"PARALLEL" TRANSPORT IN FIBRE SPACES

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1. Introduction

Of fundamental importance in differential geometry is the notion of parallel transport along curves (for technical reasons, the equivalent "connection" is often given greater emphasis). Roughly speaking, the idea is this: Given a curve λ from p to q, a tangent vector at p can be *transported* along λ to a tangent vector at q, called the *transport* of v along q. This transport is required to be rather well behaved. In particular, the induced map of the tangent space at p to that at q is to be a linear isomorphism. Moreover, this isomorphism should vary nicely as we vary the curve from p to q.

Now a structure of this sort need not be confined to tangent bundles. In [14], Chapter 7, J. H. C. Whitehead and O. Veblen suggested the study of corresponding operations when the tangent space is replaced by some general "associated" space. An appropriate setting for such a generalization is in the theory of fibre bundles. For vector bundles, the concept is essentially that of linear connection, while for general bundles, the corresponding concept was considered by E. H. Brown [1] under the name "lifting function". An even more appropriate setting is in the theory of fibre spaces, for the existence of such "transports" and their relation to the classification of bundles are intimately connected with the homotopy properties of the bundle.

The idea is to measure the "twist" in the fibre space by observing how a typical fibre is mapped into another as we move along a path in the base. For many purposes, it is sufficient to consider paths beginning and ending at a fixed point $* \in B$. Let ΩB denote the space of such paths with the usual topology. (The parameter is assumed to run over some finite interval $[0, r]$ and, for a path λ, the end point will be denoted by $\lambda(r)$.) Let F denote $p^{-1}(*)$. Until we are in a position to give a precise definition, we will use the word "transport" to mean a map $\theta: \Omega B \times F \to F$ such that for each $\lambda \in \Omega B$, the map $\theta(\lambda, \quad): F \to F$ is a homotopy equivalence. Since the most we can say about different fibres above a path is that they are of the same homotopy type, this is all we can ask of $\theta(\lambda, \quad)$. As we shall see (and as is well known) the fact that p is a fibre space gives rise to such a transport.

In the classical case, θ is not only continuous but suitably differentiable, and the homotopy equivalences are in fact linear isomorphisms. As one would expect, the existence of more restricted transports (in terms of differentiability, linearity, etc.) follows from the presence of appropriate structure on the fiber space in question.

A more significant fact in the classical case is that θ is transitive; i.e., displacement along successive paths, λ and then μ, is the same as the displacement along

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the total path $\mu + \lambda$. (N.B. We reverse the usual order in writing $\mu + \lambda$ so as to correspond to functional notation; in symbols, $\theta(\mu + \lambda, x) = \theta(\mu, \theta(\lambda, x))$.) The same is true in the case of (Steenrod) fibre bundles for the "lifting functions" constructed by Brown ([1], Th. (1.5)), but the proof makes significant use of the local product structure and the fact that the transition functions lie in a group; this proof will not carry over to general fibre spaces.

Although the existence of some sort of "transport" for a fibre space has been known for some time, there has been no complete analysis of its lack of transitivity; nor has its significance in determining the structure of the fibre space been fully realized. Consider the important, though usually unemphasized, role that "transports" play in the theory of bundles. In the simplest case, that of a covering space, the transport $\theta: \Omega B \times F \to F$ induces a homomorphism of $\pi_1(B)$ into the automorphism group of the fibre, and this homomorphism characterizes the covering. If B is a suspension SX and p is a bundle, then the bundle is characterized by a homotopy class of maps of X into the group of the bundle. This homotopy class can be interpreted as a class of homomorphisms of ΩSX into the group which are adjoint to transports. More generally, by looking at ΩB as a groupoid, Lashof [8] has obtained homomorphisms of ΩB into the group which characterize general bundles. Brown improves this result slightly in that he uses the usual addition of loops, although the relation to classification is never made explicit.

So much for bundles. This paper has as its main objective a description of the general situation for fibre spaces, emphasizing the homotopy structure underlying the classical concepts as well as the naturality and importance of "transport" in the classification problem. In particular, there is given a method of constructing fibre spaces using a more precisely defined "transport". In effect, this analyzes the construction of "associated fibrings" from a homotopy point of view and extends it to the case of non-transitive operations.

The more precise definition is motivated and introduced in §2 along with statements of the main results regarding its significance. Such transports are constructed in §3 and used to construct fibre spaces in §4. The classification of fibre spaces via transports is carried out in §5, and this classification is compared with other forms. This, of necessity, involves a study of "universal base spaces" and of associative H-spaces and maps. An alternate approach which brings out the naturality (in the technical sense) of the classification is discussed in §6. Finally, in §7, we return to G-bundles briefly to indicate their position as a special case in relation to fibre spaces. These last two sections are comparatively technical; although they cast additional light on the preceding sections, they are not essential to the basic understanding of "transport" at which we aim.

2. Definitions and main results

We adopt the definition of fibre space used by Dold (in [2], Def. 5.1; cf. Fuchs [5], and Weinzweig [15], Def. 5.1).

DEFINITION 2.1. *$p: E \to B$ is a* fibre space *if, for every map $f_0: X \to E$ and homotopy $h_t: X \to B$ $0 \leq t \leq 1$ such that $pf_0 = h_0$, there exists a homotopy*

$f_t: X \to E$ $0 \leq t \leq 1$ *such that* $pf_t = h_0$ $0 \leq t \leq \frac{1}{2}$, $pf_t = h_{2t-1}$ $\frac{1}{2} \leq t \leq 1$. (*It follows that any two such liftings are vertically homotopic.*)

The existence of a non-trivial transport for p arises in the following way. Consider the homotopy $h_t: \Omega B \times F \to B$ given by $h_t(\lambda, y) = \lambda(rt)$, where $\lambda: [0, r] \to B$. Let $f_t: \Omega B \times F \to E$ be a lifting with $f_0(\lambda, y) = y \in F$. A transport $\theta: \Omega B \times F \to F$ is given by f_1. That it is a transport—i.e., that $\theta(\lambda, \quad): F \to F$ is a homotopy equivalence—can be seen readily using the fact that the induced fibre space $\lambda^* p: \lambda^* E \to I$ is fibre homotopy equivalent to a product.

Remark. The restriction $\theta(\ , *): \Omega B \to F$ induces the usual boundary $\partial: \pi_n(B) \to \pi_{n-1}(F)$.

Since any two liftings are homotopic, each fibre space determines in this way a unique class of "transports" $\Omega B \times F \to F$. The extent to which the converse is true is an intriguing problem, intimately related to the fact that there is nothing to guarantee the transitivity of θ as constructed. The situation is not completely hopeless, however, for Hilton has shown [7] that the transport is "homotopy transitive" in the sense that

$$\begin{array}{ccc} \Omega B \times \Omega B \times F & \xrightarrow{1 \times \theta} & \Omega B \times F \\ \downarrow{\scriptstyle m \times 1} & & \downarrow{\scriptstyle \theta} \\ \Omega B \times F & \xrightarrow{\theta} & F \end{array}$$

is homotopy commutative, where m denotes multiplication (composition) of loops. (Remember that if $\lambda: [0, r] \to B$, and $\mu: [0, s] \to B$, then

$$m(\lambda, \mu) = \lambda + \mu: [0, s + r] \to B$$

is given by μ for $0 \leq t \leq s$ and by λ for $s \leq t \leq s + r$ so as to conform to functional notation for composition.) The full significance of the transport is brought out only in terms of higher orders of homotopy transitivity (involving more than two loops). To study this property, it is very helpful that for the space of loops, as we have defined it, m is strictly associative.

