $\epsilon > 0$, there exists an $S \in \tilde{S}[\tau_*, \tau^*]$ such that $A \in \tilde{A}(S)$ implies $|B(A) - b|_Y < \epsilon$ and b is denoted by $\int_{\tau_*}^{\tau^*} \mathfrak{D}U(\tau, t)$.*

It can be shown that if $U(\tau, t) = u(\tau)\alpha(t)$, with u continuous and α with bounded variation in $[\tau_*, \tau^*]$, then $\int_{\tau_*}^{\tau^*} \mathfrak{D}U$ exists and is equal to the integral of Riemann-Stieltjes $\int_{\tau_*}^{\tau^*} u(\tau)\, d\alpha(\tau)$. If $\partial U(\tau, t)/\partial t = u(\tau, t)$ is continuous, then $\int_{\tau_*}^{\tau^*} \mathfrak{D}U = \int_{\tau_*}^{\tau^*} u(\tau, \tau)\, d\tau$. If $U(\tau, t) = f(\tau)t$, with $Y = R^n$ and $f(\tau)$ Perron-integrable, then $\int_{\tau_*}^{\tau^*} \mathfrak{D}U = \int_{\tau_*}^{\tau^*} f(\tau)\, d\tau$ (see [1]).

In [2] the following existence theorem was proved.

Theorem 1. Let *$U(\tau, t)$ be defined and continuous on $[\tau_*, \tau^*] \times [\tau_*, \tau^*]$, and let $\psi(\eta) \geq 0$ be defined for $\eta \in [0, \sigma]$, with $\sigma > 0$, $\psi(0) = 0$, and $\sum_{j=1}^{\infty} 2^j \psi(\eta/2^j)$ uniformly convergent on $[0, \sigma]$. Suppose that*

$$(1.1) \quad |U(\tau + \eta, t + \eta) - U(\tau + \eta, t) - U(\tau, t + \eta) + U(\tau, t)|_Y \leq \psi(\eta),$$

for $0 < \eta \leq \sigma$, $(\tau + \eta, t + \eta), (\tau + \eta, t), (\tau, t + \eta), (\tau, t) \in [\tau_*, \tau^*] \times [\tau_*, \tau^*]$. Then $\int_{\tau_*}^{\tau^*} \mathfrak{D}U(\tau, t)$ exists and

$$(1.2) \quad \left|\int_{\lambda_1}^{\lambda_2} \mathfrak{D}U - U(\lambda_1, \lambda_2) + U(\lambda_1, \lambda_1)\right|_Y \leq \tfrac{1}{2}(\lambda_2 - \lambda_1)\Psi(\lambda_2 - \lambda_1)$$

and

$$(1.3) \quad \left|\int_{\lambda_1}^{\lambda_2} \mathfrak{D}U - U(\lambda_2, \lambda_2) + U(\lambda_2, \lambda_1)\right|_Y \leq \tfrac{1}{2}(\lambda_2 - \lambda_1)\Psi(\lambda_2 - \lambda_1),$$

for $\tau_* \leq \lambda_1 < \lambda_2 \leq \tau^*$, where $\Psi(\eta) = \sum_{j=1}^{\infty} (2^j/\eta)\psi(\eta/2^j)$. Moreover, $\Psi(\eta) \to 0$ for $\eta \to 0_+$.

Consider an open subset Y_1 of Y and a continuous function $F: Y_1$

$\times (T_1, T_2) \to Y$. A function $x(t)$ defined on $(t_1, t_2) \subset (T_1, T_2)$ is called a solution of the generalized differential equation

$$dx/d\tau = \mathfrak{D}F(x, t) \tag{1.4}$$

in the interval (t_1, t_2), with the initial condition $x(t_0) = x_0 \in Y_1$, $t_0 \in (t_1, t_2)$, if for every $t_3, t_4 \in (t_1, t_2)$ one has

$$x(t_4) - x(t_3) = \int_{t_3}^{t_4} \mathfrak{D}F(x(\tau), t).$$

Existence, uniqueness, and continuous dependence theorems for (1.4) were given by J. Kurzweil (in [1], [2], and [3]) for $Y = R^n$, and adequate modifications of the proofs allow the extension to Banach spaces [4].

