# GENERALIZED ORDINARY DIFFERENTIAL EQUATIONS IN BANACH SPACE AND APPLICATIONS TO FUNCTIONAL EQUATIONS

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## Introduction

In order to generalize in a certain direction the theorem of continuous dependence of solutions of ordinary differential equations with respect to parameters, J. Kurzweil introduced the notion of a generalized Perron integral for functions with values in euclidean spaces [1] and for functions with values in Banach spaces [2]. In the present work we prove in a different way some of Kurzweil's results concerning the properties of the integral, especially in connection with sufficient conditions for existence of the integral. After that—and this is our main objective—we prove a theorem on continuous dependence of solutions of ordinary differential equations in a Banach space and relate it to functional differential equations, generalizing results such as those in [3]. Of course, ordinary differential equations are to be understood in a generalized sense, based upon the concept of the above mentioned integral.

## 1. Definition of the integral

For a given pair of real numbers $t_1 < t_2$, let us say that a set $M \subset [t_1, t_2] \times [t_1, t_2]$ belongs to a family $\mathfrak{M}(t_1, t_2)$ if there exists a real positive function $\delta(t)$ defined on $[t_1, t_2]$ such that $t \in [t_1, t_2]$ and $s \in [t - \delta(t), t + \delta(t)] \cap [t_1, t_2]$ imply $(t, s) \in M$. Geometrically such sets are like neighborhoods of the diagonal in the square $[t_1, ,t_2] \times [t_1, t_2]$. Let $B$ designate a partition of the interval $[t_1, ,t_2]$ of the form $t_1 = \sigma_0 \leq \tau_1 \leq \sigma_1 \leq \tau_2 \leq \cdots \leq \tau_k \leq \sigma_k = t_2$, where $\sigma_i < \sigma_{i+1}$ for $i = 0, 1, \cdots, k - 1$. A partition $B$ is said to be subordinate to a set $M \in \mathfrak{M}(t_1, t_2)$ if $(\tau_i, s) \in M$ for $\sigma_{i-1} \leq s \leq \sigma_i$, $i = 1, \cdots, k$. For a given set $M \in \mathfrak{M}(t_1, t_2)$ we denote by $\mathfrak{B}(M)$ the class of all partitions of $[t_1, t_2]$ of the above type and subordinate to $M$.

It is immediate from the definitions that if $M_1, M_2 \in \mathfrak{M}(t_1, t_2)$ and $M_1 \subset M_2$ then $\mathfrak{B}(M_1) \subset \mathfrak{B}(M_2)$. It is also easy to prove [1] that for every $M \in \mathfrak{M}(t_1, t_2)$ the class $\mathfrak{B}(M)$ is not empty.

Let $U(t, s)$ be a function with domain in some set $M \in \mathfrak{M}(t_1, t_2)$ and values in a real Banach space $X$ with norm $\| \ \|$. To every subdivision $B \in \mathfrak{B}(M)$ we make correspond to the function $U(t, s)$ the element $L(B) \in X$ defined as

$$L(B) = \sum_{j=1}^{n} [U(\tau_j, \sigma_j) - U(\tau_j, \sigma_{j-1})]$$

using the subdivision $B$.

Definition 1.1 *The function $U: M \to X$ belongs to the class $\kappa(M)$ if with every $\epsilon > 0$ we can associate a set $M_\epsilon \in \mathfrak{M}(t_1, t_2)$ such that $M_\epsilon \subset M$ and $\| L(B_1) - L(B_2) \| < \epsilon$ whenever $B_i \in \mathfrak{B}(M_\epsilon)$, $i = 1, 2$.*

If the function $U \in \kappa(M)$, for some set $M \in \mathfrak{M}(t_1, t_2)$, we shall say that $U(t, s)$ is $\kappa$-integrable in $[t_1, t_2]$ or, simply, $U \in \kappa$. The definition seems to be dependent on the set $M$ considered; actually this dependence is only apparent, and the integrability depends on the local behaviour of $U(t, s)$ in the diagonal of the square $[t_1, t_2] \times [t_1, t_2]$.

Since the space $X$ is complete, it is not difficult to see that the following lemma is true.

LEMMA 1.1 *$U \in \kappa$ if and only if there exists a unique element $\int_{t_1}^{t_2} \mathfrak{D}U \in X$ such that with every $\epsilon > 0$ we can associate a set $M_\epsilon \in \mathfrak{M}(t_1, t_2)$ such that if the subdivision $B \in \mathcal{B}(M_\epsilon)$, then $\| \int_{t_1}^{t_2} \mathfrak{D}U - L(B) \| < \epsilon$.*

This element $\int_{t_1}^{t_2}\mathfrak{D}U \in X$ will be called the Kurzweil integral of the function $U(t, s)$ on $[t_1, t_2]$. If $t_1 = t_2$, define $\int_{t_1}^{t_1} \mathfrak{D}U = 0$; and if $t_1 > t_2$, define $\int_{t_1}^{t_2} \mathfrak{D}U = -\int_{t_2}^{t_1} \mathfrak{D}U$ if the last one is defined.

If one denotes by $\mathfrak{M}_\alpha(t_1, t_2)$ a subclass of $\mathfrak{M}(t_1, t_2)$ corresponding to a special choice $\delta(t) \equiv \alpha > 0$ and if, in Definition 1.1, we restrict to $\mathfrak{M}_\alpha(t_1, t_2)$, a special class $\kappa_\alpha \subset \kappa$ is obtained. This class can in some way be considered as the class of functions integrable in the generalized sense of Riemann. In fact, if one takes $U(t, s) = f(t)s$, the corresponding elements $L(B)$ represent the ordinary Riemann sums; and $\int_{t_1}^{t_2} \mathfrak{D}f(t)s$ exists if and only if $\int_{t_1}^{t_2} f(t)\, dt$ exists in the Riemann sense, and in such case they coincide.

The following properties of the Kurzweil integral have been proved in [1] and [2].

