THE GEOMETRIC DIMENSION OF REAL STABLE VECTOR BUNDLES

By S. Gitler and M. Mahowald*

Introduction

Let X be a finite t-dimensional CW-complex, and let ξ be an m-plane bundle over X, with $m > t$. We know from the classification theorem for plane bundles (see [12], 19.3), that a lower bound for the number of sections that ξ admits is $m - t$. We will say that codim $(\xi) \geq k$, if there exists a bundle η of dimension $t - k$ with $\xi = \eta \oplus (m + t - k)$. Therefore the codimension is a nonnegative function, for bundles of dimension $m > t$.

Suppose ξ is trivial over the q-skeleton of X, then it seems reasonable that ξ should have positive codimension. The main object of this paper is to prove the following results.

Theorem A. *Let* $| \log_2 t |$ *denote the integral part of* $\log_2 t$. *If* ξ *is trivial over the* $(q - 1)$*-skeleton of* X, *then*

$$\text{codim } (\xi) \geq \min (q - 2 | \log_2 t | - 5, | t/2 | - 1).$$

A sharper but more complicated result is actually proved and is stated as Theorem 7.17.

Consider now the following problem. Given a fixed integer k and a complex X of dimension t, find the least q such that any bundle ξ over X trivial over the q-skeleton of X has codim $(\xi) \geq k$.

If we apply Theorem A directly, we see that q increases as t increases.

The next theorem, then, should be viewed as a stability result. Let $\alpha(m) =$ the number of ones in the dyadic expansion of m.

Theorem B. *Given integers* a, k *and* n *with* $k \leq n$, *there exists an integer* $q = q(a, k, n)$ *such that if dimension* X *is* $t = s2^r + n$, *where* $\alpha(s) = a$, *and* $n \leq 2^{r-1}$, *any stable bundle* ξ *trivial over the* q*-skeleton of* X *has* codim $(\xi) \geq k$.

Again a much sharper result than Theorem B is obtained and is stated as Theorem 7.19.

If we let $X = S^t$, in Theorem A we obtain the following result of Barrat and Mahowald [4],

Corollary C. *If* ξ *is a* $(t + 1)$*-plane bundle over* S^t, *then* codim $(\xi) \geq | t/2 | - 1$, *if* $t > 16$.

The results of (7.17) are actually needed for $t = 20$. Corollary C implies strong results about the metastable homotopy of $O(n)$ and these implications

* M. Mahowald is an A. P. Sloan Fellow and was partially supported by the U.S. Army Research Office (Durham). The authors would also like to thank F. Nusbaum for critically reading the manuscript.

are given in [4]. Another consequence of Theorem A is the following result in differential topology.

COROLLARY D. *If M^t is a differentiable manifold such that the stable tangent bundle is trivial over the $(q-1)$-skeleton, then M^t immerses in* $R^{2t+5-(q-2|\log_2 t|)}$.

This result should be compared with the beautiful result of M. Hirsch [7] which says that if M^t is a π-manifold, then M^t immerses in $t+1$. For such manifolds our result gives only M^t immerses in $R^{3/2t+1}$.

The method used in proving Theorems A and B is a sharper version of the modified Postnikov tower introduced in [8]. That presentation lacked sufficient generality for our purposes, and since a part of [8] has to be slightly modified to get this generality, we give a short development of it here. The main new result in this direction is Theorem (2.11), and this result is the key to our applications. While we have all the machinery set up for obstruction theory we will prove the following theorem.

THEOREM E. *Real projective space RP^n for $n \equiv 0 \bmod 4$ and $n \neq 2^r$ immerses in* R^{2n-5}.

This result uses only the method of this paper and in no way involves Theorem A or B.

1. The transgression in a fibre space

Let $F \xrightarrow{i} T \xrightarrow{p} B$ be a fibre space, and set $b = p(F)$. Denote by $\bar{p}:(T, F) \to (B, b)$ the relative projection, and consider the map of the cohomology sequences induced by $\bar{p}$,

$$\begin{array}{ccccccccc}
\cdots \to & H^{k-1}(T) & \xrightarrow{i^*} & H^{k-1}(F) & \xrightarrow{\delta} & H^k(T, F) & \xrightarrow{j^*} & H^k(T) & \to \cdots \\
 & \uparrow p^* & & & & \uparrow \bar{p}^* & & \uparrow p^* & \\
\cdots \to & H^{k-1}(B) & \xrightarrow{i_1^*} & H^{k-1}(b) & \xrightarrow{\delta_1} & H^k(B, b) & \xrightarrow{j_1^*} & H^k(B) & \to \cdots .
\end{array}$$

Define, for $k > 0$,

$$T^{k-1}(F) = \delta^{-1}\bar{p}^* j_1^{*-1} H^k(B) \tag{1.1}$$

and

$$\Sigma^k(B) = j_1^* \bar{p}^{*-1} \delta H^{k-1}(F). \tag{1.2}$$

The elements of $T^{k-1}(F) \subset H^{k-1}(F)$ are called *transgressive*, those of $\Sigma^k(B) \subset H^k(B)$ are called *suspension* elements.

The map $j_1^* \bar{p}^{*-1} \delta$ induces the transgression

$$\tau: T^{k-1}(F) \to \Sigma^k(B)/j_1^* \,(\operatorname{Ker} \bar{p}^*), \tag{1.3}$$

and the mapping $\delta^{-1}\bar{p}^* j_1^{*-1}$ induces the suspension

$$\sigma: \Sigma^k(B) \to T^{k-1}(F)/i^* H^{k-1}(T). \tag{1.4}$$

σ and τ together yield an isomorphism,

$$T^{k-1}(F)/i^*H^{k-1}(E) \cong \Sigma^k(B)/j_1^* (\mathrm{Ker}\, \bar{p}^*). \tag{1.5}$$

From the diagram it is easy to identify $\Sigma^k(B)d$ as Ker p^*; thus

$$T^{k-1}(F)/i^*H^{k-1}(E) \cong (\mathrm{Ker}\, p^*/j_1^* (\mathrm{Ker}\, \bar{p}^*))^k. \tag{1.6}$$

2. Totally transgressive fibrations

By a *q-totally transgressive fibration* we will understand a fibre space $F \xrightarrow{i} T \xrightarrow{p} B$ such that, with Z_2-coefficients,

(2.1) $H^*(F)$ forms a simple system of coefficients over B and for $k \leq q$;

(2.2) $p^*: H^k(B) \to H^k(T)$ is onto; and

(2.3) $T^{k-1}(F) = H^{k-1}(F)$.

The following are examples of q-totally transgressive fibrations. Let $C = K(Z_2, q)$; then $\Omega C \to PC \to C$ is $2q$-totally transgressive. More generally, if X is $(q-1)$-connected, then $\Omega X = PX \to X$ is $2q$-totally transgressive.

More important for our purposes is the fibration

$$V_{n,k} \to BSO_{n-k} \to BSO_n, \tag{2.4}$$

where BSO_m is the classifying space for orientable m-plane bundles and $V_{n,k}$ is the Stiefel manifold of k frames in n-space. This fibration is $2(n-k)$-totally transgressive according to Borel (see [5], (18.3)).

First we show what the kernel of $p^*: H^*(B) \to H^*(T)$ is, in a q-totally transgressive fibration.

We recall the notion of an S-sequence considered by Massey and Peterson (in [10], p. 56). Let S be a graded algebra over Z_2. If $s_1, \cdots, s_n, \cdots$, are elements of S, $(s_1, \cdots, s_n)$ will denote the ideal generated by $s_1, \cdots, s_n$. The sequence $s_1, \cdots, s_n, \cdots$ is called an *(S, q)-sequence* if s_{i+1} is not a zero divisor in $S/(s_1, \cdots, s_i)$ in dimensions $\leq q$, for $i = 0, 1, \cdots, n, \cdots$; that is, if there is no $x \in S$ such that $xs_{i+1} \in (s_1, \cdots, s_i)$, where $\deg x + \deg s_{i+1} \leq q$. The ideal $(s_1, \cdots, s_n, \cdots)$ associated with an (S, q)-sequence will be called a *q-Borel ideal.*

PROPOSITION 2.5. *Suppose $F \xrightarrow{i} T \xrightarrow{p} B$ is a q-totally transgressive fibration, where F is $(n-1)$-connected and $q \leq 2n-1$. Let $X = X^{n+1} + \cdots + X^q$ be a subspace of $H^*(B)$ representing $\tau(\sum_{k=n}^{q-1} H^k(F))$. Then (X) is a q-Borel ideal and* Ker $p^* = (X)$ *in dimensions $\leq q$.*

Proof. Consider the spectral sequence of the fibration. We have $E_2^{r,s} = H^r(B) \otimes H^s(F)$. Let us restrict to the triangle $r + s \leq q$. Then since F is $(n-1)$-connected and $H^{k-1}(F)$ is transgressive for $k \leq q$, we obtain $E_{n+1}^{r,s} = E_2^{r,s}$ if $r > 0$ and $E_{n+1}^{0,q} \subset E_2^{0,q}$.

Therefore in dimensions $\leq q$ we have

$$E_{n+1} = H^*(B) \otimes (H^0(F) + \textstyle\sum_{k=n}^{q-1} H^k(F) + E_{n+1}^{0,q}).$$

Now we have $E_{n+1}{}^{0,q} = E_{q+1}{}^{0,q}$. Therefore the differential d_{n+1} is determined by its value $d_{n+1}(H^n(F)) = X^{n+1}$. Let $x \in X^{n+1}$ and $y \in H^p(B)$, with $n + 1 + p \leq q$. Then $xy \neq 0$. For, if $xy = 0$, we have $d_{n+1}(\bar{x}) = x$; so $d_{n+1}(y\bar{x}) = 0$ and $y\bar{x} \in E_{n+1}{}^{p,n}$ is a cocycle under d_{n+1} and, therefore, a d_m cocycle for $m \geq n + 1$. Moreover, $y\bar{x}$ survives to $E_\infty{}^{r,s}$; but, by (2.2), $E_\infty{}^{r,s} = 0$ if $s > 0$ and $r + s \leq q$.

