AN EXACT SEQUENCE FOR PRINCIPAL FIBRATIONS

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1. The exact sequence

Let B and C be spaces and w a map $B \to C$. Let $p : E \to B$ denote the principal fiber space with classifying map w and with fiber the loops on C, ΩC. (See §3 for details.) The purpose of this paper is to obtain an exact sequence relating the cohomology of B, E, and $\Omega C \times E$.

We proceed to define the morphisms in the sequence. Consider the maps

$$m,\ \rho : \Omega C \times E \to E,$$

where m denotes the action map for the fibration and ρ the projection. In [3, §2] we showed that there is a unique morphism

$$\mu : H^*(E) \to H^*(\Omega C \times E, E)$$

such that

$$\iota^* \circ \mu = m^* - \rho^*, \tag{1}$$

where ι denotes the inclusion $\Omega C \times E \subset (\Omega C \times E, E)$. (We take all spaces with basepoint $*$ and identify E with $* \times E$ in $\Omega C \times E$. Throughout the paper cohomology will be taken with coefficients in a fixed principal ideal domain.)

By using the mapping cylinder we can regard p as an inclusion; doing this and applying §III of [4], we obtain a morphism τ_1 from a submodule of $H^*(\Omega C \times E)$, written $T^*(\Omega C \times E)$, to a factor module of $H^*(B)$, written $H^*(B)/\Lambda^*$. Set

$$T^*(\Omega C \times E, E) = \iota^{*-1} T^*(\Omega C \times E),$$

$$\tau = \tau_1 \circ \iota^* : T^*(\Omega C \times E, E) \to H^*(B)/\Lambda^*.$$

We show in §4 that $p^*\Lambda^* = 0$, and so we can regard p^* as a morphism from $H^*(B)/\Lambda^*$ to $H^*(E)$. Moreover, we will show that

$$\text{Image } \mu \subset T^*(\Omega C \times E, E).$$

We now can state the main result.

Theorem 1. (i) *In the triangle*

$$\begin{array}{ccc} H^*(B)/\Lambda^* & \xrightarrow{p^*} & H^*(E) \\ & \nwarrow_{\tau} \quad {}_{\mu}\swarrow & \\ & T^*(\Omega C \times E, E), & \end{array}$$

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and thus δ is an epimorphism. Therefore there is a class $a \in H^*(X)$ such that

$$\delta l^* a = w.$$

Let $c \in H^*(Z)$ be a class such that

$$h^* c = i^* a,$$

and set $a' = a - k^* j^* c$. Then

$$\delta l^* a' = \delta l^* a - \delta l^* k^* j^* c = \delta l^* a - \delta j^* c = \delta l^* a = w.$$

But $i^* a' = i^* a - h^* c = 0$, and so there is a class $a'' \in H^*(X, A)$ such that $\iota^* a'' = a'$. Therefore

$$\Delta a'' = \delta l^* \iota^* a'' = \delta l^* a' = w,$$

and so Δ is an epimorphism. This completes the proof.

3. Principal fibrations

Let Z be a space (with basepoint $*$); define the path space PZ to be the space of all maps $\lambda\colon [0, 1] \to Z$ such that $\lambda(0) = *$. The map $\pi\colon PZ \to Z$ given by

$$\lambda \to \lambda(1),$$

is then an Hurewicz fibration [2]. Given a space Y and a map $f\colon Y \to Z$, we let $p\colon X \to Y$ denote the fibration induced by f from π. Thus X is the subspace of $Y \times PZ$ consisting of all pairs (y, λ) such that $f(y) = \lambda(1)$, and p is the projection $(y, \lambda) \to y$.

Suppose now that B is a subspace of Y; set $A = p^{-1}B \subset X$ and let $p' = p \mid A$. Thus we have the following commutative diagram, where g denotes the inclusion:

$$\begin{array}{ccccc} A & \subset & X & \longrightarrow & PZ \\ \downarrow p' & & \downarrow p & & \downarrow \pi \\ B & \underset{g}{\subset} & Y & \underset{f}{\longrightarrow} & Z. \end{array}$$

LEMMA 3. *Suppose that the map $f \circ g\colon B \to Z$ is null-homotopic. Then there is a fiber homotopy equivalence*

$$k\colon \Omega Z \times B \to A,$$

where $\Omega Z \times B \to B$ is the trivial fibration. Moreover, if we are given a section $q\colon B \to A$ (i.e., $p'q = 1$) then we can choose k so that the map from B to A, given by

$$b \to k(*, b),$$

is homotopic to q.