(1) If $U \in \kappa$ on $[t_1, t_2]$, $\alpha \in R^1$, then

$$\int_{t_1}^{t_2} \mathfrak{D}\alpha U = \alpha \int_{t_1}^{t_2} \mathfrak{D}U.$$

(2) If $U_1, U_2 \in \kappa$, on the same interval $[t_1, t_2]$, then $U_1 + U_2 \in \kappa$ and

$$\int_{t_1}^{t_2} \mathfrak{D}(U_1 + U_2) = \int_{t_1}^{t_2} \mathfrak{D}U_1 + \int_{t_1}^{t_2} \mathfrak{D}U_2 .$$

(3) If $t_1 \leq t_3 \leq t_4 \leq t_2$ and $U \in \kappa$ on $[t_1, t_2]$, then $U \in \kappa$ on $[t_3, t_4]$.

(4) If $t_1 < t_3 < t_2$, $U \in \kappa$ on $[t_1, t_3]$ and also on $[t_3, t_4]$, then $U \in \kappa$ on $[t_1, t_2]$ and

$$\int_{t_1}^{t_2} \mathfrak{D}U = \int_{t_1}^{t_3} \mathfrak{D}U + \int_{t_3}^{t_4} \mathfrak{D}U.$$

(5) Let $U \in \kappa$ on every interval $[t_1, t]$ with $t_1 \leq t < t_2$, and suppose that

$$\lim_{t \to t_2} [\int_{t_1}^{t} \mathfrak{D}U - U(t_2, t) + U(t_2, t_2)] = L \in X,$$

then $U \in \kappa$ on $[t_1, t_2]$ and $\int_{t_1}^{t_2} \mathfrak{D}U = L$. Reciprocally, if $U \in \kappa$ on $[t_1, t_2]$ and $t_3 \in [t_1, t_2]$, then

$$\lim_{t \to t_3} [\int_{t_1}^{t} \mathfrak{D}U - U(t_3, t) + U(t_3, t_3)] = \int_{t_1}^{t_3} \mathfrak{D}U.$$

Property 5 implies that $\int_{t_1}^{t} \mathfrak{D}U$ depends continuously on $t$ at $t = t_3$ if and only if $U(t_3, t)$ depends continuously on $t$ at $t = t_3$.

Later on, we shall need the following theorem.

THEOREM 1.1 *If $U(t, s) \in X$ is defined on a set $M \in \mathfrak{M}_\alpha(t_1, t_2)$ and $\partial U/\partial s = u(t, s)$ exists and is continuous on $M$, then $U \in \kappa$ on $[t_1, t_2]$ and*

$$\int_{t_1}^{t_2} \mathfrak{D}U = \int_{t_1}^{t_2} u(t, t)\, dt,$$

*the last one in the Riemann sense. Moreover, if $M = [t_1, t_2] \times [t_1, t_2]$, then*

$$\begin{aligned} &\lim_{i\to\infty} \textstyle\sum_{j=1}^{2^i} U(\sigma_{j-1}, \sigma_j) - U(\sigma_{j-1}, \sigma_{j-1}) \\ (1.1) \qquad &= \lim_{i\to\infty} \textstyle\sum_{j=1}^{2^i} U(\sigma_j, \sigma_j) - (U\sigma_j, \sigma_{j-1}) \\ &= \textstyle\int_{t_1}^{t_2} u(t, t)\, dt, \end{aligned}$$

*where $\sigma_j = t_1 + j/2^i(t_2 - t_1), j = 0, 1, \cdots, 2^i$.*

*Proof*. From the hypothesis we can assert that, given $\epsilon > 0$, there exists a set $M_\epsilon \in \mathfrak{M}_\alpha(t_1, t_2)$ such that

(a) $\| (t - t_4)^{-1}[U(t_3, t) - U(t_3, t_4)] - u(t_3, t_4) \| < \epsilon/[3(t_2 - t_1)]$,
for $(t_3, t), (t_3, t_4) \in M_\epsilon, t \neq t_4$;

(b) $\| u(t, t_3) - u(t_3, t_3) \| < \epsilon/[3(t_2 - t_2)]$,
for $(t, t_3) \in M_\epsilon$;

(c) $\| \sum_{j=1}^{k} u(\sigma_{j-1}, \sigma_{j-1})(\sigma_j - \sigma_{j-1}) - \int_{t_1}^{t_2} u(t, t)\, dt \| < \epsilon/3$,
for every partition $B \in \mathfrak{B}(M_\epsilon)$.

Take then any partition $B = (\sigma_0, \tau_1, \sigma_1, \cdots, \tau_k, \sigma_k) \in \mathfrak{B}(M_\epsilon)$. We have now

$$\begin{aligned} &\| \textstyle\sum_{j=1}^{k} [U(\tau_j, \sigma_j( - U(\tau_j, \sigma_{j-1})] - \int_{t_1}^{t_2} u(t, t)\, dt \| \\ &\leq \textstyle\sum_{j=1}^{k} \| U(\tau_j, \sigma_j) - U(\tau_j, \sigma_{j-1}) - u(\tau_j, \sigma_{j-1})(\sigma_j - \sigma_{j-1}) \| \\ &+ \textstyle\sum_{j=1}^{k} \| [u(\tau_j, \sigma_{j-1}) - u(\sigma_{j-1}, \sigma_{j-1})](\sigma_j - \sigma_{j-1}) \| \\ &+ \| \textstyle\sum_{j=1}^{k} u(\sigma_{j-1}, \sigma_{j-1})(\sigma_j - \sigma_{j-1}) - \int_{t_1}^{t_2} u(t, t)\, dt \| < \epsilon, \end{aligned}$$

which proves the first part of the theorem. In order to prove relations (1.1), it is sufficient to take subdivisions $B$ such that $\tau_j = \sigma_{j-1}$ or $\tau_j = \sigma_j$ and use the last inequality.

