THE REDUCED SYMMETRIC PRODUCT OF A PROJECTIVE SPACE AND THE EMBEDDING PROBLEM

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1. Studying the problem of (topological) embeddings of complexes in euclidean space. A, Shapiro has built an obstruction theory (mainly unpublished, see [3] for the first obstruction and [4]); on the other hand W. T. Wu started the study of isotopy, and Yo [5] has applied it to obtain some non-embedding theorems.

Any embedding f of a space Y in R^m gives rise to a continuous map $\bar{f}$ of the space $Y \times Y - \Delta$ (where Δ is the diagonal of $Y \times Y$) into the unit sphere $S^{m-1} \subset R^m$: for two distinct points y_1, y_2 of Y, $\bar{f}(y_1, y_2)$ is the unit vector $f(y_2) - f(y_1)/|f(y_2) - f(y_1)|$. It is clear that $\bar{f}$ is equivariant with respect to the symmetry which interchanges the factors in $Y \times Y - \Delta$ and the antipodal map of S^{m-1}. If f_1 and f_2 are two homotopic embeddings then $\bar{f}_1$ and $\bar{f}_2$ are equivariantly homotopic.

Using the Shapiro-Wu theory valid in the combinatorial case and the following approximation theorem [2],

Theorem (Haefliger)

a) Any topological embedding of a differentiable n-manifold N^n in R^m can be approximated by a differentiable embedding if $m \geq 3(n+1)/2$.

b) Any homotopy between differentiable embeddings in the category of topological embeddings can be approximated by a differentiable isotopy if $m > 3(n+1)/2$.

one obtains the following

Theorem. *The differentiable isotopy classes of differentiable embeddings of a compact manifold M^n in R^m are in 1–1 correspondence with equivariant homotopy classes of equivariant maps from $M \times M - \Delta$ into S^{m-1} provided that $m > 3(n+1)/2$ and $n > 2$.*

The equivariant homotopy classes of equivariant maps from $M \times M - \Delta$ into S^{m-1} are also in one to one correspondence with homotopy classes of maps $f: M \times M - \Delta/Z_2 \to P^{m-1}$ for which $f^*(x) = u$ where x is the generator of $H^*(P^{m-1}, Z_2)$ and u the class of the double covering. These in turn are in one-to-one correspondence with homotopy classes of non-zero sections of the bundle $m\zeta \to M \times M - \Delta/Z_2$ where ζ is the line bundle associated to the double covering.

In order to apply this theorem we study the reduced symmetric product of real projective spaces. We find that $R_*P^k = P^k \times P^k - \Delta/Z_2$ has a manifold as a deformation retract and compute the cohomology of that manifold.

2. We consider the reduced symmetric product of the real projective space $R_*P^k = P^k \times P^k - \Delta/Z_2$ where the Z_2 acts on $P^k \times P^k - \Delta$ by interchanging the two coordinates. R_*P^k can then be viewed as the set of unordered pairs of distinct points if P^k, or the set of unordered pairs of distinct points of P^k, or the set of unordered pairs of distinct lines through the origin in R^{k+1}. We thus have a fibration

$$\begin{array}{ccc} R_* P^1 & \rightarrow & R_* P^k \\ & & \downarrow \\ & & G_{k+1,2} \end{array}$$

where $G_{k+1,2}$ is the grassmanian of unoriented 2-planes in R^{k+1}. The fiber R_*P^1 is an open Möbius band.

We can deform R_*P^k onto a subspace X_k by deforming each fiber onto the generator of the Möbius band. We get in this way the bundle η:

$$\begin{array}{ccc} S^1 & \rightarrow & X_k \\ & & \downarrow \\ & & G_{k+1,2} \end{array} \tag{1}$$

It is easily seen that the deformation can be interpreted as follows: each pair of distinct lines in R^{k+1} defines a plane in R^{k+1}, we move the two lines within this plane until they become mutually orthogonal. This gives an interpretation of the bundle (1) in terms of the canonical 2-plane bundle ν over $G_{k+1,2}$: we take the associated circle bundle of ν and identify points which lie on pairs of orthogonal lines.