Proof. Let C be any space and $n\colon C \to Z$ a null-homotopic map. Then n corresponds to a map $h\colon C \to PZ$. Define a map $\bar{h}$ from C into the space of all paths on Z, by

$$\bar{h}c(t) = hc(t - 1), \qquad 0 \leq t \leq 1,\ c \in C.$$

Now take C to be the space B; a cross-section $q: B \to A$ is given by

$$q(b) = (b, hb), \qquad b \in B,$$

where $h: B \to PZ$ is a null-homotopy of $f \circ g$. Define

$$k: \Omega Z \times B \to A$$

by

$$(\omega, b) \to (b, \omega \vee hb),$$

and define

$$l: A \to \Omega Z \times B$$

by

$$(b, \lambda) \to (\lambda \vee \bar{h}b, b).$$

(Here $\vee$ denotes the usual path composition.) We leave it to the reader to check that k is a fiber homotopy equivalence (with homotopy inverse l) and that k has the desired property vis-a-vis q.

4. Proof of Theorem 1

We retain the notation given in §1. Consider the following diagram of spaces and maps:

$$\begin{array}{ccccccc}
\Omega C \times E & \xrightarrow{k} & D & \overset{j}{\subset} & \bar{M} & \underset{\bar{s}}{\overset{\bar{r}}{\rightleftarrows}} & E & & \\
\downarrow \rho & & \downarrow p' & & \downarrow \bar{p} & & \downarrow p & & \\
E & = & E & \underset{g}{\subset} & M & \underset{s}{\overset{r}{\rightleftarrows}} & B & \xrightarrow[w]{} & C
\end{array}$$

In the diagram M is the mapping cylinder of p and r is the canonical deformation retraction. $\bar{p}$ is the fibration induced from p by r and $\bar{r}$ is the natural lifting of r. s is the inclusion, which lifts to an inclusion $\bar{s}$. One has that

$$r \circ s = 1, \qquad s \circ r \simeq 1,$$

and one easily shows that $\bar{r}$, $\bar{s}$ enjoy the same properties:

$$\bar{r} \circ \bar{s} = 1, \qquad \bar{s} \circ \bar{r} \simeq 1.$$

D is $\bar{p}^{-1}(E)$ and p' is $\bar{p} \mid D$. Since $rg = p$, there is a map (see below) $q: E \to D$ such that

$$\bar{r} \circ j \circ q = 1.$$

Let k be the choice of homotopy equivalence, given in Lemma 3, corresponding to this choice of q.

We now apply Lemma 2, using the pairs $(\Omega C \times E, E)$ and $(\bar{M}, D)$. The map h is the following composite:

$$h: E \xrightarrow{i} \Omega C \times E \xrightarrow{k} D \xrightarrow{j} \bar{M}.$$

By Lemma 3, $k \circ i \simeq q$, and so,

$$\bar{r} \circ h \simeq \bar{r} \circ j \circ q = 1.$$

But $\bar{r}$ is a homotopy equivalence which means that h^* is an isomorphism in cohomology. Therefore, by Lemma 2, Δ is an isomorphism, where Δ is the composite

$$H^*(\Omega C \times E, E) \xrightarrow{\iota^*} H^*(\Omega C \times E) \xrightarrow{l^*} H^*(D) \xrightarrow{\delta} H^*(\bar{M}, D).$$

(Here l is a homotopy inverse to k.)