COROLLARY. *If $U_1(t, s)$ and $U_2(t, s)$ are defined on some $M \in \mathfrak{M}_\alpha(t_1, t_2)$, $\partial U_1/\partial s$ and $\partial U_2/\partial s$ both exist and are continuous on $M$, and $\partial U_1/\partial s = \partial U_2/\partial s$ on $s = t$, then $U_1, U_2 \in \kappa$ and*

$$\int_{t_1}^{t_2} \mathfrak{D}U_1 = \int_{t_1}^{t_2} \mathfrak{D}U_2 .$$

## 2. Sufficient conditions for the existence of the Kurzweil integral

The conditions under which Theorem 1.1 asserts the existence of the generalized integral $\int_{t_1}^{t_2} \mathfrak{D}U$ are certainly rather strong and of no interest as far as generalization is concerned. In this section we shall study existence of the integral under conditions of continuity of $U(t, s)$ with certain restrictions in the moduli of continuity.

Let $\alpha > 0$ be fixed, and let $\psi(\theta)$ be a non-negative continuous function defined on $[0, \alpha]$, $\psi(0) = 0$ and with the series

$$\sum_{j=1}^{\infty} 2^j \psi\left(\frac{\theta}{2^j}\right) \tag{2.1}$$

being uniformly convergent on $[0, \alpha]$. Let us set

$$\Psi(\theta) = \sum_{j=1}^{\infty} \frac{2^j}{\theta} \psi\left(\frac{\theta}{2^j}\right),$$

for $0 < \theta \leq \alpha$ and $\Psi(0) = 0$. Then $\Psi(\theta) \to 0$ as $\theta \to 0^+$. In fact, consider $\theta \in [\alpha/2^{k+1}, \alpha/2^k]$ for $k = 0, 1, 2, \cdots$, and write $\xi = 2^k\theta$. Obviously, $\Psi(\theta) = \sum_{j=k+1}^{\infty} (2^j/\xi)\psi(\xi/2^j) \leq \sum_{j=k+1}^{\infty} (2^{j+1}/\alpha)\psi(\xi/2^j)$, as $\xi \in [\alpha/2, \alpha]$. From the uniform convergence of (2.1) it follows the existence of a $k_0$ such that $k > k_0$ implies $\Psi(\theta) < \epsilon$ for $\theta \in [\alpha/2^{k+1}, \alpha/2^k]$, and this proves the desired result.

Related to functions like the ones above, we will later on need some other functions which we introduce here. By $\omega_i$ or $\bar{\omega}_i$, $i = 1, 2, 3$, we shall denote real positive functions, continuous and increasing in some interval $0 \leq \theta \leq \theta_0$, that vanish at $\theta = 0$ and are bounded below by linear functions. These functions will be such that if we take

$$\psi(\theta) = \omega_3[2\omega_1(\theta)]\omega_2(\theta)$$

and define $\Psi(\theta)$ as before in terms of $\psi(\theta)$, these functions satisfy all conditions mentioned there. By the symbol $\mathfrak{F}(\Omega, \omega_1, \omega_2, \omega_3, \sigma)$, we mean all functions $F(x, t)$, defined and continuous in some region $\Omega \subset X \times R$ and values in $X$, and such that

$$\| F(x, t_1) - F(x, t_2) \| \leq \omega_1(|t_1 - t_2|),$$

and

$$\| F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1) \| \leq \omega_2(|t_1 - t_2|)\omega_3(\| x_1 - x_2 \|)$$

for all admissible values of the arguments and $|t_2 - t_1| \leq \theta_0$.

We can now state an existence theorem for the generalized integral.

THEOREM 2.1 *Let $U(t, s)$ with values in a Banach space $X$ be defined and continuous on a set $Q = [t_1, t_2] \times [t_1, t_2]$, and suppose that*

$$\| U(t + \theta, s + \theta) - U(t + \theta, s) - U(t, s + \theta) + U(t, s) \| \leq \psi(\theta),$$

*for $0 \leq \theta \leq \alpha$ and all arguments in $Q$. Then the Kurzweil integral $\int_{t_1}^{t_2} \mathfrak{D}U(t, s)$ exists and the following estimates hold:*

$$\| \textstyle\int_{\lambda_1}^{\lambda_2} \int \mathfrak{D}U - U(\lambda_1, \lambda_2) + U(\lambda_1, \lambda_1) \| \leq \frac{1}{2}(\lambda_2 - \lambda_1)\Psi(\lambda_2 - \lambda_1), \tag{2.2}$$

*and*

$$\| \textstyle\int_{\lambda_1}^{\lambda_2} \mathfrak{D}U - U(\lambda_2, \lambda_2) + U(\lambda_2, \lambda_1) \| \leq \frac{1}{2}(\lambda_2 - \lambda_1)\Psi(\lambda_2 - \lambda_1), \tag{2.3}$$

*for $t_1 \leq \lambda_1 \leq \lambda_2 \leq t_2$.*

Theorem 2.1 will be proved by means of the following approximation lemma.

LEMMA 2.1 *Under the hypothesis of Theorem* 2.1, *there exist functions* $U_k : Q \to X$, $k = 1, 2, 3, \cdots$, *that converge uniformly in* $Q$ *to* $U(t, s)$, *such that* $\partial U_k/\partial s$ *exist and are continuous in* $Q$, *and fulfilling the following condition*: *given* $\epsilon > 0$ *there exists a* $K(\epsilon)$ *such that*

$$\| U_k(t + \theta, s + \theta) - U_k(t + \theta, s) - U_k(t, s + \theta) + U(t, s) \| \leq \psi(\theta)$$

*if* $k > K(\epsilon)$ *and* $t + \theta, t, s + \theta, s \in [t_1 + \epsilon, t_2 - \epsilon]$.