2. Equivalence theorems

Let $f(x_t, t)$ be as in (1) and define

$$F(x, t)(\tau) = \begin{cases} \int_{T_1+h}^{\tau} f(x_\sigma, \sigma)\, d\sigma & \text{for } \tau \in [T_1 + h, t], t \in [T_1 + h, T_2] \\ \int_{T_1+h}^{t} f(x_\sigma, \sigma)\, d\sigma & \text{for } \tau \in [t, T_2], t \in [T_1 + h, T_2] \\ 0, \text{for } \tau \text{ or } t \in [T_1, T_1 + h] \end{cases} \tag{2.1}$$

for every $x \in X_1$.

Thus, for x, t fixed, $F(x, t)$ is an element of the function space X defined in the Introduction, and $F(x, t)(\tau) \in R^n$ is the value of $F(x, t)$ at the point $\tau \in [T_1 + h, T_2]$. First of all, it will be shown that $F(x, t)$ is continuous in $X_1 \times [T_1, T_2]$. Let $(x_i, t_i) \in X_1 \times [T_1, T_2]$, for $i = 1, 2$, $| x_1 - x_2 |_X$, $| t_1 - t_2 |$ sufficiently small. Then, for a sufficiently fine partition $T_1 + h = \sigma_0 < \sigma_1 < \cdots \sigma_k = t_1$,

$$\begin{aligned} &\left| \int_{T_1+h}^{t_1} f[(x_1)_\sigma, \sigma]\, d\sigma - \int_{T_1+h}^{t_2} f[(x_2)_\sigma, \sigma]\, d\sigma \right| \\ &\quad \le \left| \sum_{i=1}^{k} \int_{\sigma_{i-1}}^{\sigma_i} \{f[(x_1)_\sigma, \sigma] - f[(x_2)_\sigma, \sigma]\}\, d\sigma \right| + \left| \int_{t_2}^{t_1} f[(x_2)_\sigma, \sigma]\, d\sigma \right| \\ &\quad \le \omega_3\{| x_1 - x_2 |_X\} \sum_{i=1}^{k} \omega_2(\sigma_i - \sigma_{i-1}) + \omega_1(| t_2 - t_1 |). \end{aligned}$$

From this the continuity of $F(x, t)$ follows easily.

THEOREM 2.1. *Let $\xi(t)$ be a solution of (1) on $[t_1, t_2] \subset [T_1 + h, T_2]$; i.e., $\xi \in X_1$ is continuous in $[T_1, T_2]$ and satisfies the equation (1) in (t_1, t_2). Define for every $t \in [T_1, T_2]$*

$$x(t)(\tau) = \begin{cases} \xi(\tau) & \textit{for} \quad \tau \in [T_1, t] \\ \xi(t) & \textit{for} \quad \tau \in [t, T_2]. \end{cases} \tag{2.2}$$

Then $x(t) \in X$ is a solution of the generalized equation (1.4), with F given by (2.1) in $[t_1, t_2]$ and $| x(t_4) - x(t_3) |_X \le \omega_1(t_4 - t_3)$, for $t_1 \le t_3 \le t_4 \le t_2$, $| t_4 - t_3 | \le \sigma^*$.