DEFINITION 2.2. *Let* (X, m) *be an associative H-space. An* A_n-action *of* X *on a space* F *is a collection of maps* $\{\theta_i: I^{i-1} \times X^i \times F \to F,\ i \leq n\}$ *such that*

$$\theta_i(t_1, \cdots, t_{i-1}, x_1, \cdots, x_i, y)$$

$$= \theta_{i-1}(\cdots, t_{j-1}, t_{j+1}, \cdots, x_j x_{j+1}, \cdots), \qquad \text{if } t_j = 1,$$

$$\theta_j(\cdots, t_{j-1}, x_1, \cdots, x_j, \theta_{i-j}(t_{j+1}, \cdots, t_{i-1}, x_{j+1}, \cdots, x_i, y)), \quad \text{if } t_j = 0.$$

If the maps exist and the conditions are satisfied for all i, *the collection is called an* A_∞-action. *This is the homotopy analogue of a transformation group (as the present paper will bring out more clearly).*

Notice that a transport, as defined so far, is an A_1-action and that Hilton's

proof of homotopy transitivity shows that it can be extended to an A_2-action. In fact things can be pushed even further.

THEOREM A. *Given a fibre space $p:E \to B$, there is an A_∞-action $\{\theta_i\}$ of ΩB on F such that $\theta_1(\lambda, \quad):F \to F$ is a homotopy equivalence, $\lambda \in \Omega B$.*

If θ_1 is transitive, the higher θ_i can be defined trivially so this is a true generalization of the classical situation. We therefore revise our terminology.

DEFINITION 2.3. *A* transport *is an A_∞-action $\{\theta_i\}$ of ΩB on F such that*

$$\theta_1(\lambda, \quad):F \to F$$

is a homotopy equivalence.

As stated, Theorem A is trivially satisfied by the A_∞-action which projects $I^{i-1} \times (\Omega B)^i \times F$ onto F, but we will prove it in such a way that the transport constructed is quite significant. Our reason for calling attention to the maps θ_i for $i > 1$ is precisely that this is the additional structure necessary to characterize fibre spaces.

DEFINITION 2.4. *A homotopy between A_∞-actions $\{\theta_i^0\}$ and $\{\theta_i^1\}$ is a continuous family of A_∞-actions $\{\theta_i^t\}$.*

DEFINITION 2.5. *Two fibre spaces $p:E \to B$ and $p':E' \to B$ are* fibre homotopy equivalent *if there exist fibre preserving maps $f:E \to E'$, $g:E' \to E$ and fibre preserving homotopies $h_t:E \to E$, $h_t':E' \to E'$ such that*

$$h_0 = gf, \quad h_0' = fg$$

$$h_1 = id_E, \quad h_1' = id_{E'}.$$

We assume henceforth that all spaces have the homotopy type of CW-complexes.

THEOREM B. *Given a transport $\{\theta_i\}$, there is a fibre space $p_\theta:E_\theta \to B$ such that, up to homotopy, $\{\theta_i\}$ can be recovered as in the proof of Theorem A. If $\{\theta_i\}$ was originally obtained, as in Theorem A, from a fibre space $p:E \to B$, then p_θ is fibre homotopy equivalent to p.*

This suggests how to classify fibre spaces.

THEOREM C. *The homotopy class of the transport determines the fibre space up to fibre homotopy equivalence. The fibre space determines the homotopy class of the transport up to the action of homotopy equivalences of F into itself.*

3. The construction of transports

The idea behind the construction of θ (in §2) is to deform the trivial action $(\lambda, y) \to y$ through maps $f_t:\Omega B \times F \to E$ which cover a rather obvious homotopy of ΩB in B. The idea of Hilton's proof of homotopy transitivity is much the same. Using the maps f_t, we see that $\theta(1 \times \theta)$ and $\theta(m \times 1)$ are homotopic in E via a

homotopy whose projection into B is deformable to the constant homotopy; hence the homotopy between $\theta(1 \times \theta)$ and $\theta(m \times 1)$ can be deformed into F. We wish to use the same approach to get the A_∞-action $\{\theta_i\}$.

Instead of homotopies $\theta_n{}^t : I^{n-1} \times (\Omega B)^n \times F \to E$, we find it more convenient to seek corresponding maps $\Sigma_n : I^n \times (\Omega B)^n \times F \to E$ such that

$$\Sigma_n(t_1, \cdots, t_n, x_1, \cdots, x_n, y)$$

is given by

$$\text{(a)} \quad \Sigma_{n-1}(\cdots, t_{i-1}, t_{i+1}, \cdots, x_{i-1}x_i, \cdots),$$

if $t_i = 1$, and by

$$\text{(b)} \quad \Sigma_{i-1}(t_1, \cdots, t_{i-1}, x_1, \cdots, x_{i-1}, \theta_{n-i+1}(t_{i+1}, \cdots, t_n, x_i, \cdots, x_n', y)),$$

if $t_i = 0$, with the convention that $\Sigma_0 : F \subset E$ and the meaningless "x_0x_1" is deleted if $t_1 = 1$. Notice that if θ_j has been defined for $j < i$, the conditions define Σ_i on all faces of I^i except $t_1 = 0$. To obtain θ_i, we seek to "fill in the box" in such a way that the fact $t_1 = 0$ will give values in F. To do this, we first construct the corresponding maps in B.

By induction, we define maps $\rho_n : I^n \times R^n \to I$. The coordinates $r_1, \cdots, r_n$ will correspond to the lengths of loops $\lambda_1, \cdots, \lambda_n$. We specify $\rho_1(t, r) = tr$ and then, on $\dot{I}^n \times R^n$, set $r = r_1 + \cdots + r_n$ and $\rho_n(t_1, \cdots, t_n, r_1, \cdots, r_n)$

$$= \begin{cases} \rho_{n-1}(\cdots t_{i-1}, t_{i+1}, \cdots, r_{i-1} + r_i, \cdots), & \text{if } t_i = 1 \quad i > 0; \\ \dfrac{r - r_1}{r}\, \rho_{n-1}(t_2, \cdots, t_n, r_2, \cdots, r_n), & \text{if } t_1 = 1; \\ \dfrac{r_1 + \cdots + r_{i-1}}{r} \cdot \rho_{i-1}(t_1, \cdots, t_{i-1}, r_1, \cdots, r_{i-1}) + \dfrac{r_i + \cdots + r_n}{r}, & \text{if } t_i = 0. \end{cases}$$

This defines ρ_n consistently on $\dot{I}^n \times R^n$. Let ρ_n be any extension to $I^n \times R^n$. Next, define $\sigma_n : I^n \times (\Omega B)^n \to B$ by

$$\sigma_n(t_1, \cdots, t_n, \lambda_1, \cdots, \lambda_n) = (\lambda_1 + \cdots + \lambda_n)(\rho_n(t_1, \cdots, t_n, r_1, \cdots, r_n) \cdot r),$$

where r_i is the length of λ_i. Figure 3.2 indicates how σ_n looks for $n = 2, 3$. (Recall the functional notation: $\lambda_1 + \lambda_2$.)

