THE EQUIVALENCE OF THE RINGS OF CONTINUOUS AND ANALYTIC COMPLEX VECTOR BUNDLES ON A REAL-ANALYTIC MANIFOLD

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The equivalence of the title is a consequence of the work of Grauert [2, 3] and Whitney and Bruhat [10], but does not seem to be recorded in the literature. In this note we indicate a proof of it. The need for such a result arose when attempting to prove statements such as the corollaries which are presented at the end. The corresponding result for differentiable manifolds was proved by Steenrod in [9], while the deeper result for holomorphically complete manifolds (Stein manifolds) was proved by Grauert in [2, 4].

To state the theorem let M^n in the rest of this paper denote a paracompact n-dimensional real-analytic manifold having the homotopy type of a finite CW complex, and let $K_C{}^a(M)$ (respectively $K_C(M)$) denote the rings of analytic (respectively continuous) complex vector bundles on M. The assumption on the homotopy type of M is made here primarily because it is made in the developments of K-theory that we refer to. Finally, let $F(M) : K_C{}^a(M) \rightarrow K_C(M)$ be the forgetful homomorphism defined by disregarding the analytic structure of analytic bundles. For definitions such as these see [5], p. 171, and the references given in that appendix. Then if the domain of the functors $K_C{}^a$ and K_C is taken to be the category of paracompact real-analytic manifolds or complex-analytic holomorphically complete manifolds and analytic maps, one has:

Theorem A: *The natural transformation of functors* $F : K_C{}^a \rightarrow K_C$ *is an equivalence.*

Proof: That F is a natural transformation is clear; and since $F(H) : K_C{}^a(H) \rightarrow K_C(H)$ is by [2] an isomorphism for holomorphically complete manifolds H, in order to prove the theorem we need only show that $F(M)$ is an isomorphism for real-analytic $M = M^n$ as above.

First we show that F is a surjection. For this, it follows from [10] that M can be embedded analytically in a complex analytic manifold $C(M)$ of complex dimension n, and it is clear that we may assume M is a retract of $C(M)$. Further, Grauert in [3] shows that $C(M)$ contains a cofinal system of holomorphically complete neighborhoods of M, and so may itself be chosen holomorphically complete. Now consider the following commutative diagram

$$\begin{array}{ccc} K_C{}^a(C(M)) & \xrightarrow{F} & K_C(C(M)) \\ \downarrow g & & \downarrow h \\ K_C{}^a(M) & \xrightarrow{F} & K_C(M) \end{array}$$

where g and h are induced by inclusions. Since M is a retract of $C(M)$, h is a surjection, and since $C(M)$ is holomorphically complete the upper F is an isomorphism. Then the fact that the lower F is a surjection follows from this together with the commutativity of the diagram.

Next the "one-one" part. For this we choose a direct proof: Let x be an element of $K_C^a(M)$. Then it follows from the definitions that x has the form $x = [V]-[E]$, where V and E are analytic bundles on M, and $[V]$ and $[E]$ are their classes in $K_C^a(M)$. Thus in order to show that F is a monomorphism, it suffices by commutativity of the diagram above to show that there are analytic bundles W and G on $C(M)$, $C(M)$ to be appropriately chosen, whose restrictions $W \mid M$ and $G \mid M$ to M are equal to V and E respectively, and that if $Fx = 0$, then $y = [W] - [G]$ is zero in $K_C^a(C(M))$. Let us show first how to construct pre-images, say W of V.

We recall from [10] that about each point t of M there exists a neighborhood N of t in $C(M)$ covered by a coordinate system of the form $(x_1 + iy_1, \cdots, x_n + iy_n)$, $x_j + iy_j$ being the usual representation of a complex number, such that the points of $M \cap N$ are precisely the points of the form $(x_1, x_2, \cdots, x_n)$. See also [3]. We shall call such a neighborhood a patch. By paracompactness of M, we may assume that there exist two open coverings $U = \{U_i\}$, $i \in I$, and $0 = \{0_i\}$, $i \in I$, of M indexed by members of the same set I of integers, and satisfying the following three conditions:

1) Each U_i intersects only finitely many members of U.
2) The closure of each 0_i is compact and is contained in the corresponding U_i.
3) Each U_i can be covered entirely by some patch.