Let $p_1 : (\bar{M}, D) \to (M, E)$ denote the map of pairs induced by p. Consider the following diagram:

$$\begin{array}{ccccccccc} \cdots \to H^i(E) & \xrightarrow{\delta_1} & H^{i+1}(M, E) & \xrightarrow{d^*} & H^{i+1}(B) & \xrightarrow{p^*} & H^{i+1}(E) \to \cdots \\ & & \downarrow p_1^* & & & & \\ (1) & & H^{i+1}(\bar{M}, D) & & & & \\ & & \uparrow \Delta & & & & \\ & & H^i(\Omega C \times E, E). & & & & \end{array}$$

Here δ_1 is the coboundary operator and d^* is the composition

$$H^*(M, E) \xrightarrow{t^*} H^*(M) \xrightarrow{s^*} H^*(B),$$

where $t: M \subset (M, E)$ is the inclusion. Define

$$L^j = \text{Kernel } p_1^* \quad \text{in dimension } j,$$
$$\Lambda^j = d^* L^j \subset H^j(B), \qquad j \geq 0.$$

Notice that $p^*\Lambda^* = p\, d^* L^* = 0$, by exactness. We will show:

(2) $$T^*(\Omega C \times E, E) = \Delta^{-1}(p_1^* H^*(M, E)).$$

Consequently we obtain an isomorphism

(3) $$p_1^{*-1} \circ \Delta : T^*(\Omega C \times E, E) \approx H^*(M, E)/L^*.$$

Furthermore, we will show:

(4) $$-\mu = \Delta^{-1} \circ p_1^* \circ \delta_1,$$

(5) $$\tau = d^* \circ p_1^{*-1} \circ \Delta,$$

which means in particular that Image $\mu \subset T^*(\Omega C \times E, E)$. Assuming (2), (4), and (5) we have:

Proof of Theorem 1. Now the top row of diagram (1) is simply the exact cohomology sequence of the map p. Thus part (i) of Theorem 1 follows at once from Lemma 1, applied to the exact triangle

$$\begin{array}{ccc} H^*(B) & \xrightarrow{p^*} & H^*(E) \\ {\scriptstyle d^*}\nwarrow & & \swarrow{\scriptstyle \delta_1} \\ & H^*(M, E), & \end{array}$$

taking $C_1 = L^* \subset H^*(M, E)$. To obtain the groups and morphisms as given in part (i), use (3), (4) and (5). (Changing the sign in (4) does not alter exactness.)

For part (ii) of Theorem 1 we appeal to a result of Serre. (See [1] and [2; 9.3.4].) Since C is n-connected the pair (M, E) is n-connected in homology. Therefore, by Serre, p_1^* is an epimorphism in dimensions $\leq 2n$ and is a monomorphism in dimensions $\leq 2n + 1$. Therefore by (3),

$$T^{j-1}(\Omega C \times E, E) \approx H^j(M, E),$$

for $0 \leq j \leq 2n$, and

$$\Lambda^j = 0, \quad \text{for} \quad 0 \leq j \leq 2n + 1.$$

This completes the proof of the Theorem.

We are left with proving statements (2), (4), and (5).

Proof of (2) *and* (5). Consider the fibration $p_1 : (\bar{M}, D) \to (M, E)$. As on page 14 of [4], we define

$$T^*(D) = \delta^{-1} p_1^* H^*(M, E),$$

and set

$$\tau_0 = p_1^{*-1} \circ \delta : T^*(D) \to H^*(M, E)/L^*.$$

We then define (see [4, p. 16]),

$$\bar{\tau}_1 = t^* \circ \tau_0 : T^*(D) \to H^*(M)/t^*L^*.$$

Using the homotopy equivalence $k \colon \Omega C \times E \to D$, and setting $T^*(\Omega C \times E) = k^*T^*(D)$, we obtain the submodule of $H^*(\Omega C \times E)$ referred to in §1. Thus, by the definition given in §1,

$$\Delta T^*(\Omega C \times E, E) = p_1^* H^*(M, E),$$

which proves (2). To prove (5), set

$$\tau_1 = s^* \bar{\tau}_1 l^*.$$

Since $d^* = s^*t^*$ and $\Delta = \delta l^* \iota^*$, and since by definition, $\tau = \tau_1 \iota^*$, we see that

$$\tau = d^* p_1^{*-1} \Delta,$$

as claimed in (5).