If ξ is a vector-space bundle over a space B let $\mathcal{P}(\xi)$ denote its projectification, *i.e.*, the space whose points are the 1-dimensional subspaces of the fibers ξ_b, $b \in B$. Thus $\mathcal{P}(\xi) \xrightarrow{\pi} B$ is a fibering over B, the fibers being $(n - 1)$-dimensional projective spaces, $n = \dim \xi_b$. Over $\mathcal{P}(\xi)$ we have the canonical line bundle S_ξ whose fiber over $l_b \in \mathcal{P}(\xi)$ consists of the points of the line $l_b \subset \xi_b$.

We recall that line bundles over a space Y are classified by their first Stiefel-Whitney classes which are contained in $H^1(Y, Z_2)$.

Recall also that if ξ is a vector space bundle over a point (*i.e.* a real vector space) then $x = w_1(S_\xi)$ generates $H^1(\mathcal{P}(\xi), Z_2)$ and hence the powers $1, x, x^2, \cdots, x^{n-1}, n = \dim \xi$ give a free additive basis for $H^*(\mathcal{P}(\xi), Z_2)$. Finally $x^n = 0$. More generally we have:

PROPOSITION. *Let ξ be a vector bundle over B. Then as an $H^*(B; Z_2)$-module $H^*(\mathcal{P}(\xi), Z_2)$ is freely generated by $1, x_\xi, \cdots, x_\xi^{n-1}$, $n = \dim \xi$ where $x_\xi \in H^1(\mathcal{P}(\xi); Z_2)$ is equal to $w_1(S_\xi)$.*

Proof. Since the restrictions of x_ξ^i, $i = 0, \cdots, (n - 1)$ to a given fiber $\mathcal{P}_b(\xi)$ of $\mathcal{P}(\xi)$ over B form a basis for $H^*(\mathcal{P}_b(\xi), Z_2)$, the fiber is totally nonhomologous to zero and the Leray spectral sequence yields the proposition.

COROLLARY. *There are unique classes $w_i(\xi) \in H^i(B, Z_2) i = 0, \cdots, \dim \xi = n$, $w_0(\xi) = 1$, such that the equation*

$$\textstyle\sum_{k=0}^{n} x_\xi^{n-k} w_k(\xi) = 0$$

holds in $H^*(\mathcal{P}(\xi), Z_2)$. This is the defining relation of $\mathcal{P}(\xi)$ and $w_k(\xi)$ are the Stiefel-Whitney classes of the bundle ξ.

For our bundle (1) it is easily seen that $\eta = \mathcal{P}(\psi^2\nu)$ where ψ^2 is the Adams operation which in this case yields a *bona fide* bundle. The mod 2 cohomology of X_k can now be easily computed from the fibration (1) and the knowledge of the mod 2 cohomology of $G_{k+1,2}$.

We find the classes $w_1 = w_1(\psi^2\nu)$ and $w_2 = w_2(\psi^2\nu)$. Then the cohomology of X_k is a module over $H^*(G_{k+1,2}\,;Z_2)$ with two generators 1, u and the relation

$$u^2 = w_2 + w_1 u$$

determines the ring structure. Since $w_2(\psi^2\nu) = 0$ and $w_1(\psi^2\nu) = w_1(\nu)$ we have $u^2 = w_1(\nu)u$. This gives a full description of $H^*(X_k, Z_2)$ in terms of $H^*(G_{k+1,2}\,;Z_2)$.

We now turn to the determination of $H^*(G_{k+1,2}\,;Z_2)$. By Borel [1]

$$H^*(G_{k+1,2}\,;Z_2) = S(x_1, \cdots, x_{k-1}) \otimes S(x_k, x_{k+1})/S^+(x_1, \cdots, x_{k+1})$$

where $S(x_1, \cdots, x_n)$ is the symmetric algebra over $x_1, \cdots, x_n$ (all generators are of dimension 1) and $S^+(x_1, \cdots, x_n)$ is the ideal of elements of positive degree.

Let $a_1 = \sum_j^{k-1} x_j$, $a_2 = \sum_{i<j}^{k-1} x_i x_j, \cdots, a_{k-1} = x_1 x_2 \cdots x_{k-1}$ and $x = x_k + x_{k+1}$, $y = x_k x_{k+1}$ be the generators of $S(x_1, \cdots, x_{k-1})$ and $S(x_k, x_{k+1})$ respectively. Then the ideal $S^+(x_1, \cdots, x_{k+1})$ is generated by the elements