Proof. Let $[t_3, t_4] \subset [t_1, t_2]$. It will be proved that $\int_{t_3}^{t_4} \mathfrak{D}F(x(s), t)$ exists and that

$$x(t_4) - x(t_3) - \int_{t_3}^{t_4} \mathfrak{D}F[x(\tau), t] = 0. \tag{2.3}$$

According to the property following Definition 1, it is sufficient to prove that for any ϵ there exists a set $S \in \tilde{S}[t_3, t_4]$ such that, if $A = \{t_3 = \sigma_0, \xi_1, \sigma_1, \cdots, \sigma_k = t_4\}$ is subordinate to $\tilde{A}(S)$, then

$$(2.4) \qquad | x(t_4) - x(t_3) - \textstyle\sum_{i=1}^k F(x(\xi_i), \sigma_i) - F(x(\xi_i), \sigma_{i-1}) |_X < \epsilon.$$

Using (2.1) one obtains, for $i = 0, 1, \cdots, k$,

$$(2.5) \qquad \begin{aligned} [F(x(\xi_i), \sigma_i) - F(x(\xi_i), \sigma_{i-1})](\tau) &= 0, \quad \text{for} \quad \tau \in [T_1, \sigma_{i-1}] \\ &= \textstyle\int_{\sigma_{i-1}}^{\tau} f\{[x(\xi_i)]_\sigma, \sigma\}\, d\sigma, \quad \text{for} \quad \tau \in [\sigma_{i-1}, \sigma_i], \\ &= \textstyle\int_{\sigma_{i-1}}^{\sigma_i} f\{[x(\xi_i)]_\sigma, \sigma\}\, d\sigma, \quad \text{for} \quad \tau \in [\sigma_i, T_2]. \end{aligned}$$

As $\xi(t)$ is a solution of the functional equation (1), it holds that

$$\xi(\tau_2) - \xi(\tau_1) = \textstyle\int_{\tau_1}^{\tau_2} f(\xi_\sigma, \sigma)\, d\sigma,$$

for $\tau_1, \tau_2 \in [t_1, t_2]$. From this it follows, in view of (2.2), that

$$(2.6) \quad [x(\sigma_i) - x(\sigma_{i-1})](\tau) = \begin{cases} 0, & \text{for} \quad \tau \in [T_1, \sigma_{i-1}] \\ \int_{\sigma_{i-1}}^{\tau} f(\xi_\sigma, \sigma)\, d\sigma, & \text{for} \quad \tau \in [\sigma_{i-1}, \sigma_i] \\ \int_{\sigma_{i-1}}^{\sigma_i} f(\xi_\sigma, \sigma)\, d\sigma, & \text{for} \quad \tau \in [\sigma_i, T_2]. \end{cases}$$

Combining (2.5) and (2.6), one can write

$$(2.7) \quad \begin{aligned} &\{x(\sigma_i) - x(\sigma_{i-1}) - F[x(\zeta_i), \sigma_i] + F[x(\zeta_i), \sigma_{i-1}]\}(\tau) \\ &= \begin{cases} 0, & \text{for} \quad \tau \in [T_1, \sigma_{i-1}] \\ \int_{\sigma_{i-1}}^{\tau} \{f[\xi_\sigma, \sigma] - f[x(\zeta_i)_\sigma, \sigma]\}\, d\sigma, & \text{for} \quad \tau \in [\sigma_{i-1}, \sigma_i] \\ \int_{\sigma_{i-1}}^{\sigma_i} \{f[\xi_\sigma, \sigma] - f[x(\xi_i)_\sigma, \sigma]\}\, d\sigma, & \text{for} \quad t \in [\sigma_i, T_2]. \end{cases} \end{aligned}$$