Now let $g_{ij} : U_i \cap U_j \to GL(m, C)$, where $GL(m, C)$ is the group of complex nonsingular m by m matrices and m is the complex dimension of a fiber in the bundle V, be the analytic coordinate transformation defining V with respect to the covering U, as in [9]. Then by 3) each g_{ij} has a unique extension to an analytic function g_{ij}^* defined on some neighborhood N_{ij} of $U_i \cap U_j$ in $C(M)$. The reason for this is that at each point of $U_i \cap U_j$, g_{ij} being analytic, has a power series expansion. The extension g_{ij}^* is obtained by 3) through replacing $x_1, \cdots, x_n$ everywhere in this series by $x_1 + iy_1, \cdots, x_n + iy_n$. The fact that this uniquely defines g_{ij}^* on a neighborhood of $U_i \cap U_j$ in $C(M)$ follows from the identity theorem for analytic functions of several complex variables. Next, some neighborhood S of M in $C(M)$ may be treated as a vector bundle $r : S \to M$, r being the projection, and M corresponding to the zero section in S. Fix a Riemannian metric on S, and let $\| \quad \|$ be the associated norm. They by 1) and 2), it is easy to see that there exists a continuous positive real valued function f on M, such that if N_1 denotes the set of all elements e of S for which $\| e \| < f(t)$ where $re = t$, then we have $N_1 \cap r^{-1}(0_i) \cap r^{-1}(0_j) \subseteq N_{ij}$ for each i and j in I. Now set $V_i = r^{-1}(0_i)$ and $\bar{g}_{ij} = g_{ij}^* \mid (V_i \cap V_j)$. Then again the identity theorem shows that the $\bar{g}_{ij}$'s are consistent, *i.e.*, $\bar{g}_{ik}\bar{g}_{kj} = \bar{g}_{ij}$, and so may be taken as the coordinate transformations with respect to the covering $\{V_i\}$, $i \in I$, of an analytic bundle $\bar{W}$ on N_1, whose restriction to M is analytically equivalent to V.

Similarly one obtains an analytic bundle $\bar{G}$ on a neighborhood N_2 of M in $C(M)$ whose restriction to M is E. Now $N_1 \cap N_2$ contains a tubular neighborhood

N about M of which M is a deformation retract. Further, as in our first application of [3], N contains a holomorphically complete neighborhood, say T, of M. One then takes $W = \bar{W} \mid T$ and $G = \bar{G} \mid T$, and for our purposes the proper choice of $C(M)$ is $C(M) = T$. Finally, if $Fx = 0$, then by commutativity of the diagram $hFy = 0$. Since $\bar{W}$ and $\bar{G}$ are defined on $N \supset T$, (Fy) is the restriction (in the obvious sense) of an element z of $K_C(N)$. Since hFy is equal to the image of z under the homomorphism $K_C(N) \rightarrow K_C(M)$, and since this homomorphism is an isomorphism by our choice of N, it follows that (Fy) is zero in $K_C(C(M))$. But [2] implies by the holomorphic completeness of $C(M)$ that F is an isomorphism. Hence y is zero in $K_C{}^a(C(M))$. Since $x = gy$, this concludes the proof.

At this point we would like to draw attention to the fact that Grauert has shown in [2] that on a paracompact complex manifold H, which is holomorphically complete, any equivalence class of continuous complex vector bundles can be represented by an analytic one, and that two analytic complex vector bundles on H which are topologically equivalent are analytically equivalent. Keeping this in mind and taking a close look at the proof of Theorem A, one sees that the corresponding statements hold for M, namely: any equivalence class of continuous complex vector bundles on M contains an analytic one; and two analytic complex vector bundles on M which are topologically equivalent are analytically equivalent. For example, assume V and E of the proof are topologically equivalent, then since M is a deformation retract of N, W and G are topologically equivalent, and hence by Grauert [2] these are analytically equivalent. Hence V and E, their restrictions, are afortiori analytically equivalent.