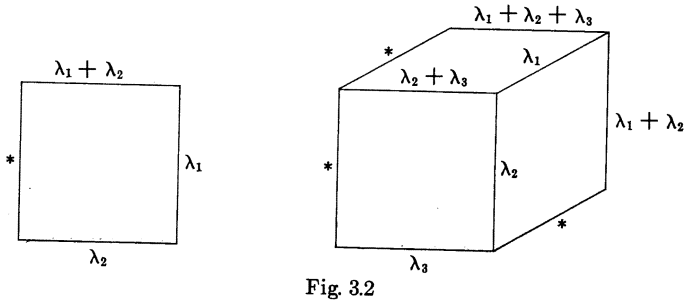

Fig. 3.2

Now assume θ_i and Σ_i defined for $i < n$ so that Σ_n is defined on all but one face of $\dot{I}^n$. Moreover by induction assume

$$p\Sigma_i(t_1, \cdots, x_i, y) = \sigma_i(t_1, \cdots, x_i); \tag{c}$$

then this is also true for Σ_n insofar as it is defined. Since σ_n is defined on all of $I^n \times (\Omega B)^n$, a lifting of σ_n can be used to complete the definition of Σ_n. Notice that, if $t_1 = 0$, $\sigma_n \mid I^{n-1} \times (\Omega B)^n = *$; and so Σ_n restricted to this face lies in F and defines a map θ_n as required. This is the significant proof of Theorem A.

Our definition of "transport" makes no reference to a fibre space. Now that we have seen how a fibre space gives rise to a transport, we can embody this insight in our next definition.

DEFINITION 3.3. *A transport* $\{\theta_i : I^{i-1} \times (\Omega B)^i \times F \to F\}$ *is a* transport within a fibre space $p : E \to B$ *if there exist maps* $\{\Sigma_i : I^i \times (\Omega B)^i \times F \to E\}$ *satisfying* (a), (b) *and* (c) *where* $\Sigma_0 : F \to p^{-1}(b)$ *is a homotopy equivalence.*

Theorem B can now be reworded.

THEOREM B. *Given a transport* $\{\theta_i\}$, *there is a fibre space* $p_\theta : E_\theta \to B$ *such that, up to homotopy,* $\{\theta_i\}$ *is a transport* within p_θ. *Given a transport* $\{\theta_i\}$ *within* p, *the constructed fibre space* p_θ *is fibre homotopy equivalent to* p.

4. Construction of fibre spaces

Dold and Lashof [3] have given a construction for a universal principal quasi-fibration $H \to E_H \to B_H$, where H is an associative H-space. In particular, this is applicable to $H = \Omega B$; and if B has the homotopy type of a CW-complex, B has the homotopy type of $B_{\Omega B}$ (see [12], II, Cor. 9.2).

In general, B_H can be defined inductively as a limit of "projective" spaces $HP(n)$. Let $HP(0)$ be a point. Let $HP(n) = I^n \times H^n \cup_{\beta_n} HP(n-1)$, where

$$\beta_n : \dot{I}^n \times H^n \to HP(n-1) \quad \text{by} \quad \beta_n(t_1, \cdots, t_n, x_1, \cdots, x_n)$$
$$= \begin{cases} \beta_{n-1}(\cdots, t_{i-1}, t_{i+1}, \cdots, x_{i-1}x_i, \cdots), & \text{if } t_i = 1, \\ \beta_{i-1}(t_1, \cdots, t_{i-1}, x_1, \cdots, x_{i-1}), & \text{if } t_i = 0, \end{cases}$$

with the meaningless "x_0x_1" being deleted.

Now, if $H = \Omega B$, a specific map $\sigma : B_{\Omega B} \to B$ is induced by the maps $\sigma_n : I^n \times (\Omega B)^n \to B$. It is a (weak) homotopy equivalence, as can readily be seen by constructing a corresponding map of $E_{\Omega B}$ into the path space $\mathcal{E}B$ which covers σ and is the identity on ΩB.

We will construct a fibre space over $B_{\Omega B}$ by using $I^n \times (\Omega B)^n \times F$ in place of $I^n \times (\Omega B)^n$ and performing certain identifications compatible with the projection which drops the factor F.

Specifically, given a transport $\{\theta_i : I^{i-1} \times (\Omega B)^i \times F \to F\}$, construct $p_n : E_n \to (\Omega B)P(n)$ by setting $E_0 = F$ and $E_n = I^n \times (\Omega B)^n \times F \cup_{\delta_n} E_{n-1}$, where $\delta_n : \dot{I}^n \times (\Omega B)^n \times F \to E_{n-1}$ by $\delta_n(t_1, \cdots, t_n, x_1, \cdots, x_n, y)$

$$= \begin{cases} \delta_{n-1}(\cdots, t_{i-1}, t_{i+1}, \cdots, x_{i-1}x_i, \cdots, y), & \text{if } t_i = 1, \\ \delta_{i-1}(t_1, \cdots, t_{i-1}, x_1, \cdots, x_{i-1}, \theta_{n-i+1}(t_{i+1}, \cdots, y)), & \text{if } t_i = 0. \end{cases}$$

Let p_n be induced by the projection of $I^n \times (\Omega X)^n \times F$ onto $I^n \times (\Omega X)^n$. At each stage p_n can be shown, by using fundamental theorems of Dold and Thom and the fact that $\theta_1(\lambda, \ )$ is always a homotopy equivalence (see [4], Thms. 2.2, 2.10, and 2.15; cf. [12], I, Thm. 10), to be a quasi-fibring. Under reasonable assumptions, p_n would actually be a fibre space; but very little has been written concerning this, and such generality is unimportant here. Let $p':E' \to B_{\Omega B}$ denote the union of the p_n, and convert p' to a true fibre space $\bar{p}:\bar{E} \to B_{\Omega B}$; i.e., $\bar{E} = \{(e, \lambda) \mid e \in E', \lambda:I \to B_{\Omega B}, \lambda(0) = p'(e)\}$. If F and B have the homotopy type of CW-complexes, $\bar{p}$ will have fibres of the homotopy type of F. Finally, let $p_\theta:E_\theta \to B$ be induced from $\bar{p}$ by an inverse to $\sigma:B_{\Omega B} \to B$.

We have constructed a fibre space $p_\theta:E_\theta \to B$; we claim that our original $\{\theta_i\}$ is a transport within p_θ. First let us define $\Sigma_i':I^i \times (\Omega B)^i \times F \to E'$ to be the identification map in the construction. Surely $\{\Sigma_i'\}$ satisfies conditions (a) and (b) of Definition 3.3. As for (c), we see that $\sigma p'\Sigma_i'(t_1, \cdots, x_i, y) = \sigma_i(t_1, \cdots, x_i)$.

$$\begin{array}{ccccccc} I^i \times (\Omega B)^i \times F & \xrightarrow{\Sigma_i'} & E' & \subset & \bar{E} & \longleftarrow & E_\theta \\ \downarrow{\scriptstyle \sigma_i} & & \downarrow{\scriptstyle p'} & \swarrow{\scriptstyle \bar{p}} & & & \downarrow{\scriptstyle p\theta} \\ B & \xleftarrow[\sigma]{} & B_{\Omega B} & & \xleftarrow[\sigma^{-1}]{} & & B \end{array}$$

Now E' can be regarded as embedded in $\bar{E}$ by $e \to (e, \lambda_e)$ where $\lambda_e(t) = p'(e)$. Since σ is a homotopy equivalence and p_θ was defined using its inverse, $\bar{p}$ is fibre homotopy equivalent to $\sigma^* p_\theta$. Let $\bar{\sigma}:E \to E_\theta$ cover σ and induce homotopy equivalences between fibres. The maps $\{\Sigma_i = \bar{\sigma}\Sigma_i'\}$ show that $\{\theta_i\}$ is a transport within p_θ. This establishes the first part of Theorem B.

