THE GEOMETRIC DIMENSION OF REAL STABLE VECTOR BUNDLES

Addendum

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In the paper referred to in the title, we established some results on totally transgressive fibrations, [1; section 2]. It is the object of this note to provide a much shorter and more conceptual proof of these results. We use Z_2-coefficients throughout.

We recall that a fibration $F \to T \xrightarrow{p} B$ is called *q-totally transgressive* if
(a) $H^*(F)$ forms a simple system of coefficients over B and for $k \leq q$
(b) $p^* : H^k(B) \to H^k(T)$ is onto and
(c) the elements of $H^{k-1}(F)$ are all transgressive.

Given a fibre pair $(\bar{F}, F) \xrightarrow{i} (\bar{T}, T) \to B$, we will say it is *q-non-homologous to zero* if $i^* : H^k(\bar{T}, T) \to H^k(\bar{F}, F)$ is onto for $k \leq q$.

Given a fibration $F \to T \xrightarrow{p} B$, we denote by $\bar{B}$ the mapping cylinder of p. Then up to homotopy we have a fibre pair $(\bar{B}, T) \to B$ with fibre $(\bar{F}, F)$ where $\bar{F}$ is contractible and $\bar{B} \to B$ is a homotopy equivalence.

PROPOSITION. *If condition* (a) *is satisfied, then condition* (c) *is equivalent with* $(\bar{F}, F) \to (\bar{B}, T) \to B$ *being a q-non-homologous to zero fibration.*

This proposition is an easy consequence of [2; Lemma 5.1].

Given a graded Z_2-vector space G, we denote by s^kG the graded Z_2-vector space which is obtained from G by increasing its degrees by k. We denote by $K(G)$ the generalized Eilenberg-MacLane space associated with G. We let $K(s^{-1}G) \to E(G) \to K(G)$ be the path space fibration over $K(G)$.

Consider now a q-totally transgressive fibration $F \to T \xrightarrow{p} B$, where F is $(n - 1)$-connected, B is simply connected and $q \leq 2n - 2$. Let $s^{-1}G$ be a Z_2-graded vector space with a basis over the Steenrod algebra A_2 of $H^*(F)$ in dimensions $\leq q - 1$. Then we have a mapping $g : F \to K(s^{-1}G)$ which induces an epimorphism in cohomology in dimensions $\leq q - 1$. For every element in the A_2-basis of $H^*(F)$, choose a representative in $H^*(B)$ of the transgression of this element. Define $f : B \to K(G)$ to be a representation of the image under transgression of the A_2-basis of $H^*(F)$. Let $K(s^{-1}G) \to E \to B$ be the fibration induced by ω. Then we can lift the map p to a map $p_1 : T \to E$ such that p_1 is a fibration, with fibre F_1 say. Then we obtain $F_1 \to (T, F) \to (E, K(s^{-1}G))$.

THEOREM 1. *The fibration* $F_1 \to T \xrightarrow{p_1} E$ *is q-totally transgressive.*

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Proof. Because of the commutative triangle

$$\begin{array}{ccc} T & \xrightarrow{p_1} & E \\ & {\scriptstyle p}\searrow \quad \swarrow & \\ & B & \end{array}$$

it follows that p_1^* is onto in the same range as p^* is, so condition (b) holds. To prove condition (c), by the above proposition, we need to prove that the fibration $(\bar{F}_1, F_1) \to (\bar{E}, T) \to E$ is q-non-homologous to zero.

Let $K(s^{-1}G) \to W \to T$ be the fibre space induced from $K(s^{-1}G) \to E \to B$ by the mapping $p : T \to B$. Then we have a fibre space equivalence $T \times K(s^{-1}G) \xrightarrow{h} W$. Let $\alpha : T \to T \times K(s^{-1}G)$ be defined by $x \to (x, y_0)$, where y_0 is a fixed point of $K(s^{-1}G)$, then $t = h\alpha$ is a cross section of $W \to T$ and is such that tF_1 is the inclusion of F_1 in F, in the fibration $F_1 \to F \to K(s^{-1}G)$. Then if we make $(T, F_1) \to (W, F)$ into a fibre space, the resulting fibre is $K(s^{-2}G)$. We therefore obtain a commutative diagram

$$\begin{array}{ccccc} K(s^{-2}G) & \to & T & \xrightarrow{t} & W \\ \downarrow & & \downarrow & & \downarrow h \\ F_1 & \longrightarrow & T & \xrightarrow{p_1} & E \\ \downarrow & & \downarrow & & \downarrow q \\ F & \longrightarrow & T & \xrightarrow{p} & B \end{array}$$

where the horizontal rows are fibrations. The map $h : W \to E$ induces a map of fibre pairs,