As a corollary to Theorem A one derives the following partial result for the case of the ring $K_R{}^a(M)$ of real analytic vector bundles on M. To set the notation, let $\bar{K}_R{}^a(M)$ and $\bar{K}_R(M)$ denote the quotient rings of $K_R{}^a(M)$ and $K_R(M)$ by their 2-torsion. Then if we restrict ourselves to the category of paracompact real-analytic manifolds and analytic maps the natural transformation $F_R(M) : K_R{}^a(M) \rightarrow K_R(M)$ induces another $\bar{F}_R(M) : \bar{K}_R{}^a(M) \rightarrow \bar{K}_R(M)$. We shall further call a homomorphism $f : A \rightarrow B$ of two abelian groups A and B n-pure, n being an integer, if and only if the following condition holds: If $f(a)$, $a \in A$, is divisible by n in B, then a is divisible by n in A. Note that if both $K_R{}^a(M)$ and $K_R(M)$ are torsion groups containing no 2-torsion, then $\bar{F}_R(M)$ is equal to $F_R(M)$ and is automatically 2-pure. Now we prove

THEOREM B: *If the forgetful homomorphism* $\bar{F}_R(M) : \bar{K}_R{}^a(M) \rightarrow \bar{K}_R(M)$ *is 2-pure, then it is an isomorphism.*

Proof: Since the real and imaginary parts of a complex analytic function defined on a real analytic manifold are analytic, the realization homomorphism $r : K_C(M) \rightarrow K_R(M)$ induces a homomorphism $r^a : K_C{}^a(M) \rightarrow K_R{}^a(M)$, which is easily seen to satisfy the usual relationships (below) with respect to the complexification transformation $c^a : K_R{}^a(M) \rightarrow K_C{}^a(M)$ which is defined in the same way as the ordinary complexification $c : K_R(M) \rightarrow K_C(M)$ is defined in [1]. Also one obtains the following two commutative diagrams:

$$\begin{array}{ccc} K_R{}^a(M) & \xrightarrow{F_R} & K_R(M) \\ {\scriptstyle C_a}\downarrow & & \downarrow{\scriptstyle c} \\ K_C{}^a(M) & \xrightarrow{F} & K_C(M) \end{array} \quad \text{and} \quad \begin{array}{ccc} K_R{}^a(M) & \xrightarrow{F_R} & K_R(M) \\ \uparrow{\scriptstyle r^a} & & \uparrow{\scriptstyle r} \\ K_C{}^a(M) & \xrightarrow{F} & K_C(M) \end{array}$$

as well as the equation $r^a c^a = 2$; and the equation $c^a r^a x = 2x$ provided x is in $F^{-1}(cK_R(M))$ as follows from the relationship $crz = 2z$ for elements z of $c(K_R(M))$, the fact that F is an isomorphism, and commutativity.

These diagrams and maps have an obvious interpretation if K is everywhere in them replaced by $\bar{K}$, and F is replaced by $\bar{F}$, and the above equations are still valid. So we assume that the diagrams have been so altered, and proceed with the proof.

To see that $\bar{F}_R$ is a monomorphism, the 2-torsion freeness of $\bar{K}_R{}^a(M)$ and the relationship $r^a c^a = 2$ imply that c^a is a monomorphism. Then the fact that $\bar{F}_R$ is a monomorphism follows from this together with the commutativity of the diagram on the left and that $\bar{F}$ is an isomorphism, by Theorem A.

To show that $\bar{F}_R$ is a surjection, note first that $r\bar{K}_C(M) \supseteq 2\bar{K}_R(M)$, so that by commutativity of the diagram on the right and the isomorphism $\bar{F}$, one obtains that $\bar{F}_R(\bar{K}_R{}^a(M)) \supseteq 2\bar{K}_R(M)$. The rest follows from the fact that if p is a prime and $f : A \to B$ is a p-pure homomorphism of two abelian groups A and B of which B is p-torsion free and if $f(A) \supseteq pB$, then $f(A) \supseteq B$, as follows readily from the uniqueness of divisibility by p in p-torsion free groups.