It follows that

$$E_{n+2} = H^*(B)/(X^{n+1}) \otimes (H^0(F) + \textstyle\sum_{k=n+1}^{q-1} H^k(F) + E_{n+2}{}^{0,q}).$$

We apply the same arguments successively, as above, until we reach

$$E_{q+1} = H^*(B)/(X^{n+1} + \cdots + X^q) \otimes (H^0(F) + E_{q+1}{}^{0,q}).$$

Now $E_{q+1}{}^{0,q}$ is totally transgressive, since $i^*:H^q(T) \to H^q(F)$ is trivial. Therefore,

$$E_{q+2} = H^*(B)/(X^{n+1} + \cdots + X^q) = E_\infty,$$

with $(X^{n+1} + \cdots + X^q)$ a q-Borel ideal, and (2.5) follows.

We have also the following proposition.

PROPOSITION 2.6. *Let F be $(n - 1)$-connected and B simply connected and $F \to T \to B$ a fibre space such that*

(a) *$T^k(F) = H^k(F)$ for $k \leq q - 2$, where $q \leq 2n - 2$.*

(b) *Let $X = X^{n+1} + \cdots + X^{q-1}$, where X^{k+1} is a subspace of $H^{k+1}(B)$ representing $\tau(H^k(F))$, and suppose that (X) is a $(q + 1)$-Borel ideal. Then*

$$T^{q-1}(F) = H^{q-1}(F).$$

Proof. Consider the spectral sequence of the fibre space. We have $H^{q-1}(F) = E_2{}^{0,q-1} \supset \cdots \supset E_{q-n+1}{}^{0,q-1} = E_q{}^{0,q-1} = T^{q-1}(F)$. Therefore we need to show that

$$d_k: E_k{}^{0,q-1} \to E_k{}^{k,q-k}$$

is trivial for $2 \leq k \leq q - n$.

Consider $k = q - n$. Then $E_2{}^{q-n,0} = \cdots = E_{n+1}{}^{q-n,0}$ and, by (a), $E_2{}^{0,n} = \cdots = E_{n+1}{}^{0,n}$; so we have a commutative diagram

$$\begin{array}{ccc} E_{n+1}{}^{q-n,0} \otimes E_{n+1}{}^{0,n} & \xrightarrow{\mu} & E_{n+1}{}^{q-n,n} \\ \downarrow{\scriptstyle 1 \otimes d_{n+1}} & & \downarrow{\scriptstyle d_{n+1}} \\ E_{n+1}{}^{q-n,0} \otimes E_{n+1}{}^{n+1,0} & \xrightarrow{\mu_0} & E_{n+1}{}^{q+1,0}. \end{array}$$

Moreover, $E_{n+1}{}^{q+1,0} = H^{q+1}(B)$. Therefore, by (b), $d_{n+1}(E_{n+1}{}^{q-n,n}) = X^{n+1}H^{q-n}(B)$, and μ is an isomorphism. Since $q - n < n$, $d_{q-n}: E_{q-n}{}^{0,q-1} \to E_{q-n}{}^{q-n,n}$ must be trivial, and $E_{n+2}{}^{q+1,0} = H^{q+1}(B)/X^{n+1}H^{q-n}(B)$.

Now make the following induction hypothesis: for $3 \leq k \leq q - n$,

1) $E_k{}^{0,q-1} = E_{k+1}{}^{0,q-1} = \cdots = E_q{}^{0,q-1} = T^{q-1}(F)$,

and, for $3 \leq k \leq q - n$,

2) $E_{q-k+2}{}^{q+1,0} = H^{q+1}(B)/(X^{n+1}H^{q-n}(B) + \cdots + X^{q-k+1}H^{k}(B))$

Consider then

$$d_{k-1}: E_{k-1}{}^{0,q-1} \rightarrow E_{k-1}{}^{k-1,q-k+1}.$$

We have $E_2{}^{k-1,0} = E_3{}^{k-1,0} = \cdots = E_{q-k+2}{}^{k-1,0}$ and $E_2{}^{0,q-k+1} = E_3{}^{0,q-k+1} = \cdots = E_{q-k+2}{}^{0,q-k+1}$ by (a). Therefore we have a commutative diagram,

$$\begin{tikzcd}
E_{q-k+2}{}^{k-1,0} \otimes E_{q-k+2}{}^{0,q-k+1} \arrow[r, "\mu"] \arrow[d, "1 \otimes d_{q-k+2}"] & E_{q-k+2}{}^{k-1,q-k+1} \arrow[d, "d_{q-k+2}"] \\
E_{q-k+2}{}^{k-1,0} \otimes E_{q-k+2}{}^{q-k+2,0} \arrow[r, "\mu_0"] & E_{q-k+2}{}^{q+1,0}.
\end{tikzcd}$$

Moreover, μ is an epimorphism and, by (b) and (2), $\mu_0(1 \otimes d_{q-k+2})$ is a monomorphism; and therefore μ is an isomorphism.

Now $3 \leq k \leq q - n$, so $k \leq n$; but $q - k \geq n$, hence d_{k-1} is trivial and, by (b), $E_{q-k+3}{}^{q+1,0} = H^{q+1}(B)/(X)^{q+1}$, and the induction is complete.

Given a cohomology class $x \in H^n(Y)$, a *representation* of this class is a mapping $f: Y \rightarrow K(Z_2, n)$ such that $f^*\gamma = x$, where $\gamma \in H^n(K(Z_2, n))$ is the fundamental class. More generally, it is a representation of a set of classes $x_1, \cdots, x_k$, where $x_i \in H^{n_i}(Y)$ is a mapping $f: X \rightarrow C$, where $C = \prod_{i=1}^{k} K(Z_2, n_i)$ such that $f^*\gamma_i = x_i$, for $i = 1, 2, \cdots, k$, and where γ_i is the image in $H^*(C)$ of the fundamental class of $K(Z_2, n_i)$.

Let now $F \xrightarrow{i} T \xrightarrow{p} B$ be a q-totally transgressive fibration, where F is $(n-1)$-connected and B is simply connected. We assume that $q \leq 2n$. Let y_i, $i = 1, \cdots, m$ be a set of generators over the Steenrod algebra A_2, of $H^*(F)$ in dimensions $\leq q - 1$. Let x_i be a representative of τy_i in $H^*(B)$ and let $f: B \rightarrow C$ be a representation of the set $\{x_i\}$. Then f induces a principal fibre space $\Omega C \xrightarrow{i_1} E \xrightarrow{p_1} B$ and we have the following diagram:

(2.7)
$$\begin{tikzcd}
 & & \Omega C \arrow[d, "i_1"] & \\
 & & E \arrow[d, "p_1"] & \\
F \arrow[r, "i"] \arrow[uurr, dashed, "v"] & T \arrow[r, "p"] \arrow[ur, dashed, "h"] & B \arrow[r, "f"] & C .
\end{tikzcd}$$

Since fp is null-homotopic, we can lift p to a mapping $h: T \rightarrow E$ such that $v = h/F: F \rightarrow \Omega C$ is a representation of the set $\{y_i\}$. We now make $(T, F) \rightarrow (E, \Omega C)$ into a fibre pair with fibre F_1.

PROPOSITION 2.8. *In dimensions* $\leq q$, *we have that*

$$\operatorname{Ker} p^* = \operatorname{Ker} p_1{}^*$$

and

$$\text{Ker } \bar{p}^* = \text{Ker } \bar{p}_1^*.$$

Proof. From (2.7), it is clear that Ker $p_1^* \subset$ Ker p^*. Now since $F \xrightarrow{i} T \xrightarrow{p} B$ is q-totally transgressive, we may apply (2.5). With the notation of (2.5), we have that Ker $p^* = (X)$ is a q-Borel ideal. Moreover, Ker p^* is an algebra over $A_2(B)$, in the sense of §1 of [6], and it has, as a system of generators over $A_2(B)$, the elements $x_1, \cdots, x_m$. But $x_i = f^*\gamma_i$, and thus $p_1^*x_i = p_1^*f^*\gamma_i = 0$ implies Ker $p^* \subset$ Ker p_1^*. Now the second part follows since Ker p^* is a q-Borel ideal.

Consider now, from (2.7),

$$\begin{tikzcd} p^*E \arrow[r, "\bar{p}"] \arrow[d] & E \arrow[d, "p_1"] \\ T \arrow[r, "p"] \arrow[ur, "h"] & B \end{tikzcd},$$

where p^*E is the induced principal fibre space over T. Because of the existence of h, p^*E is homeomorphic to $\Omega C \times T$, and we can thus form a commutative diagram,

$$\begin{tikzcd} \Omega C \times T \arrow[r, "\nu"] \arrow[d, "q"] & E \arrow[d, "p_1"] \\ T \arrow[r, "p"] \arrow[ur, "h"] & B \end{tikzcd} \qquad (2.9)$$

And we can change this diagram up to homotopy, so that both ν and p are inclusions and $(E, \Omega C \times T) \to (B, T)$ is a relative fibre space with fibre ΩC.

PROPOSITION 2.10. *In dimensions* $\leq q$, *we have that* ν^* *is a monomorphism.*

Proof. We have that (B, T) and ΩC are both $(n - 1)$-connected, so that we have the Serre-exact sequence,

$$\cdots \to H^m(B, T) \xrightarrow{j^*} H^m(E) \to H^m(\Omega C \times R) \to \cdots$$

in dimensions $\leq 2n - 1$.