Using hypothesis (3), one obtains

$$(2.8) \quad \begin{aligned} &| \textstyle\int_{\sigma_{i-1}}^{\tau} \{f[\xi_\sigma, \sigma] - f[x(\xi_i)_\sigma, \sigma]\}\, d\sigma | \\ &\quad \leq \omega_2(\tau - \sigma_{i-1}) \sup_{\sigma \in [\sigma_{i-1}, \tau]} \omega_3\{| [\xi - x(\xi_i)]_\sigma |_H\} \\ &\quad \leq \omega_2(\tau - \sigma_{i-1}) \sup_{\sigma \in [\sigma_{i-1}, \tau]} \omega_3\{\sup_{\vartheta \in [\sigma - h, \sigma]} | [\xi - x(\xi_i)](\vartheta) |\} \\ &\quad \leq \omega_2(\tau - \sigma_{i-1}) \omega_3\{\sup_{\vartheta \in [\sigma_{i-1} - h, \tau]} | [\xi - x(\xi_i)](\vartheta) |\} \\ &\quad \leq \omega_2(\tau - \sigma_{i-1}) \omega_3\{\omega_1(\tau - \sigma_{i-1})\}, \end{aligned}$$

for $\tau \in [\sigma_{i-1}, \sigma_i]$, $\sigma_i - \sigma_{i-1} \leq \sigma^*$.

The last inequality follows from the fact that $\xi(\vartheta) = x(\xi_i)(\vartheta)$, for $\vartheta \in [T_1, \xi_i]$, and $x(\xi_i)(\vartheta) = \xi(\xi_i)$, for $\vartheta \in [\xi_i, T_2]$, and from (2), where x^1 is replaced by ξ. From (2.8) it follows that the norm of the left hand side in (2.7) has the bound $\omega_3\{\omega_1(\sigma_i - \sigma_{i-1})\}\omega_2(\sigma_i - \sigma_{i-1})$, for all $\tau \in [T_1, T_2]$. Summing the expressions in the left side of (2.7) for $i = 1, 2, \cdots, k$, one obtains

$$(2.9) \quad \begin{aligned} &| \{x(t_4) - x(t_3) - \textstyle\sum_{i=1}^k F(x(\xi_i), \sigma_i) - F(x(\xi_i), \sigma_{i-1})\}(\tau) | \\ &\qquad \leq \textstyle\sum_{i=1}^k \omega_3\{\omega_1(\sigma_i - \sigma_{i-1})\}\omega_2(\sigma_i - \sigma_{i-1}), \end{aligned}$$

if the norm $\max_{i=1,\cdots,k}(\sigma_i - \sigma_{i-1})$ of A is small enough or (which is the same) if S is chosen adequately.

It will be proved that the right side of (2.9) is arbitrarily small if the norm of A is sufficiently small. Indeed

$$\begin{aligned}
&\sum_{i=1}^{k} \omega_3\{\omega_1(\sigma_i - \sigma_{i-1})\}\omega_2(\sigma_i - \sigma_{i-1}) \\
&\quad \leq \max_{i=1,\cdots,k} \frac{\omega_3\{\omega_1(\sigma_i - \sigma_{i-1})\}\omega_2(\sigma_i - \sigma_{i-1})}{\sigma_i - \sigma_{i-1}}(t_4 - t_3) \\
&\quad \leq \max_{i=1,\cdots,k} \frac{\omega_3\{\omega_1(\sigma_i - \sigma_{i-1})\}}{\sigma_i - \sigma_{i-1}}\omega_2(\sigma_i - \sigma_{i-1})(t_4 - t_3) \\
&\quad = \max_{i=1,\cdots,k} \frac{\psi(\sigma_i - \sigma_{i-1})}{\sigma_i - \sigma_{i-1}}(t_4 - t_3).
\end{aligned} \tag{2.10}$$

But from Theorem 1 it follows that

$$\Psi(\eta) = \textstyle\sum_{j=1}^{\infty}(2^j/\eta)\psi(\eta/2^j)$$

tends toward zero for $\eta \to 0_+$, which implies $\psi(\eta)/\eta \to 0$, for $\eta \to 0_+$. Thus (2.3) is proved.

Finally, by (2.2) and (2),

$$\begin{aligned}
|x(t_4) - x(t_3)|_X &= \sup_{\tau\in[T_1,T_2]} |[x(t_4) - x(t_3)](\tau)| \\
&= \sup_{t\in[t_3,t_4]} |\xi(t) - \xi(t_3)| \leq \omega_1(t_4 - t_3), \quad \text{for} \quad t_4 - t_3 \leq \sigma^*,
\end{aligned}$$

which proves Theorem 2.1.

