ON SECONDARY OPERATIONS WHICH DETECT HOMOTOPY CLASSES

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Given a relation in A_2, the mod 2 Steenrod algebra, there is a rather well understood procedure which defines a secondary cohomology operation based on the relation. (For details, see Chapter 3 of [1].) Let α be an element of the stable j-stem π_j^S. We say that a cohomology operation Φ (of any order) detects α if there is a two-cell complex $S^n \cup_\alpha e^{n+j+1}$ on which Φ is defined and non-zero. Adams showed that Sq^1, Sq^2, Sq^4 and Sq^8 are the only primary operations which detect homotopy. In [1] Adams constructed a basis of secondary cohomology operations, $\Phi_{i,j}$ ($i < j$ but $i \neq j - 1$), raising dimension by $2^i + 2^j - 1$. In this note we wish to prove the following.

THEOREM A. *Suppose* $1 < i < j$. *Then* $\Phi_{i,j}$ *does not detect a homotopy element, except for* $\Phi_{2,4}$ *and* $\Phi_{2,5}$ *and possibly* $\Phi_{3,6}$. *The conclusion also holds when* $i = 0$ *provided that* $j > 3$.

It is known ([2], [3]) that $\Phi_{2,4}$ and $\Phi_{2,5}$ detect homotopy. We conjecture that $\Phi_{3,6}$ does not. The situation with $\Phi_{1,j}$ and with $\Phi_{j,j}$ is more difficult, but these two cases seem to be related. For connections with Whitehead products see [2]. (The cases not covered above, namely $i = 0$ with $j \leqq 3$, all detect homotopy; the proof is easy.)

The proof is obtained by calculating directly a sufficient portion of the Adams spectral sequence. We prefer to work in $\mathrm{Ext}_A^{s,t}(Z_2, Z_2)$, whereas cohomology operations are elements of $\mathrm{Tor}_{s,t}^A(Z_2, Z_2)$. Since $h_i h_j$ is dual to $\Phi_{i,j}$, we are done when we have proved the following restatement of Theorem A.

THEOREM A′. *Suppose* $1 < i < j$. *Then* $h_i h_j$ *does not detect a homotopy element, except for* $h_2 h_4$, $h_2 h_5$, *and possibly* $h_3 h_6$. *The conclusion also holds when* $i = 0$ *provided* $j > 3$.

Adams has shown that

$$\delta_2(h_k) = h_0 h_{k-1}^2 \tag{1}$$

for all $k \geqq 4$. Thus our task is essentially to show that $h_0 h_i h_{j-1}^2$ is non-zero in E_2 of the Adams spectral sequence, *i.e.* in Ext, under appropriate hypotheses on i and j.

PROPOSITION (2). *The product* $h_0 h_a^2 h_b$ *is non-zero in* E_2 *provided that* $a > 2$, $b \neq 1$, $(a, b) \neq (3, 0)$, *and* $|a - b| > 2$.

The product is known to be zero if $a = 1$ or 2, if $b = 1$, if $(a, b) = (3, 0)$, if

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$|a - b| = 1$, or if $a - b = 2$. The cases $a = 0$, $a = b$, and $a - b = -2$ are not required for the theorem.

Theorem A′ follows immediately from Proposition (2) and Equation (1).

A proof of Proposition (2) could be based on the May spectral sequence, [4], which converges to Ext $= E_2$ of the Adams spectral sequence. May has obtained a basis theorem for E_2 of his spectral sequence when $s \leqq 4$. The products we need are present and non-zero in E_2 of May's spectral sequence, and by means of the basis theorem we can verify that none of them are coboundaries. However, this basis theorem rests on a tedious and specialized calculation, and the details are not suitable for publication We will give a different proof, using the infinite sequence of spectral sequences given by Adams ([1], Chapter 2). We will be able to obtain Proposition (2) using only Adams' published work and some arithmetic.

The remainder of this note is devoted to the proof of Proposition (2).

We must assume that the reader is familiar with Chapter 2 of [1] and we will use freely the notation established there. The n^{th} spectral sequence (where $n \geqq 2$) has $K_n \otimes Q_{n-1}$ as its E_2 term, and converges to Q_n ; here K_n is a polynomial algebra over Z_2 in generators $\{h_{n,i}\}$ $(i \geqq 0)$, $Q_1 = K_1$, and Q_n is obtained inductively. In Q_1 the generator $h_{1,i}$ is denoted h_i.

We will exploit the tri-grading of the elements in each of these spectral sequences and will show arithmetically that only certain kinds of elements appear in the tri-gradings required in order to hit an element of the form $h_0 h_a^2 h_b$ with a coboundary. The exceptional cases are easily checked, using Adams' calculations, and in this way we verify that $h_0 h_a^2 h_b$ survives, *i.e.* projects to a non-zero element in Ext.

We will use without much comment some obvious lemmas about binary arithmetic, of the following sort. If $2^w + 2^x = 2^y + 2^z$ (all exponents will be non-negative integers), then the exponents on the right match those on the left. If $2^v + 2^w = 2^x + 2^y + 2^z$, then some pair of exponents on the right must match, and the other exponent on the right appears also on the left. We will use such arguments in these and in more complicated cases, but usually under restrictions on the exponents (derived from the hypotheses of Proposition (2)) which greatly reduce the number of cases to consider.