*Proof.* Let $\{\rho_k\}$ be a sequence of positive numbers tending to zero with $k \to \infty$. Let $f_k(t, s) \geq 0$, for $k = 1, 2, 3, \cdots$, be real functions with continuous partial derivatives in $R^2$ and support in the $\rho_k$ neighbourhood of the origin and such that $\int_{R^2} f_k(t, s)\, dt\, ds = 1$. Such functions are easily constructed. Let $W(t, s)$ be a continuous extension of $U(t, s)$ to all $R^2$. For every $k$ and $(t, s) \in Q$, let

$$U_k(t, s) = \int_{R^2} f_k(t - \tau, s - \sigma) W(\tau, \sigma)\, d\tau\, d\sigma.$$

Then one has

$$\| U_k(t, s) - U(t, s) \| = \| \int_{R^2} f_k(t - \tau, s - \sigma)[W(\tau, \sigma) - W(t, s)]\, d\tau\, d\sigma \|$$
$$\leq \sup_{(t-\tau)^2 + (s-\sigma)^2 \leqq \rho_k} \| W(\tau, \sigma) - W(t, s) \|,$$

which tends to zero with $k \to \infty$ uniformly with respect to $(t, s) \in Q$ in view of the continuity of $W$. It is an easy matter to check that

$$\frac{\partial U_k(t, s)}{\partial s} = \int_{R^2} \frac{\partial f_k(t - \tau, s - \sigma)}{\partial s} W(\tau, \sigma)\, d\tau\, d\sigma.$$

Finally, let $\epsilon > 0$; by the definition of $U_k(t, s)$ we have

$$\| U_k(t + \theta, s + \theta) - U_k(t + \theta, s) - U_k(t, s + \theta) + U_k(t, s) \|$$
$$= \| \int_{R^2} f_k(t - \tau, s - \sigma)[W(\tau + \theta, \sigma + \theta) - W(\tau + \theta, \sigma)$$
$$- W(\tau, \sigma + \theta) + W(\tau, \sigma)]\, d\tau\, d\sigma \| \leq \int_{R^2} f_k(t - \tau, s - \sigma)\psi(\theta)\, d\tau\, d\sigma,$$

for $t + \theta, t, s + \theta, s \in [t_1 + \epsilon, t_2 - \epsilon]$ and $k$ large enough so that $\rho_k \leq \epsilon$, and the lemma is proved.

*Proof of Theorem* 2.1 Let $U_k(t, s), k = 1, 2, \cdots$, be as in Lemma 2.1. We know from Theorem 1.1 that, for any $\lambda_1, \lambda_2, t_1 \leq \lambda_1 \leq \lambda_2 \leq t_2$, the integrals $\int_{\lambda_1}^{\lambda_2} \mathfrak{D}U_k(t, s)$ exist. We also know from the same theorem that if we put $L_i(U_k, \lambda_1, \lambda_2) = \sum_{j=1}^{2^i} [U_k(\sigma_{j-1}, \sigma_j) - U_k(\sigma_{j-1}, \sigma_{j-1})]$ for a convenient partition of $[\lambda_1, \lambda_2]$, then

$$\int_{\lambda_1}^{\lambda_2} \mathfrak{D}U_k(t, s) = \lim_{i \to \infty} L_i(U_k, \lambda_1, \lambda_2).$$

It is not difficult to see that

$$L_{i+1}(U_k, \lambda_1, \lambda_2) - L_i(U_k, \lambda_1, \lambda_2)$$
$$= \sum_{j=0}^{2^i - 1} [U_k(\mu_j, \mu_j + \theta) - U_k(\mu_j, \mu_j) - U_k(\mu_j - \theta, \mu_j + \theta) + U_k(\mu_j - \theta, \mu_j)],$$

where $\mu_j = \lambda_1 + (\lambda_2 - \lambda_1) j/2^i + (\lambda_2 - \lambda_1)/2^{i+1}$, $\theta = (\lambda_2 - \lambda_1)/2^{i+1}$. If $t_1 < \lambda_1 \leq \lambda_2 < t_2$, then, using Lemma 2.1, we know that for $k > K(\epsilon)$, where $\epsilon = \max\{\lambda_1 - t_1, t_2 - \lambda_2\}$, we have

$$\| L_{i+1}(U_k, \lambda_1, \lambda_2) - L_i(U_k, \lambda_1, \lambda_2) \| \leq 2^i \psi\left(\frac{\lambda_2 - \lambda_1}{2^{i+1}}\right)$$

and, hence,

$$\begin{aligned} \| \textstyle\int_{\lambda_1}^{\lambda_2} \mathfrak{D} U_k(t, s) - U_k(\lambda_1, \lambda_2) + U_k(\lambda_1, \lambda_1) \| &= \| \lim_{i\to\infty} L_i - L_0 \| \\ \leq \sum_{i=0}^{\infty} \| L_{i+1} - L_i \| \leq \sum_{i=1}^{\infty} 2^i \psi\left(\frac{\lambda_2 - \lambda_1}{2^{i+1}}\right) &= \frac{1}{2}(\lambda_2 - \lambda_1)\Psi(\lambda_2 - \lambda_1). \end{aligned} \tag{2.4}$$

In a similar way one gets

$$\| \textstyle\int_{\lambda_1}^{\lambda_2} \mathfrak{D} U_k(t, s) - U_k(\lambda_2, \lambda_2) + U_k(\lambda_2, \lambda_1) \| \leq \frac{1}{2}(\lambda_2 - \lambda_1)\Psi(\lambda_2 - \lambda_1). \tag{2.5}$$

Both estimates (2.4) and (2.5) are valid for any $\lambda_1, \lambda_2$ with $t_1 < \lambda_1 \leq \lambda_2 < t_2$, as long as $k$ is big enough depending on $\lambda_1, \lambda_2$.

Let $0 < \delta \leq \alpha$, and let $B = (\sigma_0, \tau_1, \sigma_1, \cdots, \tau_k \sigma_p)$ be a partition of $[\lambda_1, \lambda_2]$ with

$$\sigma_0 = \lambda_1 < \sigma_1 < \cdots < \sigma_p = \lambda_2, \quad \tau_i \leq \sigma_i, \quad \tau_i - \sigma_{i-1} < \delta, \quad \sigma_i - \tau_i < \delta. \tag{2.6}$$