Now suppose $x \in (\text{Ker } \nu^* \cap \text{Ker } h^*)^m$, where $m \leq q$. Then, there exists $y \in H^m(B, T)$ with $j^*y = x$. But j^* is the composition $H^m(B, T) \xrightarrow{k^*} H^m(B) \xrightarrow{p_1^*} H^m(E)$, so that $p_1^*(k^*y) = x$. Now $h^*x = h^*p_1^*(k^*y) = p^*(k^*y) = 0$. But, by (2.8), we have $p_1^*(k^*y) = x = 0$.

We may now state the main theorem of this section.

THEOREM 2.11. *Suppose that* $F \xrightarrow{i} T \xrightarrow{p} B$ *is a* q*-totally transgressive fibration, where* B *is simply connected,* F *is* $(n - 1)$*-connected and* $q \leq 2n - 1$. *Then the associated fibre space* $F_1 \to T \xrightarrow{h} E$ *is also* q*-totally transgressive.*

Proof. First note that in (2.7) we have that $F_1 \to F \xrightarrow{v} \Omega C$ is $(q - 1)$-totally transgressive, since v^* is an epimorphism in dimensions $\leq q - 1$, by construction, and both F and ΩC are $(n - 1)$-connected. Therefore we have exact sequences

$$0 \to H^{k-1}(F_1) \to H^k(\Omega C) \xrightarrow{v^*} H^k(F) \to 0,$$

for $k \leq q - 1$, and

$$0 \to H^{q-1}(F_1) \to H^q(\Omega C) \to H^q(F) \to \cdots .$$

Using the cohomology sequence of the pair $(\Omega C, F)$, we obtain

$$\tau : H^{k-1}(F_1) \cong H^k(\Omega C, F), \quad \text{for} \quad k \leq q. \tag{2.12}$$

Again referring to diagram (2.7), we next show that $i_1 : \Omega C \to E$ induces a homomorphism

$$i_1^* : (\operatorname{Ker} h^*/j_2^*(\operatorname{Ker} \bar{h}^*))^k \to H^k(\Omega C, F), \tag{2.13}$$

for $k \leq q$, where $j_2 : E \to (E, *)$, is the inclusion. From $i_1 v = hi$, it follows that $v^* i_1^*(\operatorname{Ker} h^*) = 0$; so $i_1^*(\operatorname{Ker} h^*) \subset H^*(\Omega C, F)$. Moreover, in the diagram

$$\begin{tikzcd} H^k(E, *) \arrow[r, "\bar{h}^*"] \arrow[d, "\bar{i}_1"] & H^k(T, F_1) \arrow[d, "\bar{i}^*"] \\ H^k(\Omega C, *) \arrow[r, "\bar{v}^*"] & H^k(F, F_1), \end{tikzcd}$$

we have that $\bar{v}^*$ is an isomorphism for $k < 2n$. Therefore, $\bar{h}^*(x) = 0$ implies $\bar{i}_1^* x = 0$, and, therefore, $i_1^*(j_2^* \operatorname{Ker} \bar{h}^*) = 0$. Now (2.12) and (2.13) give a commutative diagram,

$$\begin{tikzcd} T^{k-1}(F_1) \arrow[r, "\tau_1", "\cong"'] \arrow[d, "\alpha"] & (\operatorname{Ker} h^*/j_2^*(\operatorname{Ker} \bar{h}^*))^k \arrow[d, "i_1^*"] \\ H^{k-1}(F_1) \arrow[r, "\tau_1", "\cong"'] & H^k(\Omega C, F), \end{tikzcd} \tag{2.14}$$

for $k \leq q$, where α is the inclusion.

We first prove that $F_1 \to T \to E$ satisfies, for $k \leq q$,

(c) $h^* : H^k(E) \to H^k(T)$ is onto, and

(d) $H^{k-1}(F_1) \cong T^{k-1}(F_1)$.

The assertion (c) follows clearly from the hypothesis and the fact that $p = p_1 h$.

To prove (d) it suffices to show that i_1^* in (2.14) is an epimorphism, for $k \leq q$. The fibrations $\Omega C \to E \xrightarrow{p_1} B$ and $F \to T \to B$ give a commutative diagram,

$$\begin{tikzcd} H^{k-1}(\Omega C) \arrow[r, "\tau'"] \arrow[d, "v^*"] & (\operatorname{Ker} p_1^*/j_1^*(\operatorname{Ker} \bar{p}_1^*))^k \arrow[d, "\cong"] \\ H^{k-1}(F) \arrow[r, "\tau", "\cong"'] & (\operatorname{Ker} p^*/j_1^*(\operatorname{Ker} \bar{p}^*)), \end{tikzcd} \tag{2.15}$$

where the vertical isomorphism follows from (2.8) and the horizontal one, from the hypothesis and (1.5). Now take $x \in H^{k-1}(\Omega C, F)$; then its image, $y \in H^{k-1}(\Omega C)$, is non-zero if x is non-zero. Now $v^*y = 0$, so $\tau'(y) = 0$, by (2.15). But by (1.5) we have

$$H^{k-1}(\Omega C)/i_1^*H^{k-1}(E) \overset{\tau'}{\cong} (\mathrm{Ker}\ p_1^*/j_1^*(\mathrm{Ker}\ \bar{p}_1^*))^k,$$

and therefore $y = i_1^*z$, where $z \in H^{k-1}(E)$. If $h^*z = 0$, we are finished, for then $i_1^*[z] = x$. However, if $h^*z \neq 0$, choose $w \in H^{k-1}(B)$, with $p^*w = h^*z$; so $h^*(z - p_1^*w) = 0$ and $i_1^*(z - p_1^*w) = i_1^*z = y$, and thus $i_1^*[z - p_1^*w] = x$. Therefore (d) is satisfied.

In order to prove (2.11) it suffices to show that

$$(2.16) \qquad T^{q-1}(F_1) = H^{q-1}(F_1).$$

Notice that condition (d) is condition (a) of (2.6). Let $X = X^{n+1} + \cdots + X^{q-1}$ be as in (2.6) for the fibre space $F_1 \to T \to E$. Then we need to verify that (X) is a $(q-1)$-Borel ideal in $H^*(E)$.

Now notice that $H^*(\Omega C \times T)$ is an $A_2(B)$-algebra. Take a basis $w_1, \cdots, w_r$ of X with $\dim w_i \leq \dim w_{i+1}$. Then in (2.9) we have, since $\dim w_i \leq q - 1 \leq 2n - 1$,

$$(2.17) \qquad \nu^*(w_i) = \sum^m (\alpha_{ij} + \beta_{ij})\gamma_j \neq 0,$$

where $\alpha_{ij} \in A \times 1$ is non-zero for at least one j, $\beta_{ij} \in A \otimes H^+(B)$, and $H^+(B)$ denotes the positive elements of $H^*(B)$. Moreover, the elements ν^*w_i and $\sum \alpha_{ij}\gamma_j = i_1^*w_i$ are linearly independent in $H^*(\Omega C \times T)$ and $H^*(\Omega C)$ respectively because of (2.10) and (2.12). Therefore the elements $i_1^*w_i$ satisfy no relation

$$\sum i_1^*w_i \times t_i \to 0, \qquad t_i \in H^*(T)$$

in $\bar{H}^*(\Omega C) \otimes H^*(T)$, which is a free $A_2(B)$-module in dimensions $\leq q + 1$ on generators $\gamma_i \times 1$, $i = 1, \cdots, m$.

Now we cannot have a relation

$$(2.18) \qquad \sum_{i=1}^s a_i w_i = 0 \qquad \text{in dimensions} \qquad \leq q + 1,$$

where the w_i all have the same dimension p, because, if we apply (2.17), we obtain

$$\nu^*(\textstyle\sum_{i=1}^s a_i w_i) = \sum_{i=1}^s a_i(\sum_{j=1}^m (\alpha_{ij} + \beta_{ij})\gamma_j) = 0;$$

and, since $\beta_{ij} \in A \otimes H^+(B)$, this implies

$$\sum_{i=1}^s i^*w_i \times a_i = 0.$$

Also, we cannot have a relation

$$(2.19) \qquad \sum_{i=1}^s a_i w_i = \sum_{k=1}^t b_k w_k,$$

where $\dim w_i = p$ and $\dim w_k < p$ for all $k = 1, 2, \cdots, t$, because, again upon application of (2.17), we would obtain

$$\sum_{i=1}^{s,m} {}_{j=1} a_i(\alpha_{ij} + \beta_{ij})\gamma_j = \sum_{k=1}^{t,s} {}_{j=1} b_k(\alpha_{kj} + \beta_{kj})\gamma_j$$

or

$$\sum_{i=1}^{s} a_i(\alpha_{ij} + \beta_{ij})\gamma_j = \sum_{k=1}^{t} b_k(\alpha_{kj} + \beta_{kj})\gamma_j,$$

for $j = 1, \cdots, m$. Now $\dim \alpha_{kj} < \dim \alpha_{ij}$ for each k and each i, and

$$\beta_{jk} \in A \otimes H^+(B);$$

so we cannot have a relation among the $\alpha_{ij}\gamma_j$ and the $\beta_{kj}\gamma_j$. Therefore we would obtain

$$\Sigma a_i w_i = 0, \qquad \dim w_i = p,$$

which contradicts the previous statement (2.18) unless the $a_i = 0$.

Now clearly $a_1 w_1 = 0$ implies $a_1 = 0$, and (2.18) and (2.19) carry the induction to show (X) is a $(q + 1)$-Borel ideal. Thus (2.16) holds and the proof of (2.11) is complete.

COROLLARY 2.20. *Let $F \xrightarrow{i} T \xrightarrow{p} B$ be a q-totally transgressive fibration, where F is $(n - 1)$-connected, B is simply connected and $q \leq 2n - 1$. Then there exists a sequence of principal fibrations*

$$\rightarrow E_k \xrightarrow{p_k} E_{k-1} \rightarrow \cdots \rightarrow E_1 \xrightarrow{p_1} E_0 = B$$

with ΩC_k, the fibre of p_k, a product of $K(Z_2, m), m \leq q - 1$, and fibre maps $h_k: T \rightarrow E_k$ with fibres F_k, where $F_0 = F$, $h_0 = p$ which are q-totally transgressive. Moreover $p_k h_k = h_{k-1}$ and $F_k \rightarrow F_{k-1} \rightarrow \Omega C_k$ are $(q - 1)$-totally transgressive fibrations.