The t-grading of h_k is 2^k and that of $h_{n,k}$ is $2^k(2^n - 1)$. Whenever $k > 0$ this grading is 0 mod 2, and for this technical reason it is convenient to run through the proof first under the hypothesis $b \neq 0$, in which case the t-grading of $h_0 h_a^2 h_b$, namely $T = 1 + 2^{a+1} + 2^b$, is 1 mod 2. The case $b = 0$ can be treated separately. (Of course the cases $b \neq 0$ and $b = 0$ correspond to the cases $i > 1$ and $i = 0$, respectively, in the theorem.)

We begin by looking for elements in the n^{th} spectral sequence which lie "entirely in the fibre," *i.e.* elements of the form $h_{n,i} h_{n,j} h_{n,k}$, which have t-grading of the form $1 + 2^{a+1} + 2^b$ (with $b \geqq 2$). We can suppose $i \leqq j \leqq k$. The t-grading of such an element is

$$T(i, j, k) = (2^n - 1)(2^i + 2^j + 2^k)$$

and if this is to be 1 mod 2 then $i = 0$. Now for Proposition (2) we assume $a > 2$, and if $b \geqq 2$ then $1 + 2^{a+1} + 2^b = T \equiv 1 \bmod 4$; but $n \geqq 2$ so that in order that $T(0, j, k) \equiv 1 \bmod 4$ we must have $2^j + 2^k \equiv 2 \bmod 4$. Thus either $j = k = 0$ or else $j = 1$ and $k > 1$.

In the case $i = j = k = 0$, we have $T(0, 0, 0) = 3(2^n - 1) = T$. This can be written

$$2^n + 2^{n+1} = 4 + 2^{a+1} + 2^b$$

which forces us to take $b = 2$, $n = 3$, and $a = 3$, but $b - a = -1$ is excluded, so this case is eliminated.

Otherwise we must consider $T(0, 1, k) = (2^n - 1)(3 + 2^k) = T$ with $k > 1$. This can be written

$$2^n + 2^{n+1} + 2^{n+k} = 4 + 2^{a+1} + 2^b + 2^k$$

and then either $n = 2$ or else some exponent on the right is 2. If $n = 2$ then

$$8 + 2^{k+2} = 2^{a+1} + 2^b + 2^k$$

and there must be some match on the right: either $k = b$ or else $k = a + 1$. (The case $b = a + 1$ is ruled out by hypothesis.) Since $a > 2$, $k = b$ implies $k = b = 2$ and then $2^{a+1} = 2^{k+2} = 16$ so that $a = 3$, but $b - a = -1$ is excluded. If on the other hand $k = a + 1$, then necessarily $2^{a+1} + 2^k = 8$ so that $a = 1$, which is also excluded. This disposes of the case $n = 2$. Thus either $b = 2$ or $k = 2$. If $k = 2$, we violate the hypothesis $| a - b | > 2$. If $b = 2$ then

$$2^n + 2^{n+1} + 2^{n+k} = 8 + 2^{a+1} + 2^k$$

and $b = 2$ implies $a \geqq 5$, so that $n = 3$, $k = 4$, and $a = 6$. This produces our first exceptional element: $h_{3,0}h_{3,1}h_{3,4}$ has the same t-grading as $h_0h_2h_6^2$. However, the differential on this element is known, from Adams' calculations in Chapter 2 of [1]; it hits a non-zero element, nothing else can hit that element (by the arithmetical argument we have just given), hence we need not be concerned with it.

This completes the argument for elements lying in the "fibre" K_n, when $b \neq 0$. We next look for elements in the n^{th} spectral sequence having tri-grading $(1, 2, T)$, *i.e.* elements of the form $h_i \otimes h_{n,j}h_{n,k}$. (Here we use Lemma 2.5.3 of [1] to assert that Q_{n-1} has only the $\{h_i\}$ in dimension 1.)

The t-grading of such an element is $U(i; j, k) = 2^i + (2^n - 1)(2^j + 2^k)$. If this is to be 1 mod 2, then either $i = 0$ or $j = 0$ (assuming $j \leqq k$).

In this first case, if $i = 0$, then

$$2^{n+j} + 2^{n+k} = 2^{a+1} + 2^b + 2^j + 2^k$$

and this is seen to be impossible in the light of the restrictions $n \geqq 2$ and $| b - a | > 2$.

In the other case $j = 0$ so that

$$2^i + 2^n + 2^{n+k} = 2 + 2^{a+1} + 2^b + 2^k$$

and therefore either $i = 1$ or $k = 1$. Suppose first that $k = 1$; this gives a 4 on the right, and thus we must have $i = 2$, $n = 2$, or $b = 2$. If $b = 2$ then

$$2^i + 2^n + 2^{n+1} = 8 + 2^{a+1}$$

which is impossible since $a \geqq b + 3 = 5$. If $i = 2$ we have $2^n + 2^{n+1} = 2^{a+1} + 2^b$ which is impossible. Thus $k = 1$ implies $n = 2$, and we get

$$2^i + 8 = 2^{a+1} + 2^b$$

and $a > 2$ so necessarily $b = 3$ and $i = a + 1$. The element $h_{a+1} \otimes h_{2,0}h_{2,1}$ has the same t-grading as $h_0 h_3 h_a^{\,2}$.