Putting $L(U, B) = \sum_{i=1}^{p} [U(\tau_i, \sigma_i) - U(\tau_i, \sigma_{i-1})]$, and applying (2.4) and (2.5) to the subintervals $[\sigma_{i-1}, \tau_i]$, $[\tau_i, \sigma_i]$, one has

$$\begin{aligned} &\| \textstyle\int_{\lambda_1}^{\lambda_2} \mathfrak{D} U_k(t, s) - L(U_k, B) \| \\ &= \| \textstyle\sum_{i=1}^{p} [\int_{\sigma_{i-1}}^{\tau_i} \mathfrak{D} U_k - U_k(\tau_i, \tau_i) + U_k(\tau_i, \sigma_{i-1}) \\ &\qquad + \textstyle\int_{\tau_i}^{\sigma_i} \mathfrak{D} U_k - U_k(\tau_i, \sigma_i) + U_k(\tau_i, \tau_i)] \| \\ &\leq \textstyle\sum_{i=1}^{p} \frac{1}{2}[(\tau_i - \sigma_{i-1})\Psi(\tau_i - \sigma_{i-1}) + (\sigma_i - \tau_i)\Psi(\sigma_i - \tau_i)] \\ &\leq \frac{\lambda_2 - \lambda_1}{2} \sup_{0<t\leq\delta} \Psi(t). \end{aligned}$$

If $B_1$, $B_2$ are two partitions of $[\lambda_1, \lambda_2]$ fulfilling (2.6), then

$$\| L(U_k, B_2) - L(U_k, B_1) \| \leq (\lambda_2 - \lambda_1) \sup_{0<t\leq\delta} \Psi(t). \tag{2.7}$$

By Lemma 2.1, $U_k(t, s) \to U(t, s)$ uniformly on $Q$, and consequently

$$L(U_k, B_1) \to L(U, B_1), \qquad L(U_k, B_2) \to L(U, B_2)$$

and

$$\| L(U, B_2) - L(U, B_1) \| \leq (\lambda_2 - \lambda_1) \sup_{0<t\leq\delta} \Psi(t). \tag{2.8}$$

Since $\Psi(\theta) \to 0$ as $\theta \to 0$, it follows that $U(t, s)$ is integrable on $[\lambda_1, \lambda_2]$ (see Lemma 1.1).

From (2.7) and (2.8) we have

$$\| \textstyle\int_{\lambda_1}^{\lambda_2} \mathfrak{D} U_k(t, s) - L(U_k, B_1) \| \leq (\lambda_2 - \lambda_1) \sup_{0<t\leq\delta} \Psi(t)$$

and also

$$\| \int_{\lambda_1}^{\lambda_2} \mathfrak{D}U(t, s) - L(U, B_1) \| \leq (\lambda_2 - \lambda_1) \sup_{0<t\leq\delta} \Psi(t).$$

Since $\delta$ is arbitrarily small and $L(U_k, B_1) \to L(U, B_1)$ with $k \to \infty$, one gets

$$\int_{\lambda_1}^{\lambda_2} \mathfrak{D}U_k(t, s) \to \int_{\lambda_1}^{\lambda_2} \mathfrak{D}U(t, s).$$

Passing to the limit as $k \to \infty$ in (2.4) and (2.5), one obtains relations (2.2) and (2.3), for $t_1 < \lambda_1 \leq \lambda_2 < t_2$; and, using property 5 of §1, one gets (2.2) and (2.3) for $t_1 \leq \lambda_1 \leq \lambda_2 \leq t_2$, and the theorem is proved.

## 3. Generalized ordinary differential equations in Banach space

As before, $X$ is a real Banach space, and $\Omega$ is an open subset of $X \times R^2$ with the property that with every point $(x, t, t) \in \Omega$ we can associate a number $\delta > 0$ such that $(x, t, s) \in \Omega$ for $| t - s | < \delta$. Let $F(x, t, s)$ be a function defined on $\Omega$ and with range in $X$.

DEFINITION 3.1 *We say that a function $x(t)$, defined on $(t_1, t_2)$ and with range in $X$, is a solution of the generalized differential equation*

$$\frac{dx}{dt} = \mathfrak{D}F(x, t, s) \tag{3.1}$$

*if $(x(t), t, t) \in \Omega$, for all $t_1 < t < t_2$, and*

$$x(t_4) = x(t_3) + \int_{t_3}^{t_4} \mathfrak{D}F(x(t), t, s) \tag{3.2}$$

*for all $t_1 < t_3 < t_4 < t_2$.*

Let $\Omega^*$ be an open subset of $X \times R^1$ with the property that $(x, s), (x, t) \in \Omega^* \Leftrightarrow (x, t, s) \in \Omega$, and let $f(x, t)$ be a continuous function on $\Omega^*$ with range in $X$. Consider now the classical equation

$$\frac{dx}{dt} = f(x, t), \tag{3.3}$$

where $dx/dt$ means the derivative of $x$ at $t$. A continuously differentiable function $x(t)$, defined on $(t_1, t_2)$ and with range in $X$, is a solution of (3.3) if $(x(t), t) \in \Omega^*$ and if $dx(t)/dt = f(x(t), t)$ for $t_1 < t < t_2$. In order to relate equation (3.3) with a generalized equation, let us set up the following generalized equation:

$$\frac{dx}{dt} = \mathfrak{D}[\int_t^s f(x, \tau)\, d\tau], \tag{3.4}$$

where, of course, the integral sign is to be interpreted as a Riemann integration. We then have the following.

THEOREM 3.1 *A function $x(t)$ defined on $(t_1, t_2)$ and with range in $X$ is a solution of (3.3) on $(t_1, t_2)$ if and only if it is a solution of (3.4) on $(t_1, t_2)$.*

*Proof*. Suppose $x(t)$ is a solution of (3.3). Let $(t_3, t_4)$ be a bounded subinterval of $(t_1, t_2)$. Clearly, for every $\epsilon > 0$, there exists a $\delta > 0$ such that $| \tau_2 - \tau_1 | < \delta$

and that $\tau_1, \tau_2 \in (t_3, t_4)$ and $(x(\tau_2), \tau_1) \in \Omega^*$ imply

$$\| f(x(\tau_2), \tau_1) - f(x(\tau_1), \tau_1) \| < \frac{\epsilon}{t_4 - t_3}. \tag{3.5}$$