3. Modified Postnikov towers

Let $F \xrightarrow{i} T \xrightarrow{p} B$ be a fibre space. Then a modified Postnikov tower through dimension t of this fibre space, in short a t-M.P.T. (see [8], §2), is a sequence of fibre spaces,

$$E_n \xrightarrow{q_n} E_{n-1} \rightarrow \cdots \rightarrow E_1 \xrightarrow{q_1} B,$$

and maps $p_i: T \rightarrow E_i$ such that

(3.1) $q_i p_i = p_{i-1}$;

(3.2) the fibre of q_i, C_i is a product of Eilenberg-Maclane spaces $K(\Pi, k)$, where $\Pi = Z$ or Z_p, where p is a prime and $k < t$; and

(3.3) the fibre of p_i, F_i is $t(i)$-connected, where $t(n) \geq t - 1$, and if $i: F_{i+1} \subset F_i$, then $i^*: H^k(F_i) \rightarrow H^k(F_{i+1})$ is trivial for $k \leq t$.

Let $[X, Y]$ denote the set of homotopy classes of maps $X \rightarrow Y$. It follows from (3.1), (3.2) and (3.3) that if X is any CW-complex of dimensions $\leq t - 1$, then $[X, T] \cong [X, E_n]$; and if dimension of X is t, then $[X, T] \xrightarrow{p_n^*} [X, E_n]$ is onto.

Since the fibre space $C_m \rightarrow E_m \xrightarrow{q_m} E_{m-1}$ is a principal fibre space, it is classified by a map $f_m: E_{m-1} \to D_m$, where $C_m = \Omega D_m$. If $D_m = \Pi K(\Pi_j, j)$, let $\gamma_j^m \in H^j(D_m, \Pi_j)$ be the image of the fundamental class of $K(\Pi_j, j)$. The k-invariants of the fibration $C_m \to E_m \to E_{m-1}$ is the set of cohomology classes $\{k_j^m\}$, where $k_j^m = f_m^* \gamma_j^m$. A mapping $g: X \to E_{m-1}$ can be lifted to E_m if and only if $g^* k_j^m = 0$, for all j.

Now we show that under special circumstances a t-M.P.T. can be constructed in a manner slightly different from that given in [8].

Let us take Z_2 as coefficients and assume that in the fibre space $F \xrightarrow{i} T \xrightarrow{p} B$ the fibre F is $(n-1)$-connected and that $H^*(F)$ is free over the algebra generated by Sq^0 and Sq^1 in dimensions $\leq t - 1 \leq 2n - 2$.

THEOREM 3.4. *Suppose $F \xrightarrow{i} T \xrightarrow{p} B$ is t-totally transgressive, then it has a t-modified Postnikov tower.*

Proof. Consider the sequence of fibre spaces

$$E_m \xrightarrow{q_m} E_{m-1} \to \cdots \to E_1 \xrightarrow{q_1} B \tag{3.5}$$

constructed in Theorem (2.20). Then we have that $C_k \to E_k \xrightarrow{q_k} E_{k-1}$ is a principal fibre space, with C_k a product of Eilenberg-Maclane spaces $K(Z_2, m)$, with $m \leq t - 1$. Moreover, we have fibre spaces $F_k \to T \xrightarrow{p_k} E_k$ and

$$F_k \to F_{k-1} \xrightarrow{v_{k-1}} C_{k-1},$$

which are respectively t-totally transgressive and $(t-1)$-totally transgressive. In particular we have exact sequences

$$0 \to H^{q-1}(F_k) \xrightarrow{\tau} H^q(C_{k-1}) \xrightarrow{v_k^*} H^q(F_{k-1}) \to 0,$$

through dimension $t - 1$.

Moreover, v_k was chosen so that

$$0 \to \Pi_{n_k}(F_{k+1}) \to \Pi_{n_k}(F_k) \xrightarrow{v_k^*} \Pi_{n_k}(C_{k+1}) \to 0,$$

where n_k is the least integer $n \leq n_k \leq t - 1$, with $\Pi_{n_k}(F_k) \neq 0$. Therefore, starting with $F_0 = F$ and $n_0 = n$, we have a sequence of proper inclusions

$$\Pi_n(F_k) \subset \Pi_n(F_{k-1}) \subset \cdots \subset \Pi_n(F),$$

and since $\Pi_n(F)$ is finite by hypothesis, this sequence must terminate in zero; i.e., there exists k_1 with $n_{k_1} \geq n + 1$. If we continue this process, we obtain an integer m, so that (3.5) is a t-modified Postnikov tower of $F \xrightarrow{i} T \xrightarrow{p} B$.

4. Determination of the k-invariants

In this section we recall the connection between successive k-invariants in a modified Postnikov tower. We assume that all the cohomology groups are taken with Z_2-coefficients.

Let $w: B \to C$ be a map and $p: E \to B$, the principal fibre space induced by w with fibre ΩC. A mapping $f: X \to B$ can be lifted to a mapping $g: X \to E$ if and only if wf is null-homotopic. Now such a g is not unique, in fact, given any map $h: X \to \Omega C$, we obtain another lifting, $g': X \to E$, as the composite

$$(4.1) \qquad X \xrightarrow{d} X \times X \xrightarrow{h \times g} \Omega C \times E \xrightarrow{\mu} E,$$

where d is the diagonal mapping and μ is the multiplication given by the principal structure. Moreover, up to homotopy, all liftings $X \to E$ can be obtained from g as we vary h in the homotopy classes of maps $X \to \Omega C$, (see [11], (2.9)).

Now let $w_1, \cdots, w_m$ be classes in $H^*(B)$, with dim $w_k = q_k + 1$, and let $w: B \to C$ be a representation of this set of classes. Form the associated principal fibre space $p: E \to B$. Then a mapping $f: X \to B$ lifts to E if and only if $f^* w_k = 0$, for $k = 1, \cdots, m$. Let $y \in H^q(E)$, where $q < 2 \min (q_k)$. Then, as we vary the liftings $g: X \to E$ of $f: X \to B$, we obtain a set of classes $g^* y$ in $H^q(X)$. We denote this set by $\Phi(y, f)$. Thus $\Phi(y, f) \subset H^q(X)$. We begin by determining the structure of this set.

Let $A_2(B)$ be the split extension algebra of A_2 and $H^*(B)$ as considered in §1 of [6]. Since we have, $\Omega C \times E \xrightarrow{\mu} E \xrightarrow{p} B$, $H^*(E)$ and $H^*(\Omega C \times E)$ are A_2-(B)-modules. Consider the mapping $\mu^*: H^*(E) \to H^*(\Omega C \times E)$; then we can write

$$(4.2) \qquad \mu^*(y) = 1 \otimes y + i^* y \otimes 1 + \Sigma y_i' \otimes y_i''$$

where $i: \Omega C \to E$ is the inclusion and the $y_i' \times y_i''$ are positive dimensional classes of $H^*(\Omega C)$ and $H^*(E)$, respectively. Let γ_k, for $k = 1\ 2, \cdots, m$, be the fundamental classes of ΩC, and let $\bar{H}^*(\Omega C)$ be the subspace of positive dimensional classes of $H^*(\Omega C)$. Then it is easy to see that in dimensions $< 2 \min (q_k) = 2 \min (\dim \gamma_k)$, $\bar{H}^*(\Omega C) \otimes H^*(E)$ is a free $A_2(B)$-module on generators $\gamma_1, \cdots, \gamma_m$. Therefore (4.2) can be rewritten as

$$(4.3) \qquad \mu^* y = 1 \otimes y + \textstyle\sum_{k=1}^m (\alpha_k + \beta_k) \gamma_k,$$

where the α_k are elements of $A_2 \otimes 1 \subset A_2(B)$ and the β_k are elements of $A_2 \otimes \bar{H}^*(B)$.

The mapping $f: X \to B$ induces $f^*: A_2(B) \to A_2(X)$. Let $g: X \to E$ be a lifting of $f: X \to B$ and $g' = \mu(h \times g)d$, another such lifting; then it follows that

$$(4.4) \qquad g^*(y) - g'^*(y) = \textstyle\sum_{k=1}^m (f^* \alpha_k + f^* \beta_k) h^* \gamma_k ;$$

but $f^* \alpha_k = \alpha_k$.

Therefore if we let

$$(4.5) \qquad Q(y, f) = \textstyle\sum_{k=1}^m (\alpha_k + f^* \beta_k) H^{p_k}(X)$$

be the subgroup generated by all the elements of the form $\alpha_k x + f^* \beta_k x$, for all $x \in H^{p_k}(X)$ and $k = 1, \cdots, m$, we have obtained the following proposition.

PROPOSITION 4.6. *The set $\Phi(y, f)$ is actually a coset, namely the coset of $g^*(y)$ in $H^q(X)$ modulo $Q(y, f)$, where g is any lifting $X \to E$ of $f: X \to B$.*

Because of (4.6), $Q(y, f)$ is called the *indeterminacy of y induced by f*.

Now let us consider a t-totally transgressive fibration $F \to T \xrightarrow{q} B$, where F is

$(n-1)$-connected and $t \leq 2n-1$. Form, as in (2.20), the principal fibre space $\Omega C \to E \xrightarrow{p} B$, and let $h:T \to E$ be a lifting of q, with fibre F_1. Then $F_1 \to T \xrightarrow{h} E$ is again t-totally transgressive. Let $\mu:\Omega C \times E \to E$ be the multiplication in the principal structure. We have then the following theorem.

