ON THE COMMUTATIVITY OF RINGS

POR ROLANDO E. PEINADO

A ring is said to be a P-ancestral ring if all proper non-zero subrings of R have property P. In a previous paper [3] we study the structure of several P-ancestral rings. For R an arbitrary ring and n a fixed element of Z, (now and throughout this note Z will denote the ring of rational integers) the subring $nR = \{nr : r \in R\}$ is called an integral multiple of the ring R. nR is called S-maximal if n is the smallest positive integer for which $nR \subseteq S$, S a subring of R.

F. Szász in [4] considered P'-ancestral rings, where if S is a proper non-zero subring of R, $P' : S = nR$ for some n in Z.

His main result is: *R is a P'-ancestral ring if and only if the additive group of R is a cyclic group.*

In this note we consider the condition P, where if S is a proper non-zero subring of R, $P : 0 \neq nR \subseteq S$ for nR S-maximal.

Clearly every P'-ancestral ring is a P-ancestral ring. Our main result is: *If R is a P-ancestral ring, then R is a commutative ring.* The converse of this result is obviously not true, since for any n in Z, nR is an ideal of R. It is sufficient to consider an arbitrary field, of characteristic zero, to observe that there exist commutative rings that are not P-ancestral rings.

It is easy to show that if R is a P-ancestral ring then:

(i) *Any proper non-zero subring of R is a P-ancestral ring.*

(ii) *Every non-zero homomorphic image of R is a P-ancestral ring.*

By considering the usual adjunction of a unit to a ring without unit, it can be observed that in considering P-ancestral rings it is enough to consider rings with unit. Henceforth all rings will be assumed to have a unit element that we write 1.

LEMMA 1. *Let R be a P-ancestral ring such that its additive group is a torsion-free group. Then R is a commutative ring:*

Proof: The center C of R is a subring of R. $C \neq 0$ since $1 \in C$. Since R is a P-ancestral ring, then there exists $n \in Z$ such that $nR \subseteq C$. Thus for all a, b in R na, nb belongs to C and we have

$$n^2ab = na \cdot nb = nb \cdot na = n^2ba$$

Therefore

$$n^2(ab - ba) = 0$$

But R is torsion-free. Hence $ab = ba$ for all a, b in R, and R is a commutative ring.

Now consider a ring whose additive structure is a torsion group. The q-components of R, say R_q, are not only subgroups of R but also ideals of R, (R as a ring). Consequently R is ring-direct sum of q-rings, where a q-ring is a ring whose additive group is a q-group. Thus in considering P-ancestral rings whose additive structure is a torsion group, it is enough to examine P-ancestral rings that are q-rings.

LEMMA 2. *Let R be a P-ancestral ring, which is a q-ring, then R is a commutative ring.*

Proof: Let R_q be a P-ancestral ring. Then every proper subgroups of R_q contains a multiple nR of Rq and n is the smallest positive integer for which $nR \subseteq S$. This fact together with exercise 19 of [2, page 19], enables us to show that R_q is bounded, otherwise there exists a homomorphism of R_q onto $Z(p^\infty)$, then by (i) and (ii) above, every subgroup of $Z(p^\infty)$ would contain a non-zero integral multiple of $Z(p^\infty)$. But the subgroup of $Z(p^\infty)$ generated by $1/p$, p a prime number in Z, does not contain a multiple of $Z(p^\infty)$.

Now let us say that $q^k R_q = 0$ and $q^{k-1}R_q \neq 0$. Then $q^R q$, $q^2 R_q, \cdots q^{k-1}R_q$ are proper subrings of R_q which are also ideals of R_q. Consider the quotient rings $R_q{}^i = (q^i R_q/q^{i+1}R_q)\, i = 1 \cdots r - 1$. By (ii) $R_q{}^i$ is also a P-ancestral ring. Let S be a proper non-zero subring of $R_q{}^i$ which is also an ideal of $R_q{}^i$ then there exists n in Z such that nS is S-maximal in $R_q{}^i$. Since $R_q{}^i$ is a bounded group $n = q^j$ for some j in Z, which is a contradiction $(R_q{}^i = q^i R_q/q^{i+1}R_q)$. Hence $R_q{}^i$ has no proper left ideals and thus $R_q{}^i$ is a field of prime characteristics or a zero ring with a prime number of elements. In any case $R_q{}^i$ is a group of order q^k and $q^{k-1}R_q \neq 0$. Therefore R_q has an element or order q^k and thus R_q is a cyclic group which implies R_q is a commutative ring.

COROLLARY: *If R is a P-ancestral ring whose additive group is a torsion group, then R is a commutative ring.*

Proof: R is a ring-direct sum of R_q rings.

LEMMA 3. *Let R be a P-ancestral ring, then R is a ring-direct sum $F \oplus T = R$, where the additive group of F is a torsion-free group and the additive group of T is a torsion group.*

Proof: By similar arguments as in the proof of lemma 2, the maximal torsion subgroup T of R is bounded. Hence by [2, page 183] R is a direct sum of T and a torsion-free group F, but ovbiously T and F are ideals in R and the sum is a ring direct sum.

Now we are ready to prove the main result as a consequence of the above results.

THEOREM. *Any P-ancestral ring is a commutative ring.*

Proof: By lemma 3, $R = F \oplus T$ a ring direct sum. Since F is a torsion-free group and by (i) and (ii) a P-ancestral ring, lemma 1 implies F is a commutative ring. Similarly by the corollary to lemma 2 T being a P-ancestral ring whose additive structure is a torsion group, is also a commutative ring. Thus R is a commutative ring.

UNIVERSITY OF PUERTO RICO
MAYAGUEZ, PUERTO RICO

REFERENCES

[1] L. FUCHS, Abelian groups, Pergamon Press, New York, 1960.
[2] I. KAPLANSKY, Infinite abelian groups, University of Michigan Press, Ann Arbor, 1960.
[3] R. E. PEINADO, *Ancestral Rings*, Proceedings Edinburgh Math. Soc., **15** (1967) 107–110.
[4] F. SZÁSZ, *On rings every subring of which is a multiple of the ring*, Publ. Math. Debrecen, **4** (1956) 237–238.