NON-IMMERSION THEOREMS FOR COMPLEX AND QUATERNIONIC PROJECTIVE SPACES

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1. Introduction

Let M be a differentiable m-manifold, and let f be a differentiable map of M into euclidean $(m + k)$-space. We call f an immersion if its jacobian has rank m at every point of M. We write $M \subseteq R^{m+k}$ to denote the existence of an immersion. In treating the immersion problem it is only natural to resort to K-theory, the cohomology theory of real (complex) vector bundles. Using a refinement of the methods of [2] we prove the following theorems.

THEOREM 4. *$CP_n \nsubseteq R^{4n-2\alpha(n)}$ for n odd where $\alpha(n)$ is the number of 1's in the dyadic expansion of n, $(n > 3)$.*

THEOREM 5. *$HP_n \nsubseteq R^{8n-2\alpha(n)-2}$, where $\alpha(n)$ is the number of 1's in the dyadic expansion of n, $(n > 3)$.*

Theorem 5 is related to a conjecture in the theory of immersions. Let $\tau(RP_n)$ be the tangent bundle of the real projective n-space. I. M. James [3] has proven that if $n = 2^r - 1$, then $g \cdot \dim (-\tau RP_n) > n - q$, where

$$\begin{aligned} q &= 2r && \text{if } r \equiv 1, 2 \bmod 4 \\ q &= 2r + 1 && \text{if } r \equiv 0 \quad \bmod 4 \\ q &= 2r + 2 && \text{if } r \equiv 3 \quad \bmod 4. \end{aligned}$$

On the other hand consider the fibration

$$RP_3 \to RP_{4n+3} \to HP_n .$$

Since RP_3 is parallelizable we have the inequality

$$g \cdot \dim (-\tau HP_n) \geq g \cdot \dim (-\tau RP_{4n+3}).$$

B. J. Sanderson has conjectured that

$$g \cdot \dim (-\tau HP_n) = g \cdot \dim (-\tau RP_{4n+3}).$$

Theorem 5 implies that either Sanderson's conjecture is false or the result of James is not the best possible (consider for example the fibration $RP_3 \to RP_{127} \to HP_{31}$).

2. The Grothendieck ring

We define the Grothendieck rings $KO(X)$ and $KU(X)$ for a finite, connected CW-complex X as universal solutions for homomorphisms from the semi-group

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$\mathcal{E}(X)$ of isomorphism classes of real (complex) vector bundles into abelian groups. Thus we get a group $K(X)$ and a map $\theta: \mathcal{E}(X) \to K(X)$, such that for any homomorphism φ of $\mathcal{E}(X)$ into an abelian group A, there exists a unique homomorphism ψ, making the following diagram commutative:

$$\begin{array}{ccc} \mathcal{E}(X) & \xrightarrow{\ \theta\ } & K(X) \\ & {\scriptstyle\varphi}\searrow \quad \swarrow{\scriptstyle\psi} & \\ & A & \end{array} .$$

$K(X)$ is a ring with multiplication induced by tensor product of bundles. Operations in vector bundles provide us with operations in the ring $K(X)$. We will be concerned here only with exterior powers and "spinification," a non-stable operation. For elements of $\mathcal{E}(X)$, the exterior powers have the following formal properties: (a) $\lambda^0(x) = 1$, (b) $\lambda^1(x) = x$, (c) $\lambda^i(x + y) = \sum_{j=0}^{i} \lambda^j(x)\lambda^{i-j}(y)$, and (d) $\lambda^i(x) = 0$, for $i > \dim x$. These operations extend to the ring $K(X)$. Define $\lambda_t(x) = \sum_{i=0}^{\infty} \lambda^i(x)t^i$, where t is an indeterminate. Then (a), (b), (c), and (d) imply

$$\lambda_t(x + y) = \lambda_t(x)\lambda_t(y).$$

If V is a real vector space, then $\lambda^i V \otimes C \cong \lambda^i(V \otimes C)$. This gives us the commutative diagram

$$\begin{array}{ccc} KO(X) & \xrightarrow{\ \lambda^i\ } & KO(X) \\ \downarrow{\scriptstyle \epsilon_u} & & \downarrow{\scriptstyle \epsilon_u} \\ KU(X) & \xrightarrow{\ \lambda^i\ } & KU(X), \end{array}$$

where ϵ_u denotes complexification.