We now wish to show that if $\{\theta_i\}$ is a transport within p, then p is fibre homotopy equivalent to p_θ as constructed. Let $\{\Sigma_n:I^n \times (\Omega B)^n \times F \to E\}$ show that $\{\theta_i\}$ is a transport within $p:E \to B$. Observe that $\{\Sigma_n\}$ induces a map $\sigma':E' \to E$ which covers σ and is a homotopy equivalence on fibres. Since E' is a deformation retract of $\bar{E}$, σ' can be extended to a map $\bar{\sigma}:\bar{E} \to E$ covering σ. Because the fibres of p' are mapped by homotopy equivalences into those of $\bar{p}$, $\bar{\sigma}$ is a fibre homotopy equivalence. This completes Theorem B.

5. Classification of fibre spaces

We have just seen how to construct a fibre space from a transport. Given a homotopy $\{\theta_i^t\}$ of transports as described in §2, we can construct a corresponding fibre space over $B \times I$ such that $p^{-1}(B \times t)$ is fibre homotopy equivalent to the fibre space constructed using $\{\theta_i^t\}$. Since any fibre space $p:E \to B \times I$ is fibre homotopy equivalent to $p_t \times 1:E_t \times I \to B \times I$, where $E_t = p^{-1}(B \times t)$, p_0 and p_1 are fibre homotopy equivalent (see [2], Cor. 6.6).

Conversely, suppose that, for $\epsilon = 0, 1$, we have fibre spaces $p_\epsilon:E_\epsilon \to B$ which are fibre homotopy equivalent by mutual inverses $f_\epsilon:E_\epsilon \to E_{\epsilon\pm1}$. Let $\Sigma_i^\epsilon:I^i \times (\Omega B)^i \times F \to E_\epsilon$ show that $\theta_i^\epsilon:I^{i-1} \times (\Omega B)^i \times F \to F$ are transports

within p_ϵ. Consider $\Sigma_0^{\epsilon\pm 1}$ and $f_\epsilon\Sigma_0^{\epsilon}: F \to p_{\epsilon\pm 1}^{-1}(b)$. They are both homotopy equivalences; hence there exists $h_\epsilon: F \to F$ such that $\Sigma_0^{\epsilon\pm 1}h_\epsilon$ and $f_\epsilon\Sigma_0^{\epsilon}$ are homotopic. It follows that $\Sigma_i^{\epsilon\pm 1}(1 \times \cdots \times 1 \times h_\epsilon)$ and $f_\epsilon\Sigma_i^{\epsilon}$ are vertically homotopic, since we can look at them as obtained by covering the same homotopy in B. Thus $\{\Sigma_0^{\epsilon\pm 1}\theta_i^{\epsilon\pm 1}(1 \times \cdots \times 1 \times h_\epsilon)\}$ and $\{f_\epsilon\Sigma_0^{\epsilon}\theta_i^{\epsilon}\}$ are homotopic as A_∞-actions, and, hence, so are $\theta_i^{\epsilon\pm 1}(1 \times \cdots \times 1 \times h_\epsilon)$ and $h_\epsilon\theta_i^{\epsilon}$. This is what we mean by saying the homotopy class is determined up to the action of homotopy equivalences of F into itself. This action will appear in more familiar terms shortly.

Theorems A, B, C can now be combined as follows.

CLASSIFICATION THEOREM (2). *The fibre homotopy equivalence classes of fibre spaces $p: E \to B$ with fibres of the homotopy type of F are in natural* 1–1 *correspondence with the homotopy classes of transports $\{\theta_i: I^{i-1} \times (\Omega B)^i \times F \to F\}$ under the action of homotopy equivalences of F into itself. (As always, we are assuming that F and B have the homotopy type of CW-complexes.)*

In [10] we have given an alternative form of the Classification Theorem. First let $LF(B)$ denote the functor described, i.e., fibre homotopy equivalence classes of fibre spaces $p: E \to B$ with fibres of the homotopy type of F. Let $H(F)$ denote the space of homotopy equivalences of F into itself with a suitable topology to be described later. $H(F)$ is an associative H-space; so, using the Dold-Lashof construction, we can look at the universal base space $B_{H(F)}$. For any spaces X, Y, let $[X, Y]$ denote homotopy classes of maps of X into Y (no mention being made of base points).

CLASSIFICATION THEOREM (1) (see [10]). *If F is a finite CW-complex, the functors $LF(\ )$ and $[\ , B_{H(F)}]$ are equivalent on the category of spaces of the homotopy type of CW-complexes and homotopy classes of maps.*

We wish to point out in detail how these two classification theorems coincide. The additional restriction in Classification Theorem (1) that F be a finite complex was used to make sure that $B_{H(F)}$ and, hence, the fibres of the universal example would have the homotopy type of CW-complexes. In Classification Theorem (2), this restriction was avoided by constructing directly the induced fibring over B, using ΩB, which has the homotopy type of a CW-complex if B does.

In comparing the two classifications, the first step is to look at the maps $\{\bar{\theta}_i: I^{i-1} \times (\Omega B)^i \to F^F\}$ which are adjoint to the transport $\{\theta_i\}$. If θ_1 were transitive, $\bar{\theta}_1$ would be a homomorphism. (This is why we have written the composition of loops according to functional notation; the usual notation would make $\bar{\theta}_1$ an anti-homomorphism.) That $\bar{\theta}_1$ is homotopy transitive means precisely that $\bar{\theta}_1$ is an H-map. Restricted classes of H-maps which are the homotopy analogues of homomorphisms have been studied (by Sugawara [13], Fuchs [6], and the author [12], II). The conditions for an A_n-action are easy to interpret in terms of the adjoint maps $\bar{\theta}_i: I^{i-1} \times X^i \to F^F$.

PROPOSITION 5.1. *$\{\theta_i, i = 1, \cdots, n\}$ is an A_n-action if and only if $\{\theta_i, i = 1, \cdots, n\}$ shows that $\bar{\theta}_1$ is an A_n-map.*

The conditions correspond precisely, except that $t_i = 0$ and 1 have been interchanged to make things more compatible with the Dold-Lashof construction; only the continuity is in doubt. We take the point of view that the adjoint maps into F^F are a mere device. We want them to be continuous precisely because the original θ_i are. If F is locally compact Hausdorff, the compact-open topology will do. Otherwise we force our desires to be satisfied, thus defining at least a quasi-topology on F^F (see [9]). We use the same approach on $H(F)$.

The case of greatest interest is $n = \infty$, in which case Sugawara calls $\bar{\theta}_1$ "strongly homotopy multiplicative" and Fuchs calls the collection $\{\theta_i\}$ an "H-homomorphism". In light of Proposition 5.1, Classification Theorem (2) can be rephrased.