Then take $M_1 \in \mathfrak{M}(t_3, t_4)$ such that $(\tau_2, \tau_1) \in M_1$ implies $(\tau_2 - \tau_1) < \delta$, $(x(\tau_2), \tau_1) \in \Omega^*$, and let $\{\sigma_0, \alpha_1, \sigma_1, \cdots, \alpha_k, \sigma_k\} \in \mathcal{B}(M_1)$. Using (3.5) and the fact that $x(t)$ is a solution of (3.3), we can write

$$\begin{aligned} &\| x(t_4) - x(t_3) - \textstyle\sum_{j=1}^k \int_{\sigma_{j-1}}^{\sigma_j} f(x(\alpha_j), \sigma) \, d\sigma \| \\ &\leq \textstyle\sum_{j=1}^k \int_{\sigma_{j-1}}^{\sigma_j} \| f(x(\sigma), \sigma) - f(x(\alpha_j), \sigma \| \, d\sigma \\ &\leq \sum_{j=1}^k \frac{\epsilon(\sigma_j - \sigma_{j-1})}{t_4 - t_3} = \epsilon; \end{aligned}$$

and therefore $x(t)$ is a solution of (3.4).

If now $x(t)$ is a solution of (3.4), then, for every subinterval $(t_3, t_4) \subset (t_1, t_2)$, we have

$$x(t_4) - x(t_3) = \textstyle\int_{t_3}^{t_4} \mathfrak{D} \int_t^s f(x(t), \sigma) \, d\sigma;$$

on the other hand, using Theorem 1.1, we know that

$$\textstyle\int_{t_3}^{t_4} \mathfrak{D} \int_t^s f(x(t), \sigma) \, d\sigma = \int_{t_3}^{t_4} f(x(t), t) \, dt,$$

and this implies then that $x(t)$ is a solution of (3.3).

*Note.* Consider the generalized equations

$$\frac{dx}{dt} = \mathfrak{D}[f(x, t)s] \tag{3.6}$$

and

$$\frac{dx}{dt} = \mathfrak{D}[\textstyle\int_{t_0}^s f(x, \sigma) \, d\sigma], \tag{3.7}$$

where $t_0$ is a fixed point in $(t_1, t_2)$. The functions $\int_t^s f(x, \sigma) \, d\sigma$, $f(x, t)s$, and $\int_{t_0}^s f(x, \sigma) \, d\sigma$ that appear in the equations (3.4), (3.6) and (3.7) are defined on some $M \in \mathfrak{M}(t_1, t_2)$ and have continuous first derivatives, with respect to $s$, which are equal to $f(x(t), t)$ for $s = t$. Then, by the Corollary to Theorem 1.1, the corresponding integrals are equal, and therefore equations (3.3), (3.4), (3.6) and (3.7) are all equivalent.

## 4. Relation with functional differential equations of retarded type

For the sake of completeness we shall describe functional differential equations of retarded type; further references to this topic can be found in the bibliography of [3].

For any given non negative numbers $a$, $b$, and $r$, we denote by $X[r, a, b]$ and $X[r]$ respectively the spaces of continuous functions defined on $[a - r, a + b]$ and $[-r, 0]$ and with range in $R^n$. Both are real Banach spaces with the usual supremun topology. For every $y \in X[r, a, b]$ and $a \leq t \leq a + b$, denote by $y_t$ the element of $X[r]$ which, as a function of $\theta \in [-r, 0]$, is given $y(t + \theta)$. Now, if $f(\varphi, t)$ is a function with domain in some open subset of $X[r] \times R^1$ and range in $R^n$, and if $\dot{y}(t)$ denotes the right hand derivative of a function $y(t) \in R^n$ at the point $t$, then a functional differential equation of retarded type is a functional relation of the form

$$\dot{y}(t) = f(y_t, t). \tag{4.1}$$

If, given a point $(\varphi, a)$ in the domain of $f$, there exists a number $b > 0$ and a function $y \in X[r, a, b]$ such that $(y_t, t)$ is in the domain of $f$ for $a \leq t < a + b$ and

(a) $y_a = \varphi$ and

(b) $\dot{y}(t) = f(y_t, t), a \leq t < a + b$,

we say that $y$ is a solution of (4.1) on $(a, a + b)$ with initial condition $(\varphi, a)$.

The type of equations just described includes, among others, the so called difference-differential equations (see [3]).

The relation among functional equations such as (4.1) and generalized ordinary differential equations in Banach spaces can be found in [4].

Let $X_1$ be an open set contained in $X[r, a, b]$, and define

$$F(x, t)(\tau) = \begin{cases} \int_a^\tau f(x_s, s)\, ds; & a \leq \tau \leq t \leq a + b \\ \int_a^t f(x_s, s)\, ds; & a \leq t \leq \tau \leq a + b \\ 0; \quad a - r \leq \tau \leq a & \text{or} \quad a - r \leq t \leq a, \end{cases} \tag{4.2}$$

for all $x \in X_1$. Of course, in order that (4.2) is well defined, we suppose that for $x \in X_1$ we have $x_s$ in the domain of $f(\varphi, t)$ for all $a \leq s \leq a + b$ and that the resulting function is integrable. Thus, for each $(x, t)$, (4.2) defines an element $F(x, t) \in X[r, a, b]$.

Consider the generalized equation

$$\frac{dx}{dt} = \mathfrak{D}F(x, t), \tag{4.3}$$

where $F(x, t)$ is given by (4.2). And let us suppose that the function $f(\varphi, t)$ satisfies

$$\left| \int_{\tau_1}^{\tau_2} f(x_s, s)\, ds \right| \leq \omega_1(|\tau_2 - \tau_1|) \tag{4.4}$$

and

$$\left| \int_{\tau_1}^{\tau_2} [f(x_s^1, s) - f(x_s^2, s)]\, ds \right| \leq \omega_2(|\tau_2 - \tau_1|) \cdot \sup_{\tau_1 \leqq s \leqq \tau_2} \omega_3(\| (x^1 - x^2)_s \|), \tag{4.5}$$

for $a \leq \tau_1 \leq \tau_2 \leq a + b$, and $|\tau_2 - \tau_1| \leq \sigma$, for some positive $\sigma$ and $x, x^1, x^2 \in X_1$; see the beginning of §2 for the definitions of $\omega_1, \omega_2, \omega_3$.

Under these conditions we have the following theorems.