$$\text{(A)} \qquad \begin{array}{ccc} (\bar{K}(s^{-2}G), K(s^{-2}G)) & \xrightarrow{h_0} & (\bar{F}_1, F_1) \\ \downarrow & & \downarrow \\ (\bar{W}, T) & \xrightarrow{h_1} & (\bar{E}, T) \\ \downarrow & & \downarrow \\ W & \xrightarrow{h} & E \end{array}$$

where h_0 is the extension of the inclusion i of $K(s^{-2}G)$ in F_1 in the fibration $K(s^{-2}G) \xrightarrow{i} F_1 \to F$. Now F is $(n - 1)$-connected and $K(s^{-2}G)$ is $(n - 2)$-connected, so that the Serre exact sequence in this fibration is valid up to dimension $2n - 2$. By construction $i^* : H^k(F_1) \to H^k(K(s^{-2}G))$ is a monomorphism for $k \leq q - 1$, so that h_0^* is a monomorphism in dimensions $\leq q$. In the fibration $K(s^{-2}G) \to E(s^{-1}G) \to K(s^{-1}G)$ the elements of $H^k(K(s^{-2}G))$ are all transgressive for $k \leq 2n - 3$, hence also in any induced fibration, such as $K(s^{-2}G) \to T \to W$.

By the proposition, the fibration $(\bar{K}(s^{-2}G),\ K(s^{-2}G)) \to (\bar{W},\ T) \to W$ is $(2n - 2)$-non homologous to zero.

Let us look at the spectral sequences of the fibrations in the diagram (A). Let ${}_1E_r$ be the spectral sequence for $(\bar{W},\ T) \to W$ and ${}_2E_r$ the spectral sequence for $(\bar{E},\ T) \to E$. The map g induces maps $g_r : {}_2E_r \to {}_1E_r$. At the E_2 level we have

$$ {}_2E_2 = H^*(E) \otimes H^*(\bar{F}_1, F_1) $$

$$ {}_1E_2 = H^*(W) \otimes H^*(K(s^{-2}G), K(s^{-2}G)) $$

Now $W \cong T \times K(s^{-1}G)$, so that $g_1^* : H^k(E) \to H^k(W)$ is an isomorphism for $k \leq n - 1$ and a monomorphism for $k = n$. Therefore $g_2 : {}_2E_2^{a,b} \to {}_1E_2^{a,b}$ is a monomorphism for $0 \leq a \leq n$, $0 < b \leq q$. Since $(\bar{K}(s^{-2}G),\ K(s^{-2}G)) \to (\bar{W},\ T) \to W$ is $(2n - 2)$-non-homologous to zero, we have ${}_1E_2^{a,b} = {}_1E_\infty^{a,b}$ for $a + b \leq 2n - 2$. This is enough to prove that the elements of ${}_2E_2^{0,q}$ are permanent cycles if $q \leq 2n - 2$. But this implies that $H^k(\bar{E},\ T) \to H^k(\bar{F}_1, F_1)$ is onto for $k \leq q$ and by the proposition, that $H^k(F_1)$ is all transgressive for $k \leq q - 1$, in the fibration $F_1 \to T \to E$ and the proof is complete.

We observe that if $H^n(F) = Z_2$, then $H^{n-1}(K(s^{-2}G)) = Z_2$ and in $K(s^{-2}G) \to E(s^{-1}G) \to K(s^{-1}G)$, all elements of $H^k(K(s^{-2}G))$ are transgressive for $k \leq 2n - 2$. The same argument as above now gives

THEOREM 2. *Let $F \to T \to B$ be a q-totally transgressive fibration where F is $(n - 1)$-connected, B is simply connected, $H^n(F) = Z_2$ and $q \leq 2n - 1$, then $F_1 \to T \to E$ is q-totally transgressive.*

The arguments above did not use in an essential way the fact that F was $(n - 1)$-connected but only that $H^k(F) = 0$ for $k < n$.

COROLLARY 3. *Let $F \to T \to B$ be a q-totally transgressive fibration, where B is simply connected, $H^k(F) = 0$ for $k < n$. If $q \leq 2n - 2$, then there exists a sequence of principal fibrations*

$$ \to E_k \xrightarrow{p_k} E_{k-1} \to \cdots \to E_1 \xrightarrow{p_1} E_0 = B $$

and fibre maps $q_k : T \to E_k$ which are q-totally transgressive. If G_k denotes an A_2-basis for Ker (q_k^*) *in dimensions $\leq q$, then the fibre of p_k is $K(s^{-1}G_{k-1})$.*

If $H^n(F) = Z_2$, then $H^n(F_1) = Z_2$ or 0 hence we can apply necessarily either Theorem 1 or Theorem 2 to obtain:

COROLLARY 4. *If in addition to the hypothesis of Corollary 3, we assume $H^n(F) = Z_2$ then for $q \leq 2n - 1$, the same conclusion holds.*

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