CLASSIFICATION THEOREM (3). *$LF(B)$ is in natural 1–1 correspondence with homotopy classes of A_∞-maps of ΩB into $H(F)$ modulo $\pi_0(H(F))$ acting as inner automorphisms.*

To appreciate the significance of A_∞-maps, a short survey of the general theory of associative H-spaces will help. The Dold-Lashof construction is natural with respect to A_∞-maps (see [6], [13], and [12], II), so that homotopy classes of A_∞-maps give rise to homotopy classes of maps in the corresponding base spaces. Just as the Dold-Lashof construction passes from associative H-spaces to ordinary spaces, the process of passing to loop spaces goes in the other direction. As for maps, if $f: X \to Y$, then $\Omega f: \Omega X \to \Omega Y$, defined in the obvious way, is actually a homomorphism.

We have already observed that $B_{\Omega X}$ has the same homotopy type as X. We also have the relation that if H is an associative H-space of the homotopy type of a CW-complex, H has the same homotopy type as ΩB_H. If G should be a topological group, it follows from the result of Brown's applied to Milnor's universal G-bundle that the equivalence $\Omega B \to G$ can be given by a homomorphism. In the case of an associative H-space, we can only assert that the equivalences in both directions are given by A_∞-maps, and there are examples to show that A_∞-maps are essential to the study of associative H-spaces which are not groups (see the Remark following Theorem 5.2).

To use A_∞-maps more facilely, it might help to think of the category of associative H-spaces and A_∞-maps—except that this is not a category. The difficulty is with the composition operation. If $\{f_i\}$ and $\{g_i\}$ are A_∞-maps, defining the composition of $\{f_i\}$ and $\{g_i\}$ involves a decomposition of the cube, as the following simplest case indicates. Let $f: H \to J$ and $g: J \to K$ be H-maps with homotopies $h_t: H \times H \to J$ and $j_t: J \times J \to K$ such that

$$h_0(x, y) = f(xy) \quad \text{and} \quad j_0(u, v) = g(uv),$$

$$h_1(x, y) = f(x)f(y) \quad \text{and} \quad j_1(u, v) = g(u)g(v).$$

Then a homotopy $k_t: H \times H \to K$ showing that gf is an H-map is given by

$$k_t(x, y) = \begin{cases} gh_{2t}(x, y), & 0 \leq 2t \leq 1 \\ j_{2t-1}(f(x), f(y)), & 1 \leq 2t \leq 2. \end{cases}$$

(For the case of three variables, see [12], p. 299.) If we use this homotopy or anything like it to define the composition of H-maps, composition will not be an associative operation: we will not have a category. Fuchs took cognizance of this difficulty and avoided it by passing to homotopy classes [6].

Let $\mathfrak{K}$ denote the category of associative H-spaces and homotopy classes of A_∞-maps, i.e., homotopies $\{\bar{\theta}_i^t\}$ which for each t show $\bar{\theta}_i^t$ to be an A_∞-map. Let $\mathfrak{F}$ denote the category of spaces and homotopy classes of maps, while $\mathfrak{F}_0$ denotes the category of based spaces and base point preserving homotopy classes of maps. For any category $\mathfrak{C}$ and objects X, Y in $\mathfrak{C}$, we will denote by $\mathrm{Hom}_{\mathfrak{C}}(X, Y)$ the maps from X to Y in $\mathfrak{C}$, which means homotopy classes if $\mathfrak{C}$ is a category of spaces and homotopy classes. Note that if Y is connected, the obvious map $\mathrm{Hom}_{\mathfrak{F}_0}(X, Y) \to \mathrm{Hom}_{\mathfrak{F}}(X, Y)$ amounts to factoring out by the action of $\pi_1(Y)$.

We have mentioned that the Dold-Lashof construction is essentially a functor $B: \mathrm{Hom}_{\mathfrak{K}}(H, H') \to \mathrm{Hom}_{\mathfrak{F}_0}(B_H, B_{H'})$. Similarly we have $\Omega: \mathrm{Hom}_{\mathfrak{F}_0}(X, Y) \to \mathrm{Hom}_{\mathfrak{K}}(\Omega X, \Omega Y)$. Fuchs proves (in [6], §7) the following theorem.

THEOREM 5.2. *If all the spaces in question have the homotopy type of CW-complexes, then* $B: \mathrm{Hom}_{\mathfrak{K}}(H, H') \to \mathrm{Hom}_{\mathfrak{F}_0}(B_H, B_{H'})$ *is* 1–1 *onto, as is*

$$\Omega: \mathrm{Hom}_{\mathfrak{F}_0}(X, X') \to \mathrm{Hom}_{\mathfrak{K}}(\Omega X, \Omega X').$$

The same method of proof shows that $\mathrm{Hom}_{\mathfrak{F}}(B_H, B_{H'})$ (i.e., free homotopy classes) is in 1–1 correspondence with $\mathrm{Hom}_{\mathfrak{K}}(H, H')$ modulo the action of $\pi_0(H')$. Thus Classification Theorems (1) and (3) are equivalent in terms of the correspondences between $\mathrm{Hom}_{\mathfrak{K}}(\Omega B, H(F))/\pi_0(H(F))$, $\mathrm{Hom}_{\mathfrak{F}}(B_{\Omega B}, B_{H(F)})$ and $\mathrm{Hom}_{\mathfrak{F}}(B, B_{H(F)})$.

Fuchs comments further that B is not necessarily an isomorphism if we replace $\mathfrak{K}$ by the category with the same objects and homotopy classes of strict homomorphisms. For example, the unit interval with its usual multiplication is *not* contractible through homomorphisms (0 and 1 would both remain fixed), although it is through A_∞-maps (therefore B_I is contractible, as can be seen directly). This justifies our emphasis on A_∞-actions rather than transitive actions; they arise not only naturally but perhaps essentially in the theory of fibre spaces.

The correspondence between the two forms of classification is of greater significance if examined in detail. For a given fibre space $p: E \to B$ with F a finite complex, let $\chi: B \to B_{H(F)}$ be the classifying map constructed in [10]. Let $B_{\bar{\theta}}: B_{\Omega B} \to B_{H(F)}$ be the map of classifying spaces constructed using the adjoints $\{\bar{\theta}_i: I^{i-1} \times (\Omega B)^i \to H(F)\}$. We then claim that $B_{\bar{\theta}} \simeq \chi \circ \sigma$; i.e.,

$$\begin{array}{ccc} B_{\Omega B} & \xrightarrow{\sigma} & B \\ & {\scriptstyle B_{\bar\theta}}\searrow \quad \swarrow{\scriptstyle \chi} & \\ & B_{H(F)} & \end{array}$$

is homotopy commutative. By virtue of Classification Theorem (1), it is sufficient to show that p is fibre homotopy equivalent to $(\sigma^{-1} \circ B_{\bar\theta})^* u$, the fibre space induced from the universal example by $B_{\bar\theta}$ composed with an inverse to σ. Actually what we will do is to construct a map of fibre spaces,

$$\begin{array}{ccc} E_\theta & \longrightarrow & UE \\ {\scriptstyle p_\theta}\downarrow & & \downarrow{\scriptstyle u} \\ B_{\Omega B} & \xrightarrow[B_{\bar\theta}]{} & B_{H(F)} \end{array},$$

which induces homotopy equivalences between corresponding fibres. This is good enough, since p_θ is fibre homotopy equivalent to $\sigma^* p$ and σ is a homotopy equivalence.