THEOREM 4.1 *Let $y(t)$ be a solution of* (4.1) *in the interval* $(a, a + b)$, *and define*

$$x(t)(\tau) = \begin{cases} y(\tau); & a - r \leq \tau \leq t \leq a + b \\ y(t); & a - r \leq t \leq \tau \leq a + b. \end{cases}$$

*Then $x(t) \in X[r, a, b]$ is a solution of* (4.3) *in* $(a, a + b)$ *and satisfies*

$$\| x(t_1) - x(t_2) \| \leq \omega_1( | t_1 - t_2 | )$$

*for all* $t_1, t_2 \in (a, a + b)$, $| t_1 - t_2 | \leq \sigma$.

THEOREM 4.2 *Let $x(t)$ be a solution of* (4.3) *in* $(a, a + b)$ *with*

$$\| x(t_1) - x(t_2) \| \leq \omega_1( | t_1 - t_2 | ), \qquad t_1, t_2 \in (a, a + b),$$

*and* $| t_1 - t_2 | \leq \sigma$; *then the function*

$$y(\tau) = \begin{cases} x(a)(\tau); & a - r \leq \tau \leq a \\ x(\tau)(\tau); & a \leq \tau \leq a + b \end{cases}$$

*is a solution of* (4.1) *in* $(a, a + b)$, *and* $| y(t_1) - y(t_2) | \leq \omega_1( | t_1 - t_2 | )$, $t_1, t_2 \in (a, a + b)$, *and* $| t_1 - t_2 | < \sigma$.

For the proofs of these theorems see [4].

## 5. Continuous dependence of solutions

We shall now consider the problem of continuous dependence of solutions for equations of the form

$$\frac{dy}{d\tau} = \mathfrak{D}F(y, t), \tag{5.1}$$

with respect to initial conditions and parameters.

Together with equation (5.1), consider equation

$$\frac{dx}{d\tau} = \mathfrak{D}G(x, t); \tag{5.2}$$

and let us suppose that

$$F \in \mathfrak{F}(\Omega, \omega_1, \omega_2, \omega_3, \sigma)$$

and

$$G \in \mathfrak{F}(\Omega, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \sigma),$$

with $\omega_3$ linear.

Define the function $\Psi^*(\eta)$ for $0 < \eta < \sigma$ to be

$$\Psi^*(\eta) = \sum_{j=1}^{\infty} \frac{2^j}{\eta} \left\{ 2\omega_1 \left(\frac{\eta}{2^j}\right) \omega_2 \left(\frac{\eta}{2^j}\right) + \bar{\omega}_3 \left[ 2\bar{\omega}_1 \left(\frac{\eta}{2^j}\right) \right] \bar{\omega}_2 \left(\frac{\eta}{2^j}\right) \right\} \tag{5.3}$$

(see the beginning of §2).

LEMMA 5.1 *Let $y(t)$ and $x(t)$ be regular solutions (solutions with modulus of continuity $\omega_1$ and $\bar{\omega}_1$ respectively) respectively of (5.1) and (5.2) in an interval $[t_0, t_1]$. Let $m$ be a natural number such that $\Delta = (t_1 - t_0)/m < \sigma$, and suppose that* $\sup\{\|F(x,t) - G(x,t)\|; (x,t) \in \Omega\} = \beta < \infty$; *then*

$$\|x(t_1) - y(t_1)\| \leq \|x(t_0) - y(t_0)\| (1 + \omega_2(\Delta))^m$$

$$(5.4) \qquad + \left[\frac{\Delta}{2}\Psi^*(\Delta) + 2\beta\right] \frac{(1 + \omega_2(\Delta))^m - 1}{\omega_2(\Delta)}.$$

*Proof*. Consider a subdivision $t_0 = \xi_0 < \xi_1 < \xi_2 < \cdots < \xi_n = t_1$ with $|\xi_i - \xi_{i-1}| = \Delta$. For any $t_0 \leq \xi \leq t_1$, we have

$$x(\xi) = x(t_0) + \textstyle\int_{t_0}^{\xi} \mathfrak{D}G[x(\tau), s]$$

and

$$y(\xi) = y(t_0) + \textstyle\int_{t_0}^{\xi} \mathfrak{D}F[y(\tau), s].$$

Let us first estimate (using Theorem 2.1):

$$\begin{aligned}\|\textstyle\int_{\xi_i}^{\xi_{i+1}} \mathfrak{D}\{G[x(\tau), s] - F[y(\tau), s]\}\| &\leq \|\textstyle\int_{\xi_i}^{\xi_{i+1}} \mathfrak{D}\{G[x(\tau), s] - F[y(\tau), s]\} \\ &- G[x(\xi_i), \xi_{i+1}] + G[x(\xi_i), \xi_i] + F[y(\xi_i), \xi_{i+1}] - F[y(\xi_i), \xi_i]\| \\ &+ \|F[y(\xi_i), \xi_i] - F[y(\xi_i), \xi_{i+1}] - F[x(\xi_i), \xi_i] + F[x(\xi_i), \xi_{i+1}]\| \\ &+ \|F[x(\xi_i), \xi_i] - F[x(\xi_i), \xi_{i+1}] + G[x(\xi_i), \xi_{i+1}] - G[x(\xi_i), \xi_i]\| \\ &\leq \frac{\Delta}{2}\Psi^*(\Delta) + \|x(\xi_i) - y(\xi_i)\| \omega_2(\Delta) + 2\beta.\end{aligned}$$

Therefore,

$$\begin{aligned}\|x(\xi_1) - y(\xi_1)\| &\leq \|x(t_0) - y(t_0)\| + \|\textstyle\int_{t_0}^{\xi_1} \mathfrak{D}G[x(\tau), s] - \mathfrak{D}F[y(\tau), s]\| \\ &\leq \|x(t_0) - y(t_0)\| (1 + \omega_2(\Delta)) + \frac{\Delta}{2}\Psi^*(\Delta) + 2\beta.\end{aligned}$$