Before stating certain applications let us motivate them briefly: Given a system of differential equations of the form $\epsilon\dot{X} = A(t)X$, $t \in M$ and $A(t)$ being an m by m matrix valued function, the substitution $S(t)Y(t) = X(t)$ transforms the given system to $\epsilon\dot{Y} = (S^{-1}AS - \epsilon S^{-1}\dot{S})Y$. Often the associated system $\epsilon\dot{Y} = (S^{-1}AS)Y$ is a good approximation of the original if ϵ is small enough, and if $S^{-1}AS$ is "simple", is easier to solve. Observe also for example that the natural domain of definition of a periodic analytic function defined on the real line is the circle, etc. For an actual application we cite [6]. Thus Sibuya in [8] and Hsieh and Sibuya in [7] were led to seek conditions under which an analytic function $A : M \to L(m, C)$, $L(m, C)$ being the ring of m by m complex matrices, can be simplified through a similarity transformation. We shall now give some corollaries which together with their differentiable and holomorphic analogues in [6] complete that discussion of the various cases touched upon in [7] and [8]. These corollaries are proved exactly as their counterparts in [6] by using Theorem A where necessary in connection with the parts involving analyticity.

Given an m by m matrix A, the characteristic polynomial of A shall be the monic polynomial given by the determinant $|xI - A|$, I being the identity matrix. Finally, let $\tilde{K}_C(M)$ denote the reduced ring of complex vector bundles on M as defined for instance in [1]. Then we state

COROLLARY A: *Let* $A : M^n \to L(m, C)$ *be an analytic function such that the characteristic polynomial* $p(t, x)$ *of* $A(t)$, t *in* M, *is the product of two relatively prime monic polynomials* $r(t, x)$ *and* $q(t, x)$ *of degrees* k *and* $m - k$ *respectively, each of which when regarded as a polynomial in* x *has coefficients from the ring of analytic complex valued functions on* M. *Then if minimum* $(k, m - k) \geq n/2$ *and if* $\tilde{K}_C(M) = 0$, *it follows that there exists an analytic function* $S : M \to GL(m, C)$ *such that* $S^{-1}AS$ *is the direct sum of an analytic map* $M \to L(k, C)$ *and an analytic map* $M \to L(m - k, C)$ *having characteristic polynomials* $r(t, x)$ *and* $q(t, x)$ *respectively. Conversely, if the conclusion holds for every* A *with the above properties, then* $\tilde{K}_C(M) = 0$.

COROLLARY B: *Suppose* M *is 2-connected, that* $A : M \to L(m, C)$ *is analytic, and that the characteristic polynomial* $p(t, x)$, $t \in M$ *always (i.e. for each* t*) has distinct roots. Then there exists an analytic map* $S : M \to GL(m, C)$ *such that* $S^{-1}AS$ *is always diagonal.*

The following corollary considers the reduction of quadratic forms defined on manifolds through congruence (instead of similarity). For convenience we shall assume in it that M is connected and that $A : M \to L(m, C)$ is nonsingular. This will ensure that the index of A is the same at all points of M, and so we let it be k. If one wishes to allow A to be singular, and assumption such as A has constant rank, and minimum (nullity, index, m-(index + nullity)) $\geq n/2$ will do.

COROLLARY C: *Assume that* A *is analytic, that* $A(t)$ *is always hermitian, that minimum* $(k, m - k) \geq n/2$ *and that* $\tilde{K}_C(M) = 0$. *Then there exists an analytic map* $S : M \to GL(m, C)$ *such that* SAS^* *is always diagonal, where* S^* *denotes the conjugate transpose of* S. *Conversely if the conclusion holds for every* A *with the above properties then* $\tilde{K}_C(M) = 0$.

In conclusion, it appears that attempts at proving a satisfactory version of Theorem B are more likely to succeed if they are made along the basic methods of Grauert [3] for the holomorphically complete case, perhaps by extending those to cover analytic bundles over a complex space with an involution.

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