Now it will be proved that the converse of Theorem 2.1 is also true.

THEOREM 2.2. *Let $x(t)$ be a solution of the generalized equation* (1.4), *with F given by* (2.1), *in the interval $[t_1, t_2] \subset [T_1 + h, T_2]$ with the property $|x(t_4) - x(t_3)|_X \leq \omega_1(t_4 - t_3)$ for $t_1 \leq t_3 \leq t_4 \leq t_2$, $t_4 - t_3 \leq \sigma^*$ and with the initial condition*

$$x(t_1)(\tau) = x(t_1)(t_1) \quad \textit{for} \quad \tau \in [t_1, T_2]. \tag{2.11}$$

Define

$$\xi(\tau) = \begin{cases} x(t_1)(\tau), & \textit{for} \quad \tau \in [T_1, t_1] \\ x(\tau)(\tau), & \textit{for} \quad \tau \in [t_1, t_2] \\ x(t_2)(\tau), & \textit{for} \quad \tau \in [t_2, T_2]. \end{cases} \tag{2.12}$$

Then $\xi(\tau)$ is a solution of (1) *in $[t_1, t_2]$, $\xi(\tau) = x(t_2)(\tau)$ for $\tau \in [T_1, T_2]$, and $|\xi(t_4) - \xi(t_3)| \leq \omega_1(t_4 - t_3)$ for $t_1 \leq t_3 \leq t_4 \leq t_2$, $t_4 - t_3 \leq \sigma^*$.*

For the proof, the two following lemmas will be needed.

LEMMA 2.1. *Under the hypothesis of Theorem 2.2 it holds that*

$$x(t)(\tau) = x(t)(t), \quad \textit{for} \quad \tau \geq t, \ \tau \in [T_1, T_2], t \in [t_1, t_2]; \tag{2.13}$$

$$x(t)(\tau) = x(\tau)(\tau), \quad \textit{for} \quad t \geq \tau, \ \tau \in [T_1, T_2]\ t \in [t_1, t_2. \tag{2.14}$$

Proof. To prove (2.13) it is sufficient to show that

$$[x(t_4) - x(t_3)](\tau_2) = [x(t_4) - x(t_3)](\tau_1),$$

for $t_3 \leq t_4 \leq \tau_1 \leq \tau_2$, $\tau_1, \tau_2 \in [T_1, T_2]$; $t_3, t_4 \in [t_1, t_2]$, for $t_3 \leq t_4 \leq t_1 \leq t_2$, $t_1, t_2 \in [T_1, T_2]$; $t_3, t_4 \in [t_1, t_2]$, taking into account (2.11). But

$$x(t_4) - x(t_3) = \int_{t_3}^{t_4} \mathfrak{D}F(x(\xi), s),$$

and, for every $\epsilon > 0$, there exists an $S \in \tilde{S}[t_3, t_4]$ such that $A = \{t_3 = \sigma_0 \leq \xi_1 \leq \sigma_1 \leq \cdots \leq \sigma_k = t_4\} \in \tilde{A}(S)$ implies that

$$(2.15) \quad | x(t_4) - x(t_3) - \textstyle\sum_{i=1}^{k} F(x(\xi_i), \sigma_i) - F(x(\xi_i), \sigma_{i-1}) |_X < \epsilon/2.$$

Further, by (2.1), the sum in (2.15) evaluated at τ_2 is equal to that at τ_1, and consequently,

$$\begin{aligned} &| [x(t_4) - x(t_3)](\tau_2) - [x(t_4) - x(t_3)](\tau_1) | \\ &\quad \leq | [x(t_4) - x(t_3)](\tau_2) - \textstyle\sum_{i=1}^{k} [F(x(\xi_i), \sigma_i) - F(x(\xi_i), \sigma_{i-1})](\tau_2) | \\ &\quad + | \textstyle\sum_{i=1}^{k} [F(x(\xi_i), \sigma_i) - F(x(\xi_i), \sigma_{i-1})](\tau_1) - [x(t_4) - x(t_3)](\tau_1) | < \epsilon, \end{aligned}$$

if $\tau_i \geq t_4$, which proves (2.13).