Proof of (4). Consider the following commutative diagram:

$$\begin{array}{ccc} H^*(E) & \xrightarrow{\rho^*} & H^*(\Omega C \times E) \\ \| & & \downarrow l^* \\ H^*(E) & \xrightarrow{p'^*} & H^*(D) \\ \downarrow \delta_1 & & \downarrow \delta \\ H^*(M,E) & \xrightarrow{p_1^*} & H^*(\bar{M}, D). \end{array}$$

In other words:

$$(*) \qquad \delta l^* \rho^* = p_1^* \delta_1 .$$

Now by definition and (1) in §1,

$$\Delta\mu = \delta l^* \iota^* \mu = \delta l^*(m^* - \rho^*).$$

In a moment we show:

$$(**) \qquad \delta l^* m^* = 0.$$

Assuming this, we have

$$\Delta\mu = -\delta l^* \rho^* = -p_1^* \delta,$$

by $(*)$. Thus,

$$-\mu = \Delta^{-1} p_1^* \delta_1 ,$$

as claimed in (4).

To prove $(**)$, consider the diagram given below:

$$\begin{array}{ccccc} \Omega C \times E & & \xrightarrow{\quad m \quad} & & E \\ \| & & & & \uparrow \bar{r} \\ \Omega C \times E & \xrightarrow{k} & D & \xrightarrow{j} & \bar{M}. \end{array}$$

By using the definitions of the spaces and maps, one easily shows that the diagram commutes, *i.e.*,

$$m = \bar{r} \circ j \circ k.$$

(For this, we take the map $h : E \to PC$ to be given simply by $(b, \lambda) \to \lambda$, where $b \in B, \lambda \in PC, wb = \lambda(1)$.) Thus,

$$\delta l^* m^* = \delta l^* k^* j^* \bar{r}^* = \delta j^* \bar{r}^* = 0,$$

since $\delta j^* = 0$ by exactness. This proves $(**)$ and so completes the proof of (4).

5. Addendum

We have used Theorem 1 in two different papers: [3, §2] and [5, §4]. In each case we need some additional information about the exact sequence given in Theorem 1. In this section we sketch in this additional material.

Going back to the notation of §1, suppose that B_0 is a subspace of B such that $w(B_0) = *$ in C. Set $E_0 = p^{-1}B_0$ in E. Then $p_0\ (= p \mid E_0)$ maps E_0 homeomorphically onto B_0. Consider the triple (M, E, E_0) (see §4), with exact cohomology sequence

$$\cdots \to H^*(E, E_0) \xrightarrow{\delta_1} H^*(M, E) \to H^*(M, E_0) \to H^*(E, E_0) \to \cdots$$

Let r be the retraction $M \to B$ given in §4. Then r gives a map of pairs $r_0 : (M, E_0) \to (B, B_0)$, where $r_0 \mid E_0 = p_0$. Thus $r_0{}^*$ is an isomorphism:

$$r_0{}^* : H^*(B, B_0) \approx H^*(M, E_0).$$

Let $d_0{}^*$ denote the composite

$$H^*(M, E) \to H^*(M, E_0) \overset{r_0{}^{*-1}}{\approx} H^*(B, B_0);$$

we then have an exact sequence

$$\to H^*(E, E_0) \xrightarrow{\delta_1} H^*(M, E) \xrightarrow{d_0{}^*} H^*(B, B_0) \to \cdots$$

Using this exact sequence for the top row of diagram (1) in §4, we obtain the exact sequence given in (2.4) of [3]. The morphisms given in (2.4) are the following respective composites:

$$H^*(E, E_0) \to H^*(E) \xrightarrow{\mu} H^*(\Omega C \times E, E),$$

$$T^*(\Omega C \times E, E) \xrightarrow{p_1{}^* \circ \Delta^{-1}} H^*(M, E)/L^* \xrightarrow{d_0{}^*} H^*(B, B_0)/d_0{}^*L^*.$$

Finally, the commutative diagram given in [3, 2.5] follows at once from the definitions. We leave the details to the reader.

