ON THE CLOSED GRAPH THEOREM

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It is our purpose here to give a short proof of a duality theorem between the Closed Graph Theorem and the Open Map Theorem.

Theorem. *Let $\mathfrak{A}$ and $\mathfrak{B}$ be collections of Hausdorff topological vector spaces such that if $A \in \mathfrak{A}$ and $B \in \mathfrak{B}$, $F \subset A$ is a closed subspace and $F' \subset B$ is a closed subspace, then $A/F \in \mathfrak{A}$ and $B/F' \in \mathfrak{B}$. Then the following propositions are equivalent*:

(*a*) *any closed map from any $A \in \mathfrak{A}$ into any $B \in \mathfrak{B}$ is continuous*;

(*b*) *any closed map from any $B \in \mathfrak{B}$ into any $A \in \mathfrak{A}$ is open.*

We recall that a map T from A into B is closed if $G(T) = \{(x, Tx), x \in D(T)\}$ is closed in $A \times B$, and it is open if it maps open sets in the relative topology into open sets in the relative topology. (We do not assume the operators to be everywhere defined.)

Lemma 1. (i) *If T from A into B is closed and injective, then T^{-1} is also closed*; (ii) *if T is closed, then $Ker(T)$ is closed.*

Proof of Lemma 1. (i) Let $h: A \times B \to B \times A$ be defined by $h(a, b) = (b, a)$. Then h is bicontinuous and $G(T^{-1}) = h(G(T))$.

(ii) Let $a \in \overline{\text{Ker}(T)}$, and let $\{a_\gamma\}$ be a net in $\text{Ker}(T)$ converging to a. Then $T(a_\gamma) \to 0$; so $(a, 0) \in \overline{G(T)} = G(T)$, and $T(a) = 0$.

Lemma 2. *Let X be a topological vector space, and let $F \subset X$ be a subspace. Let $H \supset F$ be a closed subspace and let $\pi: x \to X/F$ be the natural projection. Then $\pi(H)$ is closed in X/F.*

Proof of Lemma 2. Let us show that $X/F - \pi(H)$ is open in X/F. Now $\pi^{-1}(X/F - \pi(H)) = X - \pi^{-1}(\pi(H))$; but $\pi^{-1}(\pi(H)) = \bigcup_{y \in F} y + H = H$ is closed, so $X/F - \pi(H)$ is open.

Lemma 3. *Let T be an operator from X into Y, where X and Y are topological vector spaces. Consider*

$$X \xrightarrow{\pi} X/\text{Ker}(T) \xrightarrow{\tilde{T}} Y,$$

where $D(\tilde{T}) = \pi(D(T))$, $\tilde{T}(\pi x) = Tx$. Then T is closed if and only if $\tilde{T}$ is closed.

Proof of Lemma 3. Let $h: X \times Y \to X/\text{Ker}(T) \times Y$ be defined by $h(x, y) = (\pi x, y)$, $x \in D(T)$. Now, $\text{Ker}(h) = \text{Ker}(T) \times \{0\}$ and

$$G(\tilde{T}) = \{(\pi x, Tx), x \in D(T)\} = h(G(T)).$$

Since $\text{Ker}(h) \subseteq G(T)$ we see (Lemma 2) that $G(T)$ is closed if $G(T)$ is closed, and conversely.

Proof of the theorem. Let us assume (a) and let $T: B \to A$ be closed, $D(T) \subseteq B$. Let us consider $B \xrightarrow{\pi} B/\mathrm{Ker}(T) \xrightarrow{\tilde{T}} A$. By Lemma 3, $\tilde{T}$ is closed, and, by Lemma 1, we have $B/\mathrm{Ker}(T) \in \mathfrak{B}$ and $(\tilde{T})^{-1}$ is closed. Thus $(\tilde{T})^{-1}$ is continuous and $\tilde{T}$ is open. Since π is open, it follows that $T = \tilde{T} \cdot \pi$ is open.

Let us assume (b) and let $T: A \to B$ be closed, $D(T) \subseteq B$. Let us consider $A \xrightarrow{\pi} A/\mathrm{Ker}(T) \xrightarrow{\tilde{T}} B$. Again, by Lemma 3, $\tilde{T}$ is closed. Also $(\tilde{T})^{-1}$ is closed, so, by (b), $(\tilde{T})^{-1}$ maps relative open sets into relative open sets; i.e., $\tilde{T}$ is continuous, hence T is continuous.

This theorem holds if $\mathfrak{A}$ is the class of barreled spaces and $\mathfrak{B}$ is the class of fully complete spaces (see [1]).

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Reference

[1] A. P. Robertson, *Topological vector spaces*, Cambridge University Press (1964), 114–16.