First, with some change in notation, we reconstruct $u: UE \to B_{H(F)}$. Let $q_0: D_0 = F \to *$, a point. Define $D_n = I^n \times H(F)^n \times F \cup_{\eta_n} D_{n-1}$, where $\eta_n: \dot{I}^n \times H(F)^n \times F \to D_{n-1}$, by

$$\eta_n(t_1, \cdots, t_n, \varphi_1, \cdots, \varphi_n, y) = \begin{cases} \eta_{n-1}(\cdots, t_{i-1}, t_{i+1}, \cdots, \varphi_i\varphi_{i+1}, \cdots, y), & \text{if } t_i = 1, \\ \eta_{i-1}(t_1, \cdots, t_{i-1}, \varphi_1, \cdots, \varphi_{i-1}, \varphi_i \circ \cdots \circ \varphi_n(y)), & \text{if } t_i = 0, \end{cases}$$

and let $q_n: D_n \to H(F)P(n)$ be induced by the projection of $I^n \times H(F)^n \times F$ onto $I^n \times H(F)^n$. The universal example is then obtained from the limit of the q_n by turning it into a fibring in the standard way. Since p_θ was constructed in a similar way from fibrings $p_n: E_n \to (\Omega B)P(n)$, we need only construct appropriate maps of p_n into q_n. (Similar maps for the associated principal fibrings are constructed by Fuchs in [6] and Sugawara in [13] in different ways, but neither will suffice for our purposes.) Since $\bar\theta_1$ is not a strict homomorphism, we need some elbow room to utilize the various homotopies; we obtain this by subdividing I^n and stretching the pieces appropriately. Specifically we will subdivide I^n into prisms determined by $t_2, \cdots, t_i \geq t_1 \geq t_{i+1}, \cdots, t_n$. Further let $t_i' = t_i/t_1$ and $\bar t_i = 1 - (1 - t_i)/1 - t_1$. Now let $\psi_n: E_n \to D_n$ be defined in terms of the preimage $I^n \times (\Omega B)^n \times F$ by $\psi_n(t_1, \cdots, t_n, x_1, \cdots, x_n, y) = (t_1, \bar\theta_i(\bar t_2, \cdots, \bar t_i, x_1, \cdots, x_i) \mid \psi_{n-i}(t_{i+1}', \cdots, t_n', x_{i+1}, \cdots, x_n, y))$, for $t_2, \cdots, t_i \geq t_1 \geq t_{i+1}, \cdots, t_n$, where the vertical bar indicates that the t-coordinates of $\psi_{n-i}(t_{i+1}', \cdots, y)$ are to be brought out between t_1 and $\bar\theta_i$.

To verify that ψ_n is well defined requires checking that when $t_i = t_1$ the alternate expressions indicated above agree and that when $t_i = 0$ or 1 the definition respects the identifications under δ_n and η_n. This verification will be left to the interested reader; it follows straightforwardly from the inductive definition of ψ_n and the behavior of δ_n, η_n, and θ_i on the boundary of I^{n-1}. By induction one

also verifies that ψ_n respects fibres and hence induces maps $(\Omega B)P(n) \to H(F)P(n)$. In the limit, this is the map $B_{\bar{\theta}}$ referred to. (The parameters used by Fuchs or Sugawara are somewhat different but can easily be seen to give homotopic maps.)

This completes our comparison of the various forms of the Classification Theorem.

As presented, the Classification Theorems involve the equivalence of functors. The emphasis is on naturality with respect to maps of the base space. Changes in the fibre are somewhat more subtle and involve natural transformations of the functors involved. We are led to questions of considerable interest: the classification's independence of the representative of the homotopy type of F (§6) and the naturality of the classification with respect to restricted structures on the fibre (§7).

6. Naturality of transports

Since we have already become involved with A_∞-maps which are adjoint to transports, it is perhaps worth remarking how the theory of A_∞-maps can be used to streamline the construction of transports and to show how intrinsic is the deviation from transitivity. This leads us to a study of the naturality of the classification with respect to changes in F.

First we consider a special case. If $f:A \to B$ is any map, we can construct an equivalent fibre space $\bar{f}:\bar{A} \to B$ by setting $\bar{A} = \{(a, \lambda) \mid a \in A, \lambda:[0, r] \to B, \lambda(0) = f(a)\}$, and $\bar{p}(a, \lambda) = \lambda(r)$. A rather obvious transport for $\bar{p}$ is $\vartheta(\mu, (a, \lambda)) = (a, \mu + \lambda)$ which is obviously transitive. Now let us take $f = p:E \to B$. We wish to use ϑ to construct a transport within p. There is always the map $j:A \to \bar{A}$ given by $j(a) = (a, \lambda_a)$ where $\lambda_a:[0, 0] \to f(a)$ which is a homotopy equivalence. If f is a fibre space, then j is a fibre map and hence a fibre homotopy equivalence (see [2], Th. 6.1). In particular, corresponding fibres F of p and F' of $\bar{p}$ are homotopy equivalent, which allows us to compare $H(F)$ and $H(F')$ rather neatly (as Fuchs has also realized independently in [6], §2.4).

PROPOSITION 6.1. *If K and L are spaces of the same homotopy type, then $H(K)$ and $H(L)$ are equivalent in the category $\mathfrak{K}$.*

Proof. Let $\alpha:K \to L$ and $\beta:L \to K$ be mutual homotopy inverses. Define $\chi:H(K) \to H(L)$ by $\chi(\varphi) = \alpha\varphi\beta$ and $\rho:H(L) \to H(K)$ by $\rho(\psi) = \beta\psi\alpha$. That χ, for example, is an H-map but not a homomorphism is readily seen: $\chi(\varphi_1)\chi(\varphi_2) = \alpha\varphi_1\beta\alpha\varphi_2\beta \simeq \alpha\varphi_1\varphi_2\beta = \chi(\varphi_1\varphi_2)$. (For the higher order conditions, see Fuchs [6].) That χ and ρ are mutual inverses is also easy to see: $\rho\chi(\varphi) = \beta\alpha\varphi\beta\alpha \simeq \varphi$. Since the obvious homotopy has the same form as χ, the same proof shows it is a homotopy through A_∞-maps.

Thus for $\bar{p}$, we have the transitive transport $\vartheta:\Omega B \times F' \to F'$ and an A_∞-map $\chi:H(F') \to H(F)$. The composition $\chi\vartheta$ is an A_∞-map and is therefore adjoint to an A_∞-action $\{\theta_i\}$ of ΩB on F. That this construction agrees (up to homotopy

modulo $\pi_0(H(F))$) with our earlier one will follow if we show that this $\{\theta_i\}$ is a transport within p. We first observe that ϑ is a transport within $\bar{p}$. $\Sigma_1'(t, \mu, (a, \lambda)) = (a, \mu_t + \lambda)$ where μ_t is the restriction of μ to $[0, rt]$ and the higher Σ_i' are defined analogously. To obtain the corresponding maps Σ_i for θ_i in p, we must use an analogue of χ. The adjoints to $\Sigma_n' : I^n \times (\Omega B)^n \times F' \to \bar{E}$ have values in the associated principal fibre space Prin $\bar{E} = \{\varphi : F' \to \bar{E} \mid \varphi$ is a homotopy equivalence with some fibre $\bar{p}^{-1}(b)\}$. Composition makes Prin $\bar{E}$ a "module" over $H(F')$; i.e., there is a transitive action Prin $\bar{E} \times H(F') \to$ Prin $\bar{E}$. Now the fibre homotopy equivalence $j : \bar{E} \to E$ and an inverse induce a fibre homotopy equivalence which we also call χ: Prin $\bar{E} \to$ Prin E. Defining A_∞-maps of modules in the obvious way, we use essentially the same proof as for Proposition 6.1 to assert that χ is an A_∞-map. Composing this A_∞-map with the adjoints to $\{\Sigma_n\}$ then gives the adjoints to the corresponding maps $\Sigma_n : I^n \times (\Omega B)^n \times F \to E$.