THEOREM 4.7. *Let $x \in H^{q-1}(F_1)$, where $q < t-1$; then $\tau(x) \neq 0$ and, for any class $y \in H^q(E)$ representing $\tau(x)$, we hav*

$$\mu^*(y) = 1 \otimes y + \textstyle\sum_{k=1}^{m} (\alpha_k + \beta_k)\gamma_k, \tag{4.8}$$

where $\alpha_k \in A_2 \otimes 1$, $\beta_k \in A_2 \otimes \bar{H}^*(B)$, and the α_k only depend on the coset $\tau(x)$. Moreover

$$\textstyle\sum_{k=1}^{m} (\alpha_k + \beta_k)\cdot w_k = 0 \tag{4.9}$$

is a relation in $H^*(B)$, where the w_k induce the principal structure $E \to B$. Conversely, given the relation (4.9), there is a unique class $y \in H^q(E)$ with $h^*(y) = 0$ and $\mu^*(y)$ satisfying (4.8). Furthermore, if we vary the elements β_k but leave the α_k fixed in (4.9), the corresponding element $y' \in H^q(E)$ differs from y by an element of ker $\bar{h}^*$.

Theorem 4.7 is the precise relation between the successive k-invariants in a t-modified Postnikov tower.

Proof. Since $F_1 \to T \xrightarrow{h} E$ is t-totally transgressive, $\tau(x) \neq 0$. Now, we are in the situation of (4.3); therefore, for $y \in \tau(x)$, $\mu^*(y)$ has the form (4.8). Since F_1 is the fibre of the relative fibre space $(T, F) \to (E, \Omega C)$, it follows that if $y, y' \in \tau(x)$, $i_1^*y = i_1^*y'$ and the α_k in (4.8) only depend on the coset $\tau(x)$.

To see that (4.9) follows, we note that we have the following commutative diagram

$$\begin{array}{ccc} \Omega C \times E & \xrightarrow{\mu} & E \\ \downarrow{\scriptstyle\pi} & & \downarrow{\scriptstyle p} \\ E & \xrightarrow{p} & B \end{array}$$

where π is the projection onto the second factor. We can make the above diagram into a fibre pair $(E, \Omega C \times E) \to (B, E)$ with fibre ΩC (all up to homotopy type). But both ΩC and (B, E) are $(n-1)$-connected and μ^* is a monomorphism, so that we obtain short exact sequences,

$$0 \to H^q(E) \xrightarrow{\mu^*} H^q(\Omega C \times E) \xrightarrow{\tau_0'} H^{q+1}(B, E) \to 0.$$

Let $j:B \to (B, E)$ be the inclusion. Then $j^*\tau_0'\mu^*(y) = 0$; but this is relation (4.9), as can be easily checked. Now we prove the converse. Recall from (2.9), that we have a fibre pair $(E, \Omega C \times T) \to (B, T)$ with fibre ΩC. Consider the element

$$\sigma = \textstyle\sum_{k=1}^{m} (\alpha_k + q^*\beta_k)\gamma_k \quad \text{in} \quad H^q(\Omega C \times T).$$

We have the exact sequence

$$\to H^q(E) \xrightarrow{\nu^*} H^q(\Omega C \times T) \xrightarrow{\tau_0} H^{q+1}(B, T) \to H^{q+1}(E) \to,$$

and we now show that $\tau_0\sigma = 0$. Let $j_0: B \to (B, T)$ be the inclusion; then j_0^* is a monomorphism in dimensions $\leq t$, and $j_0^*\tau_0\sigma = \Sigma(\alpha_k + \beta_k)\cdot w_k = 0$. Therefore there exists $y \in H^q(E)$, with $\nu^* y = \sigma$. Thus in particular $h^* y = 0$, and, hence, by (2.10), y is unique. Now $\nu^* = (1 \times h)^*\mu^*$, and $(1 \times h)^*$ is an isomorphism $\bar{H}^*(\Omega C) \otimes H^*(E) \to \bar{H}^*(\Omega C) \otimes H(T)$ in dimensions $\leq t - 1$. Therefore $\mu^* y$ satisfies (4.8).

Finally, suppose that

$$\sum_{k=1}^{m} (\alpha_k + \beta_k')w_k = 0$$

is another relation in $H^*(B)$. Let y' be the unique element in $H^q(E)$ associated with this relation. It is clear that $i_1^*(y - y') = 0$. Now, in the fibration $F_1 \to T \to E$, if we let x, x' be the unique classes in $H^{q-1}(F_1)$, with $\tau(x) = [y]$ and $\tau(x') = [y']$, we have $i_1^*\tau(x - x) = \tau_{F_1}(x - x') = 0$, where τ_{F_1} is the transgression in the fibration $F_1 \to F \to \Omega C$; but τ_{F_1} is a monomorphism, so that $x = x'$ and $y - y' \in \operatorname{Ker} \bar{h}^*$. Therefore the proof of (4.7) is complete.

We finally mention how one determines the relations (4.9). Suppose that we have two steps

$$\begin{tikzcd} E_2 \arrow[r, "p_2"] & E_1 \arrow[r, "p"] & B \\ & T \arrow[ul, "h_2"] \arrow[u, "h_1"] \arrow[ur, "q"'] & \end{tikzcd},$$

where h_i has fibre F_i and p_i has fibre ΩC_i, for $i = 1, 2$. Assume that we know the k^1-invariants, i.e., that we know $\mu_1^*(k_r^1)$ and, therefore, $\nu_1^*(k_r^1)$. Take now an element $x \in H^q(F_2)$; then in the fibre space $F_2 \to F_1 \to \Omega C_2$, $\tau_{F_1}(x) = \Sigma\alpha_k\gamma_k$. Therefore it follows from (4.9) that the element

$$\sum_{r=1}^{m} \alpha_r k_r^1 \in H^{q+1}(E_1)$$

can be completed to a relation of the form

$$\sum_{r=1}^{m} \alpha_r k_r^1 + \sum_{r=1}^{m} \beta_r k_r^1 = 0, \tag{4.10}$$

where the $\beta_r \in A \otimes \bar{H}^*(B)$. Since the elements in the $A_2(B)$-module generated by the k_r^1, $r = 1, \cdots, m$ are in the Kernel of h_1^*, it follows from (2.10) that it suffices to complete $\sum_{r=1}^{m} \alpha_r\nu_1^*(k_r^1)$ to a relation in $H^*(\Omega C_1 \times T)$. This we accomplish if we assume that we know the structure of $H^*(T)$ as an A_2-module. The relation must be of the form

$$\sum_{r=1}^{m} \alpha_r\nu_1^*(k_r^1) + \Sigma\beta_r\nu_1^*(k_r^1) = 0, \tag{4.11}$$

where the $\beta_r \in A \otimes \bar{H}^*(B)$; therefore (2.10) implies that (4.10) follows from (4.11).

This process will be clarified when we actually apply it to the determination of the k-invariants for the fibration $BSO_{8k-1} \to BSO$ in §6.

5. Resolutions over the Steenrod algebra

Let F be an $(n - 1)$-connected space, and let t be an integer such that $t < 2n$. Let

$$(5.1) \qquad 0 \leftarrow H^*(F) \leftarrow C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \leftarrow \cdots \xleftarrow{d_r} C_r \leftarrow \cdots$$

be an A_2-free resolution of $H^*(F)$ in dimensions $\leq t$. We assume all maps in the resolution to be of degree zero.

With (5.1) there is associated a sequence of spaces and maps

$$(5.2) \qquad \to F_{r+1} \xrightarrow{p_{r+1}} F_r \to \cdots \xrightarrow{p_1} F_0 = F$$

such that

$$F_{k+1} \xrightarrow{p_{k+1}} F_k \xrightarrow{q_k} K_k$$

is a fibre space

$$C_k \cong H^*(K_k)$$

through $2n - 1$ dimensions as A_2-modules, with a shift in dimension of $-k$, and

$$q_k^*: H^*(K_k) \to H^*(F_k)$$

is an epimorphism in dimensions $\leq t$. In addition, the short exact sequences

$$(5.3) \qquad 0 \to H^*(F_{k+1}) \xrightarrow{\tau_k} H^*(K_k) \xrightarrow{q_k^*} H^*(F_k) \to 0$$

are the short exact sequences that correspond to (5.1) under $C_k \cong H^*(K_k)$.

Indeed, K_k is a product of Eilenberg-Maclane spaces of type $K(Z_2, m)$ one for each A_2-generator of C_k and in the dimension of that generator minus k.

It is now clear that $\epsilon: C_0 \to H^*(F)$ determines a mapping $g_0: F \to K_0$ and F_1 is the fibre of q_0. Since $q_0^* = \epsilon$ is onto, we have the short exact sequence of the fibre space $F_1 \xrightarrow{p_1} F \xrightarrow{q_0} K_0$ in dimensions $\leq t$. Therefore, from the following diagram,

$$\begin{tikzcd}
 & 0 \arrow[r] & H^*(F_1) \arrow[r, "\tau_0"] & H^*(K_0) \arrow[r, "q_0^*"] & H^*(F) \arrow[r] & 0 \\
C_2 \arrow[r, "d_2"] & C_1 \arrow[rr, "d_1"] \arrow[ur, phantom, ""] & & C_0 \arrow[r, "\epsilon"] & H^*(F) \arrow[r] & 0
\end{tikzcd}$$

$$\begin{array}{ccccccc} 0 \to & H^*(F_1) & \xrightarrow{\tau_0} & H^*(K_0) & \xrightarrow{q_0^*} & H^*(F) & \to 0 \\ & \uparrow \psi_1 & & \varphi_0 \uparrow \cong & & \uparrow = & \\ C_2 \xrightarrow{d_2} & C_1 & \xrightarrow{d_1} & C_0 & \xrightarrow{\epsilon} & H^*(F) & \to 0 \end{array}$$

it follows that we can define $\psi_1: C_1 \to H^*(F_1)$ of degree -1, making the left square commutative. In turn, ψ_1 determines a mapping $q_1: F_1 \to K_1$. Note that ψ_1 is an epimorphism, so that we can continue the process described.