If a real vector bundle ξ is such that $w_1(\xi) = 0$ and $w_2(\xi) = 0$, then it admits a spin representation $\Delta(\xi)$. $\Delta(\xi)$ is then a complex, self-conjugate bundle, and representation theory (e.g. [6]) provides us with the following relation:

$$a_k\Delta(\xi)\cdot\Delta(\xi) = \lambda^0(\xi_u) + \lambda^1(\xi_u) + \cdots + \lambda^k(\xi_u) = \lambda_1(\xi_u),$$

where $\xi_u = \epsilon_u(\xi)$ is the complexification of ξ, k is the dimension of ξ, and $a_k = 1, 2$ for k even or odd respectively.

This relation immediately implies the following theorem.

THEOREM 1. *If $f: M^m \to R^{m+k}$ is an immersion, $\nu = (m + k) - \tau(M)$ its normal bundle, and $w_1(\nu) + w_2(\nu) = 0$, then there exists a self-conjugate element $x \in KU(M)$ such that $a_k x^2 = \lambda_1(\nu_u)$.*

3. Applications to CP_n and HP_n

We now apply Theorem 1 to the cases when M is the complex projective space CP_n and the quaternionic projective space HP_n.

Let H be the canonical Hopf bundle over CP_n and let $y = H - 1$; then $KU(XP_n) = Z[y]/y^{n-1}$ (e.g. [7]). The tangent bundle of CP_n satisfies the following complex bundle equation:

$$\tau(CP_n) \oplus 1 = (n+1)H.$$

If $CP_n \subseteq R^{2n+2k}$, then $\nu = 2n + 2k + 2 - (n+1)H$. If n is odd, then $w_2(\nu) = 0$ ($w_1(\nu) = 0$ because CP_n is simply connected). As in [2], we let $x = H + \bar{H} - 2$. x is then the generator of the subalgebra of self-conjugate bundles in $KU(CP_n)$ and $x^{[n/2]+1} = 0$. Since n is odd, we have $x^{(n+1)/2} = 0$; since H is a complex vector bundle, $\epsilon_u(H) = H + \bar{H}$. Thus we have

$$\lambda_t(\nu_u) = (1+t)^{2n+2k+2}(1+tH)^{-(n+1)}(1+t\bar{H})^{-(n+1)},$$

and, using the fact that $H\bar{H} = 1$, we get

$$\lambda_t\,(\nu_n) = (1+t)^{2k}\left(1 + \frac{t}{(1+t)^2}\,x\right)^{-(n+1)}.$$

Substituting $t = 1$, we have

$$\lambda_1(\nu_n) = 2^{2k}\left(1 + \frac{x}{4}\right)^{-(n+1)},$$

and Theorem 1 implies that $\Delta(\nu) = 2^k(1 + x/4)^{-(n+1)/2}$ is an element of $KU(CP_n)$. We check the coefficient of $x^{(n-1)/2}$ and get

$$\pm 2^k \binom{n-1}{\frac{n-1}{2}} 2^{1-n}.$$

This must be an integer; and, since the highest power of 2 dividing $\binom{a+b}{a}$ is $\alpha(a) + \alpha(b) - \alpha(a+b)$, we must have

$$k + 1 - n + \alpha\left(\frac{n-1}{2}\right) + \alpha\left(\frac{n-1}{2}\right) - \alpha(n-1) \geq 0.$$

Here of course $\alpha(i)$ is the number of 1's in the dyadic expansion of i. We have also $\alpha(n - 1/2) = \alpha(n-1) = \alpha(n) - 1$; thus the inequality becomes

$$k \geq n - \alpha(n).$$

Since this calculation would not change if we assumed $CP_n \subseteq R^{2n+2k+1}$ (because $a_{2k+1} = 2$), we have the following theorem.