In the same way, and using the former estimates,

$$\begin{aligned}\|x(\xi_2) - y(\xi_2)\| &\leq \|x(\xi_1) - y(\xi_1)\| + \|\textstyle\int_{\xi_1}^{\xi_2} \mathfrak{D}G[x(\tau), s] - \mathfrak{D}F[y(\tau), s]\| \\ &\leqq \|x(t_0) - y(t_0)\| (1 + \omega_2)\Delta)) + \frac{\Delta}{2}\Psi^*(\Delta) + 2\beta \\ &+ \frac{\Delta}{2}\Psi^*(\Delta) + \|x(\xi_1) - y(\xi_1)\| \omega_2(\Delta) + 2\beta \\ &\leq \|x(t_0) - y(t_0)\| (1 + \omega_2(\Delta))^2 + \left[\frac{\Delta}{2}\Psi^*(\Delta) + 2\beta\right][1 + (1 + \omega_2(\Delta))].\end{aligned}$$

Following this procedure, it is easy to see that

$$\begin{aligned}\|x(t_1) - y(t_1)\| = \|x(\xi_m) - y(\xi_m)\| &\leq \|x(t_0) - y(t_0)\| (1 + \omega_2(\Delta))^m \\ + \left[\frac{\Delta}{2}\Psi^*(\Delta) + 2\beta\right] [1 + (1 + \omega_2(\Delta)) &+ (1 + \omega_2(\Delta))^2 \\ &+ \cdots + (1 + \omega_2(\Delta))^{m-1}];\end{aligned}$$

and, upon summing the geometric progression that appears at the end, one obtains the desired conclusion (5.4).

It is clear from the way the proof has been carried out that estimate (5.4) is actually valid in all the interval $[t_0, t_1]$; that is, we can replace $t_1$ in the left hand side of (5.4) by any $\iota \in [t_0, t_1]$ and the same estimate holds.

Let us now consider equation (5.1) together with equations

$$\frac{dy_n}{d\tau} = \mathfrak{D}F_n(y_n, t), \quad n = 1, 2, 3, \cdots, \tag{5.5}$$

where the functions $F_n \in \mathfrak{F}(\Omega, \bar{\omega}_1, \bar{\omega}_2, \bar{\omega}_3, \sigma)$ (this in particular could be the same class as that of $F$) and where $F_n(x, t)$ converge uniformly to $F(x, t)$ in $\Omega$ for $n \to \infty$. Then the following theorem, which is a weak form of continuous dependence on parameters, follows directly from the lemma just proved.

THEOREM 5.1 *Let $y(t)$ be a regular solution of* (5.1) *in $[t_0, t_1]$, with $y(t_0) = y^0$; and suppose that, for a given sequence $y_n^0 \to y^0$ $(n = 1, 2, \cdots)$, equations* (5.5) *have regular solutions $y_n(t)$ in $[t_0, t_1]$ with $y_n(t_0) = y_n^0$, if the condition*

$$\eta \Psi^*(\eta) \frac{(1 + \omega_2(\eta))^{\eta^{-1}} - 1}{\omega_2(\eta)} \to 0,$$

*with $\eta \to 0^+$, is fullfilled; then $y_n(t) \to y(t)$ uniformly in $[t_0, t_1]$.*

The condition on $\Psi^*(\eta)$ that appears in the theorem is of course natural in terms of our lemma. Similar conditions have been imposed, for example, by Kurzweil [5].

Observe that Theorem 5.1, implies that the initial value problem (5.1) with $y(t_0) = y^0$ has at most one regular solution.

## 6. Application to functional equations

Let us consider functional differential equations

$$\dot{y}(t) = f(y_t, t) \tag{6.1}$$

and

$$\dot{y}^n(t) = f_n(y_t^n, t), \quad n = 1, 2, 3, \cdots, \tag{6.2}$$

where $f$ and $f^n$ satisfy conditions such as (4.4) and (4.5) in some convenient interval, with functions $\bar{\omega}_1$, $\bar{\omega}_2$, and $\bar{\omega}_3$ for all $f_n$ and functions $\omega_1$, $\omega_2$, $\omega_3$ ($\omega_3$ linear) for $f$. We suppose that the functions $f$ and $f_n$ have common domain in some open subset of $X[r] \times R^1$ and range in $R^n$ and that

$$\int_{\tau_1}^{\tau_2} f_n(y_s, s)\, ds \to \int_{\tau_1}^{\tau_2} f(y_s, s)\, ds \tag{6.3}$$

uniformly in all variables.

Let $\varphi, \varphi_n \in X[r]$ with $\varphi_n \to \varphi$; and suppose that equations (6.1) and (6.2) have solutions $y(t)$ and $y^n(t)$ with initial conditions $(a, \varphi)$ and $(a, \varphi_n)$ respectively in some common interval $(a, a + b)$. Let $\Psi^*(\eta)$ be defined as in (5.3).

As a consequence of Theorems 5.1, 4.1 and 4.2 we have the following theorem.

THEOREM 6.1 *Under the former conditions, if*

$$\eta\Psi^*(\eta)\,\frac{(1+\omega_2(\eta))^{\eta^{-1}}-1}{\omega_2(\eta)}\longrightarrow 0$$

*with* $\eta \longrightarrow 0^+$, *then the solutions* $y^n(t)$ *of equations* (6.2) *converge uniformly to the solution* $y(t)$ *of* (6.1) *in* $(a - r, a + b)$.

As a very simple illustration consider the following equations

$$\dot{y}^n(t) = y^n(t-1) + n^{1-\alpha}\sin nt, \qquad n = 1, 2, 3, \cdots, \tag{6.4}$$

where $0 < \alpha < 1$, and

$$\dot{y}(t) = y(t-1). \tag{6.5}$$

It is easy to see that solutions of equations (6.4) with initial condition $y^n(t) = 0$ in $-1 \leq t \leq 0$ converge uniformly to $y(t) \equiv 0$ in any finite interval. Certainly, this case is not covered by continuous dependence theorems like that in [3]. Nevertheless, conditions of Theorem 6.1 are satisfied if one takes $\omega_1(\eta) = \eta^\alpha$, $0 < \eta < \sigma$, and $\omega_2$, $\omega_3$ linear.

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