Under reasonable circumstances, in the sequence (5.2) the 2-torsion in the homotopy groups in dimensions $\leq t$ is being killed.

For instance we have the following result, essentially due to Adams (see [1], Th. 3, p. 62). Let $\mathcal{E}$ be the exterior algebra generated by Sq^1; then $\mathcal{E} \subset A_2$, and we have the following proposition.

PROPOSITION 5.4. *Let F be an $(n-1)$-connected space such, that in dimensions less than or equal to t, where $t < 2n$, we have $H^*(F; Zp) = 0$, for p an odd prime, and $H^*(F)$ is $\mathcal{E}$-free. Then F_r is $\min(t, n-1+q(r))$-connected, where $q(r)$ is given by*

$$q(r) = \begin{cases} 2r & \text{if } r \equiv 0 \mod 4 \\ 2r-1, & \text{if } r \equiv 1 \mod 4 \\ 2r-2, & \text{if } r \equiv 2, 3 \mod 4. \end{cases}$$

For our main application, we will work with $V_{n+k,k}$. There is a sharper form of (5.4) for these, which depends on k and is given as Proposition (7.12).

We now use the resolutions in the situation envisaged in Theorem (3.4) as follows. Given an A_2-free resolution of $H^*(F)$ as in (5.1), the fibres of $p_k: T \to E_k$ can then be chosen to be the F_k constructed from the resolution as above. This gives us two things. First, it tells us how rapidly they will converge to E; second, it gives us the information required to obtain the relations among the k^i-invariants which produce the k^{i+1}-invariants as in (4.7).

Namely, assume that we know $\nu_j^*(k^i)$; then look at $d_{i+1}: C_{i+1} \to C_i$. Let c be an A_2-generator of C_{i+1}, so that $d_{i+1}c = \alpha_k' c_k'$, where $\alpha_k' \in A_2$ and the c_k' are the A_2-generators of C_i. Then take $\nu_i^*(\alpha_k' k_k^i)$ and add to it in order to make a relation in $\mathrm{Im}(\nu_i^*)$. This gives a relation among the k^i, as in (4.7), which produces the k^{i+1}.

6. Applications to vector bundles over real projective spaces

To illustrate the techniques developed earlier in this paper, we will outline how one determines the k-invariants of $V_{8k+3} \to BSO_{8k+3} \to BSO$ and give an application to RP^n, the real projective n-space.

First note that the homotopy of V_{8k+3} is finite and has only two torsion in dimensions less than $16k+6$. Moreover, $H^*(V_{8k+3}) = H^*(RP^\infty/RP^{8k+2})$ in the same range. We start by constructing a resolution of $H^*(RP^\infty/RP^{8k+2})$ in dimensions $\leq 8k+8$,

$$0 \leftarrow H^*(RP^\infty/RP^{8k+2}) \xleftarrow{\epsilon} C_0 \xleftarrow{d_1} C_1 \xleftarrow{d_2} C_2 \xleftarrow{d_3} C_3 .$$

Using the action of A_2 in $H^*(RP^\infty)$, it is not hard to show that we can take C_0 to have generators a_1, of dimension $8k+3$ and a_2, of dimension $8k+7$. Now, C_1 has three generators: b_1, of dimension $8k+6$; b_2, of dimension $8k+7$; and b_3, of dimension $8k+8$. Moreover, $d_1b_1 = \mathrm{Sq}^2\mathrm{Sq}^1a_1$, $d_1b_1 = \mathrm{Sq}^4a_1$, $d_1b_3 = \mathrm{Sq}^1a_2 + \mathrm{Sq}^4\mathrm{Sq}^1a_1$. Similarly, C_2 has two generators: e_1, of dimension $8k+8$, and e_2, of dimension $8k+9$. The differential operator is given by $d_2e_1 = \mathrm{Sq}^2b_1$, $d_2e_2 = \mathrm{Sq}^1b_3 + \mathrm{Sq}^1\mathrm{Sq}^2b_1$. Now, C_3 has one generator f, of dimension $8k+10$, and $d_3f = \mathrm{Sq}^1e_2 + \mathrm{Sq}^2e_1$.

Therefore we must have two k'-invariants, three k^2-invariants, two k^3-invariants, and one k^4-invariant.

It is clear that the k^1-invariants are

$$k_1^1 = W_{8k+4} \quad \text{and} \quad k_2^1 = W_{8k+8}\,. \tag{6.1}$$

Now, to determine the k^2-invariants, we use d_1 and the Wu formulae to obtain the following relations:

$$\mathrm{Sq}^2\mathrm{Sq}^1 W_{8k+4} = W_2 W_{8k+8} = (W_2\mathrm{Sq}^1)W_{8k+4}\,;$$

$$\mathrm{Sq}^4 W_{8k+4} = W_4 W_{8k+4}\,; \quad \text{and}$$

$$\mathrm{Sq}^1 W_{8k+8} + \mathrm{Sq}^4\mathrm{Sq}^1 W_{8k+4} = (W_4 + W_2^2)W_{8k+5} + (W_3\mathrm{Sq}^2 + W_2\mathrm{Sq}^3)W_{8k+4}\,.$$

Therefore the relations that produce the k^2-invariants are the following:

$$\begin{aligned} &k_1^2 : (\mathrm{Sq}^2 + W_2)\mathrm{Sq}^1 k_1^1 = 0; \quad k_2^2 : (\mathrm{Sq}^4 + W_4)k_1^1 = 0; \quad \text{and} \\ &k_3^2 : \mathrm{Sq}^1 k_2^1 + [(\mathrm{Sq}^4 + W_4 + W_2^2)\mathrm{Sq}^1 + \mathrm{Sq}^1(W_2\mathrm{Sq}^2)]k_1^1 = 0. \end{aligned} \tag{6.2}$$

Now, to determine the k^3-invariants, we consider

$$\begin{aligned} \nu_2^*(k_1^2) &= \mathrm{Sq}^2\mathrm{Sq}^1\gamma_1 \otimes 1 + \mathrm{Sq}^1\gamma_1 \otimes W_2\,, \\ \nu_2^*(k_2^2) &= \mathrm{Sq}^4\gamma_1 \otimes 1 + \gamma_1 \otimes W_4\,, \text{ and} \\ \nu_2^*(k_3^2) &= \mathrm{Sq}^1\gamma_2 \otimes 1 + \mathrm{Sq}^4\mathrm{Sq}^1\gamma_1 \otimes 1 + \mathrm{Sq}^1\gamma_1 \\ &\qquad \otimes (W_4 + W_2^2) + \mathrm{Sq}^1(\mathrm{Sq}^2\gamma_1 \otimes W_2) \end{aligned}$$

And we use $d_2 : C_2 \to C_1$ in order to know that

$$\nu_2^*(\mathrm{Sq}^2 k_1^2) = \mathrm{Sq}^2\mathrm{Sq}^1\gamma_1 \otimes W_2 + \mathrm{Sq}^1\gamma_1 \otimes W_2^2$$

satisfies a relation in $\mathrm{Im}(\nu_2^*)$, namely,

$$\nu_2^*(W_2 k_1^2) = \mathrm{Sq}^2\mathrm{Sq}^1\gamma_1 \otimes W_2 + \mathrm{Sq}^1\gamma_1 \otimes W_2\,,$$

so that $\nu_2^*((\mathrm{Sq}^2 + W_2)k_1^2) = 0$ and, hence,

$$k_1^3 : (\mathrm{Sq}^2 + W_2)k_1^2 = 0. \tag{6.3}$$

Similarly,

$$\nu_2^*(\mathrm{Sq}^1 k_3^2 + \mathrm{Sq}^2\mathrm{Sq}^1 k_1^2) = \mathrm{Sq}^2\mathrm{Sq}^1\gamma_1 \otimes W_3 + \mathrm{Sq}^1\gamma_1 \otimes W_2 W_3$$

satisfies a relation in $\mathrm{Im}(\nu_2^*)$; namely, we have

$$\nu_2^*(W_3 k_1^2) = \mathrm{Sq}^2\mathrm{Sq}^1\gamma_1 \otimes W_3 + \mathrm{Sq}^1\gamma_1 \otimes W_2 W_3\,,$$

so that $\nu_2^*[\mathrm{Sq}^1 k_3^2 + (\mathrm{Sq}^2\mathrm{Sq}^1 + W_3)k_1^2] = 0$ and, hence,

$$k_3^3 : \mathrm{Sq}^1 k_3^2 + (\mathrm{Sq}^2\mathrm{Sq}^1 + W_3)k_1^2 = 0. \tag{6.4}$$

In the same way, one easily shows that

$$k_1^4 : \mathrm{Sq}^1 k_2^3 + (\mathrm{Sq}^2 + W^2)k_1^3 = 0. \tag{6.5}$$

We can thus write a table for the k-invariants.

TABLE OF k-INVARIANTS FOR $BSO_{8k+3} \to BSO$ (IN DIMENSIONS $\leq 8k + 8$)

$$k_1^1 = W_{8k+4}, \quad k_2^1 = W_{8k+8}$$

$$k_1^2 : (\mathrm{Sq}^2 + W_2)\mathrm{Sq}^1 k_1^1 = 0, \quad k_2^2 : (\mathrm{Sq}^4 + W_4)k_1^1 = 0$$

$$k_3^2 : \mathrm{Sq}^1 k_2^1 + [(\mathrm{Sq}^4 + W_4 + W_2^2)\mathrm{Sq}^1 + \mathrm{Sq}^1(W_2\mathrm{Sq}^2)]k_1^1 = 0$$

$$k_1^3 : (\mathrm{Sq}^2 + W_2)k_2^2 = 0; \quad k_2^3 : \mathrm{Sq}^1 k_3^2 + (\mathrm{Sq}^2\mathrm{Sq}_1 + W_3)k_1^2 = 0$$

$$k_1^4 : \mathrm{Sq}^1 k_2^3 + (\mathrm{Sq}^2 + W_2)k_1^3 = 0$$

In a similar way, one obtains the following table.