We now discuss several refinements of Theorem 1 above that are needed in [5]. Given spaces X and Y (with basepoint) we let $X \# Y$ denote the smash product $X \times Y/X \vee Y$. Recall that we have an exact sequence

$$0 \to H^*(X \# Y) \xrightarrow{\iota^*} H^*(X \times Y)$$

where ι is the collapsing map $X \times Y \to X \# Y$. Using the notation of §§1, 4 we prove:

Lemma 4. *Let $T^*(\Omega C \# E) = \iota^{*-1}T^*(\Omega C \times E)$, and let $i : \Omega C \to E$ denote the fiber inclusion. We then have an exact sequence*

$$H^*(E) \cap \text{Kernel } i^* \xrightarrow{\mu} T^*(\Omega C \# E) \xrightarrow{\tau} H^*(B)/\Lambda^*.$$

Moreover, $p^*(H^*(B)/\Lambda^*) \subset \text{Kernel } i^*$.

Proof. Because $i^*p^* = 0$, clearly Image $p^* \subset H^*(E) \cap \text{Kernel } i^*$. Let $j : \Omega C \to \Omega C \times E$ denote the inclusion. Then (see [3]), $m \circ j \simeq i$. Therefore given a class $u \in H^*(E)$,

$$j^* m^* u = i^* u,$$

and so $j^*m^*u = 0$ iff $i^*u = 0$. Thus $\mu(H^*(E) \cap \text{Kernel } i^*) \subset T^*(\Omega C \,\#\, E)$, and exactness is preserved.

A second fact needed in [5] is the behavior of μ on products. Suppose, as in §1, that C is n-connected; assume that B is q-connected.

LEMMA 5. *Let u and v be classes in $H^*(E)$, each with degree $\leq n + q$, and suppose that $i^*v = 0$. Then,*

$$\mu(uv) = i^*u \otimes v \quad in \quad H^*(\Omega C \,\#\, E).$$

Proof. From the hypotheses we see that

$$m^*(uv) = 1 \otimes uv + i^*u \otimes v,$$

from which the result follows.

Finally, consider the special case (in §1) where $C = K(Z_2, n + 1)$. Thus $w \in H^{n+1}(B; Z_2)$. Assume that $w^2 \neq 0$.

LEMMA 6. (a) $\Lambda^{n+2} = \{w^2\}$ *in* $H^{2n+2}(B; Z_2)$. (b) *The sequence*

$$H^{2n+1}(B) \xrightarrow{p^*} H^{2n+1}(E) \cap \text{Kernel } i^* \xrightarrow{\mu} T^{2n+1}(\Omega C \,\#\, E),$$

is exact

Proof. Following the notation in §4 we replace B by the mapping cylinder M. Set $w_1 = r^*w$ in $H^{n+1}(M)$. Since $\bar{p}^*w_1 = 0$, there is a class $\bar{w}$ in $H^{n+1}(M, E)$ such that $t^*\bar{w} = w_1$. Moreover, since $\Omega C = K(Z_2, n)$, $H^{n+1}(M, E) \approx Z_2$. Now $\bar{w}^2 \neq 0$ since $w_1^2 \neq 0$. But

$$p_1^*(\bar{w}^2) = p_1^*(\bar{w} \smile w_1) = p_1^*\bar{w} \smile \bar{p}^*w_1 = 0.$$

Thus $\bar{w}^2$ generates the kernel of p_1^* in dimension $2n + 2$, and so $\Lambda^{2n+2} = \{w^2\}$ as claimed in (a).

To prove (b), let x be a class in $H^{2n+1}(E) \cap \text{Kernel } i^*$ such that $\mu(x) = 0$. Using (4) in §4, this implies that $p_1^*\delta_1(x) = 0$, which means by (a) above that $\delta_1(x) = \lambda\bar{w}^2$, $\lambda \in Z_2$. By exactness, $d^*\delta_1 = 0$; but $d^*\bar{w}^2 = w^2 \neq 0$ and so $\lambda = 0$. Thus, by exactness of diagram (1) in §4, $x \in$ Image p^* as claimed.

Now Lemmas 4 and 6, prove 6.5 in [5], while Lemma 5 proves 6.6. Finally, 6.7 in [5] follows from Properties 1 and 2 in [4].

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