THEOREM 2. *$CP_n \not\subseteq R^{4n-2\alpha(n)-1}$ for n odd.*

When M is a quaternionic projective space HP_n, the condition on the Stiefel-Whitney classes is always fulfilled. Let h_H denote the complex bundle associated to the canonical quaternion line bundle by the inclusion $Sp(1) \subset U(2)$. Let $z = h_H - 2$; then $KU(HP_n) = Z[z]/z^{n+1} = 0$ (e.g., [7]).

The tangent bundle of HP_n satisfies the following bundle equation (e.g. [9]):

$$\tau(HP_n) \oplus \eta \oplus 1 = (n + 1)h_H,$$

where η is the 3-dimensional real vector bundle associated to h_H by the double covering $Sp(1) \to SO(3)$.

If $HP_n \subseteq R^{4n+2k}$, then $\nu = 4n + 2k + 1 + \eta - (n + 1)h_H$. Since h_H is a self-conjugate complex vector bundle, $\epsilon_u(h_H) = 2h_H$. Thus we have

$$\lambda_t(\nu_u) = (1 + t)^{4n+2k+1}(1 + th_H + t^2)^{-2(n+1)}\lambda_t(\eta_u),$$

and $\lambda_t(\eta_u) = 1 + t\eta_u + t^2\eta_u + t^3$. A short character computation shows that $\eta_u + 1 = (h_H)^2$; therefore,

$$\lambda_t(\eta_u) = (1 + t)^{2k-2}\left(1 + \frac{t}{(1 + t)^2} z\right)^{-2(n+1)} ((1 - t)^2 + t(z + 2)^2).$$

Substituting $t = 1$, we get

$$\lambda_1(\nu_u) = 2^{2k}\left(1 + \frac{z}{4}\right)^{-2(n+1)}\left(1 + \frac{z}{2}\right)^2,$$

and again Theorem 1 implies that $\Delta(\nu) = 2^k[1 + (z/4)]^{-(n+1)}[1 + (z/2)]$, where $z^{n+1} = 0$, is an element of $KU(HP_n)$. Hence the expression for $\Delta(\nu)$ must yield a polynomial with integral coefficients. The coefficient of z^n is 0, so we check the coefficient of z^{n-1}. This can be seen to be

$$\pm 2^k \cdot 2^{-2(n-1)}\left[\binom{2n-1}{n-1} - 2\binom{2n-2}{n-2}\right] = \pm 2^{k-2(n-1)}\binom{2n-1}{n}\frac{1}{2n-1}.$$

The highest power of 2 dividing $\binom{2n-1}{n}$ is $\alpha(n) + \alpha(n - 1) - \alpha(2n - 1) = \alpha(n) + \alpha(n - 1) - \alpha(2(n - 1) + 1) = \alpha(n) - 1$. Thus, in order for $\Delta(\nu)$ to be an element of $KU(HP_n)$, we must have

$$k - 2(n - 1) + \alpha(n) - 1 \geq 0;$$

and so

$$k \geq 2n - \alpha(n) - 1.$$

Since the calculation would not change if we assumed $HP_n \subseteq R^{4n+2k+1}$, we have proved the following.

THEOREM 3: $HP_n \not\subseteq R^{8n-2\alpha(n)-3}$.

Using the results of [1] and a technical lemma, B. J. Sanderson and R. L. E. Schwarzenberger had previously derived the results of Theorem 2 and Theorem 3 in [8]. Somewhat better theorems were also obtained by Mayer in [5].