Supposing finally that $j = 0$ and $i = 1$, then

$$2^n + 2^{n+k} = 2^{a+1} + 2^b + 2^k$$

and either $k = b$ or $k = a + 1$. If $k = b$ then $k = b = n - 1$ and $a + 1 = n + k = 2n - 1$; we have found that $h_1 \otimes h_{n,0}h_{n,n-1}$ has the same t-grading as $h_0 h_{n-1} h_{2n-2}^{\,2}$. If $k = a + 1$, then $n = a + 2$ and $n + a + 1 = b$, and we find that $h_1 \otimes h_{n,0}h_{n,n-1}$ also has the same t-grading as $h_0 h_{2n-1} h_{n-2}^{\,2}$.

If we now consult Adams' calculations, we find that $h_{a+1} \otimes h_{2,0}h_{2,1}$ and $h_1 \otimes h_{n,0}h_{n,n-1}$ have distinct non-zero differentials and do not survive to E_3 of the appropriate spectral sequence; therefore they are not involved with $h_0 h_a^{\,2} h_b$ in any way.

This completes the discussion of elements which have dimension 1 from the base and 2 from the fibre (still assuming $b \neq 0$). We finally must consider elements of tri-grading $(2, 1, T)$, which, according to Adams' basis theorem for $H^2(Q_{n-1})$ ([1], Lemma 2.5.1), must be of the form $h_i h_j \otimes h_{n,k}$ or else of the form $g_{n-1,j} \otimes h_{n,k}$ where $g_{n-1,j}$ is the coboundary (transgression) of $h_{n,j}$ in the n^{th} spectral sequence.

In the latter case, since $g_{n-1,j}$ has the same t as $h_{n,j}$, we have

$$(2^n - 1)(2^j + 2^k) = 1 + 2^{a+1} + 2^b$$

and we can now suppose $j = 0$. This gives

$$2^n + 2^{n+k} = 2 + 2^{a+1} + 2^b + 2^k$$

which implies $k = 1$, which in turn implies $b = 2$, so that $2^n + 2^{n+1} = 8 + 2^{a+1}$ which contradicts the assumption $| b - a | > 2$.

In the former case, namely $h_i h_j \otimes h_{n,k}$, we must have $2^i + 2^j + 2^k(2^n - 1) = T$. Thus either $i = 0$ (assuming $i \leqq j$) or else $k = 0$.

If $i = 0$ then

$$2^j + 2^{n+k} = 2^{a+1} + 2^b + 2^k$$

and either $k = b = j - 1$ and $a + 1 = n + k = n + j - 1$, or else $k = a + 1 = j - 1$ and $b = n + k = n + j - 1$. These cases reduce to the observation that $h_0 h_j \otimes h_{n,j-1}$ has the same t-grading as both $h_0 h_{j-1} h_{n+j-2}^{\,2}$ and $h_0 h_{n+j-1} h_{j-2}^{\,2}$. Consulting Adams' calculations, we find that the differential on $h_0 h_j \otimes h_{n,j-1}$ does not hit either of these elements, except when $n = 2$, and we are not concerned with $n = 2$ because of our restrictions on $| b - a |$.

If $k = 0$ it follows that $i = 1$, and $2^j + 2^n = 2^{a+1} + 2^b$ which leads us, as in the last preceding case, to the observation that $h_1h_j \otimes h_{n,0}$ has the same t-grading as both $h_0h_jh_{n-1}^2$ and $h_0h_nh_{j-1}^2$. Again the differential is known and does not hit these elements except in the case $n = 2$ which does not concern us.

This concludes the argument for the case $b \neq 0$. We have found all the elements which have the tri-grading required to hit $h_0h_a^2h_b$, and checked that in fact none of them do hit it. Thus $h_0h_a^2h_b$ survives to Ext as claimed.

For the case $b = 0$, a search of the same nature must be carried out, to find all elements in the sequence of spectral sequences which have the same t-grading as $h_0^2h_a^2$. No new ideas are involved and therefore we omit the details. The following list contains all elements which could possibly hit $h_0^2h_a^2$ for appropriate choice of a: $h_{2,1}h_{2,2}h_{2,4}$; $h_4 \otimes h_{2,1}h_{2,2}$; $h_2 \otimes h_{n,0}^2$; $h_2 \otimes h_{n,1}h_{n,n+1}$; $h_1^2 \otimes h_{n,1}$; $h_1h_j \otimes h_{n,j}$; $h_2h_{n+1} \otimes h_{n,1}$.

All the differentials involved here may be checked by Adams' calculations and none of them involve $h_0^2h_a^2$. This completes the proof of Proposition (2) and hence of Theorem A′ and Theorem A.

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