TABLE OF k-INVARIANTS FOR $BSO_{8k+7} \to BSO$ (IN DIMENSIONS $\leq 8k + 12$)

$$k_1^1 = W_{8k+8}$$

$$k_1^2 : (\mathrm{Sq}^2 + W_2)\mathrm{Sq}^1 k_1^1 = 0; \quad k_2^2 : [(\mathrm{Sq}^4 + W_2\mathrm{Sq}^2 + W_4)\mathrm{Sq}^1 + \mathrm{Sq}^1(W_2\mathrm{Sq}^2)]k_1^1 = 0$$

$$k_1^3 : (\mathrm{Sq}^2 + W_2)k_1^2 = 0; \quad k_2^3 : \mathrm{Sq}^1 k_2^2 + (\mathrm{Sq}^2 + W_2)\mathrm{Sq}^1 k_1^2 = 0$$

$$k_1^4 : \mathrm{Sq}^1 k_2^3 + (\mathrm{Sq}^2 + W_2)k_1^3 = 0$$

We now prove the following theorem.

THEOREM 6.6. *If $n = 0 \bmod 4$ and n is not a power of two, then RP^n immerses in R^{2n-5}.*

Proof. We give only the proof for the case $n \equiv 0 \bmod 8$, the other case being entirely analogous.

By (1.1) of [2], it suffices to show that $(2n - 4)\xi$ has $(n + 1)$-sections. Therefore we have the problem of whether $f : RP^n \to BSO(2n - 4)$ lifts to $BSO(n - 5)$, where f is the classifying map for $\eta = (2n - 4)\xi$. We have then

$$\begin{tikzcd} & & & & RP^n \arrow[d, "f"] \\ BSO_{n-5} \arrow[r] & E_3 \arrow[r] & E_2 \arrow[r] & E_0 \arrow[r] & BSO_{2n-4}, \end{tikzcd}$$

and we need to lift f three stages. The k-invariants for this problem are those of table 1. The Stiefel-Whitney classes of η are such that $W_1(\eta) = W_2(\eta) = W_{n-4}(\eta) = W_n(\eta) = 0$ and $W_4(\eta) \neq 0$. Therefore the k^1-invariants of η are zero, and we can lift f to $f_0 : RP^n \to E_0$. The k^2-invariants are $k_1^2(\eta) \in H^{n-2}(RP^n)$, $k_2^2(\eta) \in H^{n-1}(RP^n)$, and $k_3^2(\eta) \in H^n(RP^n)$, and they are defined modulo the indeterminacy which is $\{(0, x^{n-1}, 0), (0, 0, x^n)\}$. Therefore $k_1^2(\eta)$ has zero indeterminacy and is in fact zero, since $\mathrm{Sq}^2 k_2^2(\eta) = 0$ (because of the defining relation for k_1^3), but $\mathrm{Sq}^2 x^{n-2} \neq 0$. Therefore, we may choose appropriately f_0, such that f_0 lifts to $f_1 : RP^n \to E_1$.

Now $k_1^3(\eta) \in H^{n-1}(RP^n)$ and $k_2^3(\eta) \in H^n(RP^n)$ modulo the indeterminacy $(0, x^n)$. Thus $k_1^3(\eta)$ has zero indeterminacy. Now η is induced from the bundle $(2n - 4)\xi$ over RP^{n+1}. If g is the classifying map for this bundle, it is clear that g lifts to $g_1: RP^{n+1} \to E_1$. However, in RP^{n+1}, $k_1^3((2n - 4)\xi)$ satisfies the defining relation for k_1^4; and thus $\mathrm{Sq}^2 k_1^3((2n - 4)\xi) = 0$ in $H^{n+1}(RP^{n+1})$, but $\mathrm{Sq}^2 x^{n-1} \neq 0$, Hence $k_1^3((2n - 4)\xi) = 0$ and, by naturality, $k_1^3(\eta) = 0$. Therefore a proper choice of f_1 lifts to $f_2: RP^n \to E_2$. Here we meet a single obstruction $k_1^4(\eta) \in H^n(RP^n)$ modulo $\mathrm{Sq}^1 x^{n-1} = x^n$, and thus a proper choice of f_2 lifts to $f_3: RP^n \to E_3$. But this implies then that f lifts to $\bar{f}: RP^n \to BSO_{n-5}$, and so the proof is complete.

Remark. It is clear from the preceding proof that, if η is an m-plane bundle over RP^{n+1}, with $m > n + 1$ and $n \equiv 0 \bmod 4$, such that $W_i(\eta) = 0$ for $i = 1, 2, n - 4, n$, and $W_4(\eta) \neq 0$, then, over RP^n, η' (the induce(bundle) has $m - n + 5$ sections.

7. Geometric dimension of virtual bundles

Let X be a finite CW-complex. We now recall briefly the construction of the Groethendieck-Atiyah-Hirzebruch group $\widetilde{KO}(X)$.

Let $\mathcal{E}(X)$ denote the set of equivalence classes of real vector bundles over X, two bundles being equivalent in the sense of fibre bundle equivalence. The Whitney sum gives $\mathcal{E}(X)$ the structure of an abelian monoid. Let $F(X)$ be the free abelian group with generators the elements in $\mathcal{E}(X)$. Let $R(X)$ be the subgroup of $F(X)$ generated by the elements $r = \{\xi \otimes \eta\} - \{\xi\} - \{\eta\}$, where $\otimes$ is the Whitney sum. Then $KO(X) = F(X)/R(X)$, and we have a natural mapping $O: \mathcal{E}(X) \to KO(X)$. The mapping from $\mathcal{E}(X)$ to the natural numbers which assigns to a bundle its dimension induces $d: KO(X) \to Z$ and we define $\widetilde{KO}(X) =$ *Ker* d. The elements of $\widetilde{KO}(X)$ are called *virtual bundles*. The group $\widetilde{KO}(X)$ has a natural filtration by skeletons. Let X^q be the q-skeleton of X. Then the inclusion $X^q \subset X$ induces $\widetilde{KO}(X) \to \widetilde{KO}(X^q)$, and we denote by $\widetilde{KO}_{q+1}(X)$ the kernel of this map; its elements are said to have *filtration* $q + 1$.

Following Atiyah [2], given $x \in \widetilde{KO}(X)$, we define the *geometric dimension* of x, $gd(x)$, to be the least integer n such that $x + n$ is in the image of θ, and the *codimension of* x, codim (x) by codim $(x) = \dim X - gd(x)$.

In this section we study the relation between filtration and geometric dimension in $\widetilde{KO}(X)$.

Recall that to every $x \in \widetilde{KO}(X)$ there corresponds a unique, up to homotopy, map $\varphi_x: X \to BO$. If $x \in KO_q(X)$, then the composition $X^{q-1} \to X \to BO$ is null-homotopic.

Let $BO(q)$ be the space obtained from BO by killing its first $q - 1$ homotopy groups. We have a sequence of fibrations

$$(7.1) \qquad BO(q - 1) \to BO(q) \to \cdots \to BO(2) \to BO(1) = BO,$$

and we denote by $h_{q',q}: BO(q') \to BO(q)$ the composition of the above maps. Note that $BO(2) = BSO$.

It is clear that $x \in KO_q(X)$ if and only if $\varphi_x : X \to BO$ can be lifted to $\tilde{\varphi}_x : X \to BO(q)$. Therefore we may consider the diagram

$$\begin{array}{ccc} & & BSO_n \\ (7.2) & & \downarrow \\ & BSO(q) \xrightarrow{h_{q,2}} & BSO, \end{array}$$

and since our methods are limited to determining a modified Postnikov tower for $BSO_n \to BSO$ in dimensions $\leq 2n - 2$, we may raise the following problem.

PROBLEM 7.3. *Given integers n and t, with $n \leq t \leq 2n - 2$, find the least integer $q = q(n, t)$ such that $h_{q,2}: BO(q) \to BSO$ restricted to the t-skeleton of $BO(q)$ admits a lifting to BSO_n*

It is clear that if $\dim X \leq t$ and $x \in KO_q(X)$ with $q \geq q(n, t)$, then $gd(x) \leq n$.

We will now determine an upper bound for $q(n, t)$.

R. Stong has determined the cohomology over Z_2 of $BO(q)$ in [13]. Since his results are basic for what follows, we will recall here his results.

Let $q \equiv 0, 1, 2, 4 \bmod 8$, and let D_q be the algebra

$$D_q = H^*(K(\Pi_q(BO), q)/I(Q_q\gamma_q), \tag{7.4}$$

where

$$Q_q = \begin{cases} \mathrm{Sq}^2 & \text{if} \quad q \equiv 0, 1 \mod 8 \\ \mathrm{Sq}^3 & \text{if} \quad q \equiv 2 \mod 8 \\ \mathrm{Sq}^5 & \text{if} \quad q \equiv 4 \mod 8 \end{cases}$$

and $I(Q_q\gamma_q)$ denotes the ideal generated by $Q_q\gamma_q$.

Let $\varphi(q)$ denote the number of integers s such that $1 \leq s \leq q$ and $s \equiv 0, 1, 2, 4 \bmod 8$, and let $\alpha(n)$ be the number of ones in the dyadic expansion of n. Then we set

$$E_q = Z_2[h_{q,1}{}^*\theta_i \mid \alpha(i - 1) + 1 > \varphi(q)], \tag{7.5}$$

the polynomial algebra on generators $h_{q,1}{}^*\theta_i$, where the $\theta_i \in H^i(BO)$ are certain classes such that $\theta_i \equiv W_i$ modulo decomposable elements and are defined in [13], page 528.