4. Main results

For an even-dimensional bundle ν with $w_1(\nu) + w_2(\nu) = 0$, $\Delta(\nu)$ splits into a sum $\Delta^+(\nu) + \Delta^-(\nu)$, and we have the following relation (see [6]):

$$\Delta^+(\nu)\cdot\Delta^-(\nu) = \lambda^{k-1}(\nu_u) + \lambda^{k-3}(\nu_u) + \cdots.$$

The sum ends with $\lambda^0(\nu_u)$, if k is odd, and with $\lambda^1(\nu_u)$, if k is even, where $2k$ is the dimension of ν. Since

$$\lambda_1 = \lambda^0 + \lambda^1 + \cdots + \lambda^{2k}$$

and

$$\lambda_{-1} = \lambda^0 - \lambda^1 + \cdots + \lambda^{2k},$$

for a bundle of dimension $2k$, we get the following equalities:

$$\lambda_1 - \lambda_{-1} = 4\Delta^+\Delta^-, \quad \text{if } k \text{ is even;}$$

$$\lambda_1 + \lambda_{-1} = 4\Delta^+\Delta^-, \quad \text{if } k \text{ is odd.}$$

Since $(\Delta^+ + \Delta^-)^2 = \lambda_1$, we have $(\Delta^+ - \Delta^-)^2 = \lambda_{-1}$, if k is even, and

$$(\Delta^+ - \Delta^-)^2 = -\lambda_{-1},$$

if k is odd. If $CP_n \subseteq R^{2n+2k}$, we have

$$\lambda_t(\nu_n) = (1+t)^{2k}\left(1 + \frac{t}{(1+t)^2}x\right)^{-(n+1)};$$

since $x^{(n+1)/2} = 0$, we have, after expanding the second term, $(1+t)^{n-1}$ in the denominator; and, if $2k > n-1$, then $\lambda_{-1}(\nu_n) = 0$. By Theorem 2, $k \geq n - \alpha(n)$; therefore $\lambda_{-1}(\nu_n) = 0$ if $n > 3$. For $n > 3$, we have thus $\Delta^+(\nu_n) = \Delta^-(\nu_n)$, and

$$\lambda_1(\nu_n) = 4\Delta^+(\nu_n)\Delta^+(\nu_n),$$

and

$$\Delta^+(\nu) = 2^{k-1}\left(1 + \frac{x}{4}\right)^{-(n+1)/2}.$$

Thus $k \geq n - \alpha(n) + 1$. This yields the following theorem.

THEOREM 4. *$CP_n \not\subseteq R^{4n-2\alpha(n)}$ if $n > 3$ (n odd).*
Similarly if $HP_n \subseteq R^{4n+2k}$, we have

$$\lambda_1(\nu_n) = (1+t)^{2k-2}\left(1 + \frac{t}{(1+t)^2}z\right)^{-2(n+1)}((1-t)^2 + t(z+2)^2).$$

Since, $z^{n+1} = 0$, after expanding the second term we get $(1+t)^{2n}$ in the denominator, and, if $2k - 2 > 2n$, then $\lambda_{-1}(\nu_n) = 0$. By Theorem 3, $k \geq 2n - \alpha(n) - 1$, so $\lambda_{-1}(\nu_n) = 0$, if $n > 3$. And again for $n > 3$ we have $\Delta^+ = \Delta^-$ and $\lambda_1(\nu_n) = 4\Delta^+\Delta^+$. This gives $k \geq 2n - \alpha(n)$. We have thus proved the following theorem.

THEOREM 5: *For $n > 3$, $HP_n \not\subseteq R^{8n-2\alpha(n)-2}$.*

COROLLARY. If $n = 2^q, HP_n \not\subseteq R^{8n-4} (q > 1)$. For $n = 2^r + 1$ Theorem 4 implies that $CP_n \not\subseteq R^{4n-4}$ and since $CP_n \subset R^{4n-3}$ for n odd [4] the best embedding and immersion for such a dimension coincide.

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