What we have been talking about amounts to the naturality of transports with respect to fibre homotopy equivalence. More generally, we have the following theorem.

THEOREM 6.2. *Let*

$$\begin{array}{ccc} \bar{E} & \xrightarrow{\bar{f}} & E \\ {\scriptstyle \bar{p}}\downarrow & & \downarrow{\scriptstyle P} \\ \bar{B} & \xrightarrow{\bar{f}} & B \end{array}$$

be a map of fibre spaces such that $\bar{f}$ induces homotopy equivalences between corresponding fibres. If $\{\theta_i\}$ is a transport within $\bar{p}$, then $\{\theta_i[1 \times (\Omega f)^i \times 1]\}$ is a transport within p.

Proof. $\bar{p}$ is fibre homotopy equivalent to f^*p; so first suppose $f = id : B \to B$. If $\{\Sigma_n : I^n \times (\Omega B)^n \times F \to E\}$ show that $\{\theta_i\}$ is a transport within p, then, using a fibre inverse $g : E \to \bar{E}$, the compositions $g\Sigma_n$ show that $\{\theta_i\}$ is a transport within $\bar{p}$. For a more general f, observe that Ωf is a (strict) homomorphism and, hence, that $\{\Sigma_i[1 \times (\Omega f)^i \times 1]\}$ show that $\{\theta_i[1 \times (\Omega f)^i \times 1]\}$ is a transport within f^*p and hence within $\bar{p}$.

We have thus established the "naturality" of transports (in the technical sense) very thoroughly. We have also accomplished something else. The functor $LF(\ )$ is by definition the same as $LF'(\ )$, if F and F' have the same homotopy type. Since $H(F)$ and $H(F')$ have the same A_∞-homotopy type, $B_{H(F)}$ and $B_{H(F')}$ have the same homotopy type and hence $[\ , B_{H(F)}]$ and $[\ , B_{H(F')}]$ are isomorphic, as indeed the Classification Theorem implies they should be.

This leads us to study the naturality of $LF(X)$ with respect to submonoids of $H(F)$, that is, to the study of fibre spaces with additional structure, e.g. differentiable bundles.

7. G-structures

Although general submonoids of $H(F)$ may some day be of interest, we restrict ourselves to subgroups, the maximal one being the group of all homeomorphisms of F onto itself. A fibre space with a G-structure is then automatically a fibre bundle. We find it most convenient to follow Ehresmann-Feldbau in defining G-bundles.

DEFINITION 7.1. *If G is a group of homeomorphisms of F, a* G-bundle with fibre F *is a map $p:E \to B$ together with a preferred class of maps $\varphi:F \to E$ such that*

(1) each φ is a homeomorphism with some fibre $p^{-1}(b)$;

(2) if $\varphi, \varphi':F \to p^{-1}(b)$, then $\varphi^{-1}\varphi' \in G$;

(3) if $\varphi' \in G$ and φ is in the class, so is $\varphi\varphi'$; and

(4) the Bundle Covering Homotopy Property is valid; i.e., given a commutative diagram,

$$\begin{array}{ccc} X \times F & \xrightarrow{f_0} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow[g_0]{} & B, \end{array}$$

such that $f_0(x, \ ):F \to E$ is in the class for each $x \in X$ and a homotopy, $g_t:X \to B$, there exists a covering homotopy $f_t:X \times F \to E$ such that $f_t(x, \ ):F \to E$ is in the class for each t, x and $pf_t(x, y) = g_t(x)$.

We have chosen to redefine G-bundles for several reasons: a) our definition is intrinsic rather than in terms of coordinates; b) it is more in line with our definition of fibre space; and c) it emphasizes precisely those properties of G-bundles which we wish to use. Notice that no mention is made of a topology on G. To effect a comparison with a Steenrod fibre bundle, we specify a quasi-topology on G (cf. [9]): $X \to G$ is continuous if and only if the adjoint $X \times F \to F$ is continuous. In most cases then, our definition of G-bundle will agree with that of Steenrod. In particular, p will be locally a product if B admits a numerable covering each set of which is null homotopic in B, and a Steenrod fibre bundle will satisfy our definition if it is numerable in the sense of Dold [2], e.g. if B is a connected CW-complex.

The relevance of condition (4) to transports is fairly obvious. We apply it to

$$\begin{array}{ccc} \Omega B \times F & \xrightarrow{f_0} & E \\ \downarrow & & \downarrow \\ \Omega B & \xrightarrow{g_t} & B, \end{array}$$

where as usual $F = p^{-1}(*)$, $f_0(\lambda, y) = y$ and $g_t(\lambda) = \lambda(t)$. We get $f_1 = \theta_1:\Omega B \times F \to F$ such that $\theta_1(\lambda, \ ) \in G$. As usual, we cannot guarantee the

transitivity of θ_1 obtained this way. However, Brown has shown essentially the following (see [1], Th. (1.5)).

THEOREM 7.2. *If $p:E \to B$ is a numerable G-*bundle, *there exists a transitive transport $\theta:\Omega B \times F \to F$ such that $\theta(\lambda, \ ) \in G$.*

Moreover, θ can be extended to a map $\Sigma:\mathfrak{L}B \times F \to E$ which is transitive with respect to θ and hence shows that θ is a transport within p. The construction of p_θ is easier in that the inductive step produces a G-bundle directly at each stage. Using Σ, it is easy to cover σ by a G-bundle map $p_\theta \to p$, i.e., a G-equivalence.

The rest of our discussion of transports carries over to G-bundles in an obvious way. The final result is familiar in two forms: Let $LG(X)$ denote equivalence classes of G-bundles over X. (For clarity, we assume G given as a group of homeomorphisms of F, so all bundles are assumed to have fibres homeomorphic to F. The equivalence is the categorical one: G-bundles $p_\epsilon:E_\epsilon \to B$, $\epsilon = 0, 1$, are equivalent if there exist maps

$$\begin{array}{ccc} E_\epsilon & \xrightarrow{f_\epsilon} & E_{\epsilon\pm1} \\ \downarrow & & \downarrow \\ B & = & B \end{array}$$

such that f_ϵ takes the distinguished collection $\{\varphi:F \to E_\epsilon\}$ into the distinguished collection $\{\varphi:F \to E_{\epsilon\pm1}\}$ and $f_\epsilon f_{\epsilon\pm1} = id.$)

CLASSIFICATION THEOREM. *On the category of spaces of the homotopy type of CW-complexes and homotopy classes of maps, the functor $LG(\ )$ is naturally equivalent to the functors* $\mathrm{Hom}_{\mathfrak{T}}(\ , B_G)$ *and* $\mathrm{Hom}_{\mathfrak{Gc}}(\Omega\ , G)/\pi_0(G)$.