STONG'S THEOREM 7.6. *If $q \equiv 0, 1, 2, 4$ mod 8, then as an algebra,*

$$H^*(BO(q)) \cong D_q \otimes E_q .$$

Stong also studied the action of A_2 in $H^*(BO(q))$ and he obtained (see [13], p. 543) the following.

(7.7) *In the subalgebra that corresponds to D_q under the isomorphism (7.6), every element is obtained from the image of γ_q by the action of A_2, cup products and sums.*

From (7.7) it follows easily that, if $q' > q$, and if we identify $H^*(BO(q))$ with $D_q \otimes E_q$ and let (D_q) denote the ideal generated by D_q, then

(7.8) (D_q) *lies in the kernel of* $h_{q',q}{}^*$.

We now suppose that h is *odd* and let $2^{r-1} \leq t < 2^r$. Then we consider

$$E_s \to E_{s-1} \to \cdots \to E_1 \to BSO \tag{7.9}$$

to be a modified Postnikov tower for $BSO_n \to BSO$ in dimensions $\leq t < 2n - 1$. From (3.4) we know $s \leq t - n - 1$. Let K^{i+1} denote the set of k^{i+1}-invariants of (7.9). Thus $K^{i+1} \subset H^*(E_i)$, and

(7.10) $x \in K^{i+1}$ implies $\dim x \leq t$, for $i = 0, 1, \cdots, s-1$.

(7.11) Let q_1 be the least integer such that in $H^*(BO(q_1))$, $h_{q_1}{}^*\theta_i = 0$, for $n+1 \leq i \leq t$. Notice that $r = \varphi(q_1)$.

Now consider the diagram

$$\begin{array}{ccc} & & E_1 \\ & & \downarrow p_1 \\ BO(q_1) & \xrightarrow{f_1} & BSO, \end{array}$$

where $f_1 = h_{q_1,2}$. Then $K^1 \subset (D_{q_1})$. Let q_2 be the integer following q_1, with $q_2 \equiv 0, 1, 2, 4 \bmod 8$. Then, by (7.8), $h_{q_2,q_1}{}^*(K^1) = 0$ and, hence, in the diagram

$$\begin{array}{ccc} BO(q_2) & & E_1 \\ \downarrow h_{2,1} & & \downarrow p_1 \\ BO(q_1) & \xrightarrow{f_1} & BSO \end{array}$$

we can find $f_2 : BO(q_2) \to E_1$, making the diagram commutative.

We now apply the same argument to h_2, and in this way we obtain a sequence of successive integers, congruent to 0, 1, 2, 4 modulo 8, $q_1, q_2, \cdots, q_{s+1}$, and maps $f_{i+1} . BO(q_{i+1}) \to E_i$ such that $p_k f_{k+1} = f_k h_{q_{k+1},q_k}$. Since

$$g_{q_{s+1},2} : BO(q_{s+1}) \to BSO$$

is the composition $h_{q_{s+1},q_s} \circ \cdots \circ h_{q_2,q_1} \circ h_{q_1,2}$, we have found a lifting of $h_{q_{s+1},2}$; namely, $f_{s+1} : BO(q_{s+1}) \to E_s$.

Therefore, if we restrict to the t-skeleton Y^t of $BO(q_{s+1})$, we have obtained a lifting of $Y^t \to BSO$ to $Y^t \to BSO_n$. We now need to determine the integer q_{s+1}. We have that $\varphi(q_{s+1}) = \varphi(q_1) + s \leq r + s$, so we need only determine s. The integer s is the smallest integer such that $F_s(V_n)$ is t-connected, where $F_s(V_n)$ is the space corresponding to F_s in (5.4) for an A_2-free resolution of

$H^*(V_n)$ through dimension t. Proposition (5.4) gives us a value for s, but we will use the following stronger result.

PROPOSITION 7.12. *The values of t such that $F_s(V_n)$ is $(t-1)$-connected modulo p-torsion $(p > 2)$ are given by the following table*

	$n \equiv 0$	$n \equiv 1$	$n \equiv 2$	$n \equiv 3$
$s \equiv 0$	$n + 2s - 1 (s > 0)$	$n + 2s$	$n + 2s - 1 (s > 0)$	$n + 2s$
$s \equiv 1$	$n + 2s - 2$	$n + 2s - 1$	$n + 2s - 1$	$n + 2s$
$s \equiv 2$	$n + 2s - 2$	$n + 2s - 2$	$n + 2s - 1$	$n + 2s - 1$
$s \equiv 3$	$n + 2s - 3$	$n + 2s$	$n + 2s - 1$	$n + 2s - 2$

where the congruences are taken modulo 4.

Proof. Using Theorem 5 (p. 65) of [1], it is clear that one has to verify the proposition for $t - s \le 8$. This involves a lengthy, but not hard computation; or one can check with the tables of [9] which carry out such a computation.

Set now

(7.13) $r(n, t) = \min \{\varphi(q) \mid h_{q,1}{}^*\theta_i = 0 \quad \text{for} \quad n < i \le t\}$, and if $\psi(n, s)$ if the function given by table (7.12), set

(7.14) $$s(n, t) = \min \{s \mid \psi(n, s) \ge t\}.$$ Finally put

(7.15) $$p(n, t) = s(n, t) + r(n, t).$$

Then it is easily seen that

(7.16) $$q_{s+1} = \begin{cases} 2p(n, t), & \text{if } \; p(n, t) \equiv 0 \bmod 4 \\ 2p(n, t) - 1, & \text{if } \; p(n, t) \equiv 1 \bmod 4 \\ 2p(n, t) - 2, & \text{if } \; p(n, t) \equiv 2, 3 \bmod 4, \end{cases}$$

and we have the following theorem.

THEOREM 7.17. *An upper bound for the integer $q(n, t)$ of* (7.3) *is the integer q_{s+1} of* (7.16).

Proof of Theorem A. Observe that if $2^{m-1} \le t < 2^m$, then $r(n, t) \le m$, and also that $p(n, t) \le (t - n)/2 + \frac{3}{2} + m$ and, hence, $q_{s+1} \le t - n + 3 + 2m$. But $m = |\log_2 t| + 1$, and thus we obtain the estimate

$$q_{s+1} \le 2 \,|\log_2 t\,| + t - n + 5.$$

Theorem A follows if we set $q = t - n + 2\,|\log_2 t\,| + 5$, as can be easily checked.

Now the following lemma leads to a strengthened version of Theorem B.

LEMMA 7.18. *Given integers a, b, c, d with* $f \leq c \leq 2^{d-1}$, *then the functions* $r(n, t)$ *and* $s(n, t)$ *of* (7.13) *and* (7.14) *respectively take the same value for all pairs* (n, t), *where*

$$n = s2^d + c - b$$

$$t = s2^d + c$$

and where $\alpha(x) = a$.

Proof. Because of (7.5), $h_{q_1}{}^*$, $\theta_i = 0$, provided that $\alpha(i - 1) + 1 \leq \varphi(q)$. But now it is easy to see that the conditions of (7.18) imply that, for $j = 1, \cdots, b$

$$\alpha(s2^d + c - b + j - 1) = \alpha(s'2^d + c - b + j - 1),$$

whenever $\alpha(s) = \alpha(s') = a$, and that this implies clearly the result for the function $r(n, t)$.

Now the result for the function $s(n, t)$ follows from (7.12).

THEOREM 7.19. *Given integers a, b, c, d, with* $b \leq c \leq 2^{d-1}$, *an upper bound for the integer* $q(n, t)$ *of* (7.3) *for all pairs* (n, t) *where*

$$n = s2^d + c - b$$

$$t = s2^d + c$$

and where $\alpha(s) = a$ *is the integer* q_{s+1} *obtained from* (7.13) *by taking* $n = s_0 2^d + c - b$, $t = s_0 2^d + c$ *where* s_0 *is the smallest integer with* $\alpha(s_0) = \alpha$.

This result implies clearly Theorem B. But as can be seen, is complicated to state.

We believe that theorems (7.17) and (7.19) are best possible for some values of n and t.

CENTRO DE INVESTIGACIÓN DEL I.P.N., MEXICO, D.F. AND OXFORD UNIVERSITY
NORTHWESTERN UNIVERSITY, EVANSTON, ILLINOIS

REFERENCES

[1] J. F. ADAMS, Stable homotopy theory (Lecture Notes in Mathematics, No. 3), Springer Verlag, 1964.

[2] J. ADEM and S. GITLER, *Non-immersion theorems for real projective spaces*, Bol. Soc. Mat. Mex., **9** (1964), 37–50.

[3] M. ATIYAH, *Immersions and embeddings of manifolds*, Topology, **1** (1962), 125–32.

[4] M. BARRATT and M. MAHOWALD, *The metastable homotopy of* O(n), Bull. Am. Math. Soc., **70** (1964) 758–60.

[5] A. BOREL, *Sur la cohomologie des espaces fibrés principaux et des espaces homogenes des groupes de lie compacts*, Ann. of Math., **57** (1953) 115–207).

[6] S. GITLER and J. STASHEFF, *The first exotic class of* BF, Topology, **4** (1965) 257–66.

[7] M. HIRSCH, *Immersion of manifolds*, Trans. Amer. Math. Soc., **93** (1959), 242–76.

[8] M. MAHOWALD, *On obstruction theory in orientable fibre bundles*, Trans. Amer. Math. Soc., **110** (1964) 315–49.

[9] ———, *The metastable homotopy of S^n*, to appear.

[10] W. Massey and F. Peterson, *The cohomology structure of certain fibre spaces I*, Topology, **4** (1965), 47–65.

[11] E. H. Spanier, *Secondary operations on mappings and cohomology*, Ann. of Math., **75** (1962) 260–82.

[12] N. E. Steenrod, Topology of fibre bundles, Princeton University Press, 1949.

[13] R. Stong, *Determination of $H^*(BO(k, \cdots, \infty), Z_2)$ and $H^*(BU(k, \cdots, \infty), Z_2)$*, Trans. Amer. Math. Soc., **107** (1963) 526–44.