Similarly, (2.14) is obtained if the sum in (2.15) is evaluated at $\tau = t_3$, which yields

$$| [x(t_4) - x(t_3)](t_3) | < \frac{\epsilon}{2} \quad \text{for every} \quad \epsilon > 0.$$

LEMMA 2.2. *If the hypotheses of Theorem 2.2 are satisfied, then*

$$(2.16) \quad \begin{aligned} &| \textstyle\int_{t_3}^{t_4} \mathfrak{D}F(x(\xi), s) - F(x(t_4), t_4) + F(x(t_4), t_3) |_X \\ &\qquad\qquad \leq [(t_4 - t_3)/2]\Psi(t_4 - t_3) \end{aligned}$$

for $t_1 \leq t_3 \leq t_4 \leq t_2$, $t_4 - t_3 \leq \sigma^*$.

Proof. Using Theorem 1, especially (1.3), for $U(\xi, s) = F(x(\xi), s)$, one obtains (2.16) with $\psi(\eta) = \omega_3(\omega_1(\eta))\omega_2(\eta)$.

Proof of Theorem 2.2. In view of (2.14) it is sufficient to prove that

$$(2.17) \quad | \xi(t_4) - \xi(t_3) - \textstyle\int_{t_3}^{t_4} f(\xi_\sigma, \sigma)\, d\sigma | < \epsilon,$$

for every $\epsilon > 0$ and $t_1 \leq t_3 \leq t_4 \leq t_2$.

By (2.12) and (2.1),

$$(2.18) \quad \begin{aligned} \textstyle\int_{t_3}^{t_4} f(\xi_\sigma, \sigma)\, d\sigma &= \textstyle\int_{t_3}^{t_4} f[x(t_4)_\sigma, \sigma]\, d\sigma \\ &= \textstyle\sum_{i=1}^{k} \{F[x(t_4), \sigma_i] - F[x(t_4), \sigma_{i-1}]\}(t_4), \end{aligned}$$

where $t_3 = \sigma_0 \leq \sigma_1 \leq \cdots \leq \sigma_k = t_4$. From (2.13) it follows that

$$(2.19) \quad \xi(t_4) - \xi(t_3) = [x(t_4) - x(t_3)](t_4).$$

By hypothesis,

$$x(t_4) - x(t_3) = \int_{t_3}^{t_4} \mathfrak{D}F(x(\xi), s). \tag{2.20}$$

From (2.18), (2.19), (2.20), and Lemma 2.2 it follows that

$$\begin{aligned} | \xi(t_4) - \xi(t_3) - \textstyle\int_{t_3}^{t_4} f(\xi_\sigma, \sigma)\, d\sigma | &\leq | \{\textstyle\sum_{i=1}^{k} \int_{\sigma_{i-1}}^{\sigma_i} \mathfrak{D}F(x(\xi), s) \\ &\quad - \textstyle\sum_{i=1}^{k} F[x(t_4), \sigma_i] - F[x(t_4), \sigma_{i-1}]\}(t_4) | \\ &\leq \sum_{i=1}^{k} \frac{\sigma_i - \sigma_{i-1}}{2} \Psi(\sigma_i - \sigma_{i-1}) \leq \max_{i=1,\cdots,k} \Psi(\sigma_i - \sigma_{i-1}) \frac{t_4 - t_3}{2}, \end{aligned}$$

which implies (2.17) and proves the theorem.

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