The result of Brown and others in the K-theory (of vector bundles) over classifying spaces suggest that $\mathrm{Hom}_{\mathfrak{Gc}}(\Omega\ , G)$ can be replaced by homotopy classes of strict homomorphisms. This appears to be the outstanding open problem in this theory.

Now let us look at the naturality of the classification with respect to G. Suppose G' is a group of homeomorphisms of F'.

Given a homomorphism $h:G \to G'$ (continuous with respect to the quasi-topologies), we can convert a G-bundle $p:E \to B$ into a G'-bundle $p':E' \to B$ by using the associated principal G-bundle $\mathrm{Prin}\,E$ to form $E' = \mathrm{Prin}\,E \times_G F'$. That is, $\mathrm{Prin}\,E = \{\text{distinguished } \varphi\}$ with the corresponding quasi-topology, and $\mathrm{Prin}\,E \times_G F'$ means $\mathrm{Prin}\,E \times F'$ modulo the equivalence relation $(\varphi \circ g, y) \sim (\varphi, h(g)(y))$. Now if h is not a homomorphism, this is not an equivalence relation; however, our construction shows us how to make do if h is only an A_∞-map. We could construct a transport for p and compose with h to get a new transport which could then be used to construct the equivalent of $\mathrm{Prin}\,E \times_G F'$. Alternatively, we could modify our construction as follows. There is an obvious transitive action of G on $\mathrm{Prin}\,E$ given by composition. Let $\{\theta_i:I^{i-1} \times G^i \times F' \to F'\}$

be the A_∞-action corresponding to the fact that $h:G \to G'$ is an A_∞-map. Let $D_0 = \operatorname{Prin} E \times F$, and construct $D_n = I^n \times \operatorname{Prin} E \times G^n \times F' \cup_{\delta_n} D_{n-1}$, where $\delta_n:\dot{I}^n \times \operatorname{Prin} E \times G^n \times F' \to D_{n-1}$ by

$$\delta_n(t_1, \cdots, t_n, x_0, x_1, \cdots, x_n, y) = \begin{cases} \delta_{n-1}(\cdots, t_{i-1}, t_{i+1}, \cdots, x_{i-1}x_i, \cdots), & \text{if } t_i = 1, \\ \delta_{i-1}(t_1, \cdots, t_{i-1}, x_0, \cdots, x_{i-1}, \theta_{n-i+1}(t_{i+1}, \cdots, y)), & \text{if } t_i = 0, \end{cases}$$

where $x_0 \in \operatorname{Prin} E$, $x_1, \cdots, x_n \in G$, $y \in F'$. Similarly, define B_n by omitting the factor F' throughout, and let $q_n:D_n \to B_n$ be the induced map. The maps q_n are G'-bundles as long as G' is a group of homeomorphisms. Let $q_\infty:D_\infty \to B_\infty$ denote the limit which will again be a bundle. In [11], we have proved that B_∞ has the homotopy type of B. Let $q:D \to B$ be induced from q_∞ by a homotopy equivalence. This is the bundle we seek, the G'-bundle associated to $p:E \to B$ and the A_∞-map $h:G \to G'$.

By means of the constructions just indicated, $LG(X)$ can also be regarded as a functor from the category of homeomorphism groups and A_∞-maps. Given an A_∞-map $h:G \to G'$, we can also compose it with adjoints to transports, thus inducing

$$\operatorname{Hom}_{\mathfrak{GC}}(\Omega X, G) \to \operatorname{Hom}_{\mathfrak{GC}}(\Omega X, G').$$

Of course, the corresponding map of base spaces gives

$$\operatorname{Hom}_{\mathfrak{T}}(X, B_G) \to \operatorname{Hom}_{\mathfrak{T}}(X, B_{G'}).$$

Thus it makes sense to say, and our constructions make it clear, that the equivalence between the three functors in question is also natural with respect to A_∞-maps of the structure groups.

We conclude with some remarks to further emphasize the naturality of these constructions from the point of view of homotopy theory. First, our original construction can be regarded as a special case of the later one by taking $G = \Omega B$, $G' = H(F)$, and, for $\operatorname{Prin} E$, the path space on B. Being contractible, the latter is essentially a point and hence never mentioned explicitly in the construction. In [11], we showed that in general the equivalence $B_\infty \to B$ can be given by a fibring with fibre precisely E_G. The same is true for the covering map $D_\infty \to E$, so that we have

$$\begin{array}{ccccc} E_G & \longrightarrow & D_\infty & \longrightarrow & E \\ & \searrow & \downarrow & & \downarrow \\ & & B_\infty & \longrightarrow & B. \end{array}$$

Finally, observe that B_∞ can be mapped into B_G by omitting the factor $\operatorname{Prin} E$. This gives yet another version of the classifying map. In fact $B_\infty \to B_G$ will be a G-bundle with fibre $\operatorname{Prin} E$. This is very much in keeping with the philosophy that says, up to homotopy, any three successive terms in the sequence $\cdots \to$

$\Omega B \to F \to E \to B$ can be regarded as fibre, total space, and base and that the sequence can be extended by a classifying map $B \to B_F$ if E is a principal F-fibring.

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REFERENCES

[1] E. H. BROWN, *Twisted tensor products I*, Ann. of Math., **69** (1959), 223–46.
[2] A. DOLD, *Partitions of unity in the theory of fibrations*, Ann. of Math., **78** (1963), 223–55.
[3] A. DOLD and R. LASHOF, *Principal quasifibrations and fibre homotopy equivalence of bundles*, Ill. J., **3** (1959), 285–305.
[4] A. DOLD and R. THOM, *Quasifaserungen und unendliche symmetrische Produkte*, Ann. of Math., **67** (1958), 239–81.
[5] M. FUCHS, Homotopiefaserungen, Dissertation, Saarbrücken.
[6] ———, Verallgemeinerte Homotopie-Homomorphismen und klassifizierende Raüme, Saarbrücken (mimeo).
[7] P. J. HILTON, Homotopy theory and duality, Cornell University (mimeographed).
[8] R. LASHOF, *Classification of fibre bundles by the loop space of the base*, Ann. of Math., **64** (1956), 436–46.
[9] E. H. SPANIER, *Quasi-topologies*, Duke Math. J., **30** (1963), 1–14.
[10] J. D. STASHEFF, *A classification theorem for fibre spaces*, Topology, **2** (1963), 239–46.
[11] ———, *Associated fibre spaces* (to appear).
[12] ———, *Homotopy associativity of H-spaces I, II*, Trans. A.M.S., **108** (1963), 293–312.
[13] M. SUGAWARA, *On the homotopy commutativity of groups and loop spaces*, Mem. Coll. Sci. Univ. Kyoto, Ser. A., **33** (1960), 257–69.
[14] O. VEBLEN and J. H. C. WHITEHEAD, The foundations of differential geometry, Cambridge University Press, 1932.
[15] A. I. WEINZWEIG, On fibre spaces and fibre homotopy equivalence, Northwestern University (duplicated).