$$a_1 + x$$

$$a_2 + a_1 x + y$$

$$a_3 + a_2 x + a_1 y$$

$$\cdots$$

$$\cdots$$

$$a_{k-1}x + a_{k-2}y$$

$$a_{k-1}y$$

Considering the sequence of equalities

$$a_r + a_{r-1}x + a_{r-2}y = 0 \qquad r \geq 1$$

(set $a_0 = 1$ and $a_k = 0$ if $k < 0$) we solve for a_r to get:

$$a_r = \sum_{i=0} \binom{r-i}{i} x^{r-2i} y^i$$

Proof. For $r = 1, 2$ we get $a_1 = x$, $a_2 = x^2 + y$. We proceed by induction: assume that the formula is true for $r \leq l - 1$. We have

$$a_l = xa_{l-1} + ya_{l-2} = \sum \binom{l-1-i}{i} x^{l-2i} y^i + \sum \binom{l-2-j}{j} x^{l-2-2j} y^{j+1}$$

$$= \sum \binom{l-1-i}{i} x^{l-2i} y^i + \sum \binom{l-1-i}{i-1} x^{l-2i} y^i$$

and since

$$\binom{l-1-i}{i} + \binom{l-1-i}{i-1} = \binom{l-i}{i}$$

the assertion follows.

We can now give the description of $H^*(G_{k-1,2}\,;Z_2)$ as the algebra over Z_2 with generators $1, x, y(\dim x = 1, \dim y = 2)$ with the relations

$$a_k = 0 \quad \text{and} \quad a_{k+1} = 0.$$

To determine the Steenrod squares it suffices to find $\mathrm{Sq}^1\, y$:
$\mathrm{Sq}^1\,(x_k x_{k+1}) = x_k^2 x_{k+1} + x_k x_{k+1}^2 = x_k x_{k+1}(x_k + x_{k+1}) = xy$ *i.e.* $\mathrm{Sq}^1\, y = xy$.
This determines $H^*(G_{k+1,2})$ as a module over the Steenrod algebra. The generators x and y are the Stiefel-Whitney classes of the canonical bundle ν.
Adding the 1-dimensional generator u and the relation

$$u^2 = ux$$

we obtain the cohomology of X_k.

3. We can apply the results of the previous section to obtain the classification of embeddings of P^k in R^{2k} when k is even and greater than 2.

Such embeddings are classified by the homotopy classes of non-zero cross-sections of the following bundle

$$\begin{array}{c} 2k\zeta \\ \downarrow \\ X_k \end{array}$$

where ζ is the line bundle associated to the double covering of X_k induced by the map $P^k \times P^k - \Delta \to P^k \times P^k - \Delta/Z_2$.

Since for k even $G_{2k+1,2}$ is non-orientable and the first Stiefel-Whitney class of its tangent bundle is the same as $w_1(\eta) = x$, the manifold X_k is a $(2k-1)$ orientable manifold. Moreover $2k\zeta$ is an orientable bundle, thus we have:

$$H^{2k-1}(X_k\,;\pi_{2k-1}(S^{2k-1})) = H^{2k-1}(X_k\,;Z) = Z$$

and since each element of $H^{2k-1}(X_k\,;Z)$ can be realized as an obstruction to make two different cross-sections of $2k\xi$ homotopic we have:

THEOREM. *The isotopy classes of embeddings $P^k \subset R^{2k}$ for k even $(k > 2)$ are in one-to-one correspondence with the integers.*

One would hope to obtain other results about embeddings by computing the height of the generator $u \in H^1(X_k\,;Z_2)$. Here we use the fact that equivariant maps of $P^k \times P^k - \Delta$ into S^{m-1} are in one-to-one correspondence with maps

$$f{:}R_*P^k \to P^{m-1}$$

for which $f^*(z) = u$, where z is the generator of $H^*(P^{m-1}; Z_2)$.

For $k = 2^r - 1$ we have

$$a_k = \sum \binom{k-i}{i} x^{k-2i} y^i = x^k$$

Since this is the first relation we conclude that the height of x is $(k - 1)$ in this case and the height of u is, therefore, k.

One can prove by induction, that the height of u does not change for $2^{r-1} \leq k \leq 2^r - 1$ and is equal to $2^r - 1$.

Using this information one obtains non-embedding results which are identical with those gotten from Stiefel-Whitney classes and vanishing of the Euler class of the normal bundle of an embedding.

Some further results obtained using secondary cohomology operations will appear in a sequel to this paper.

It was communicated to me that some of these results were independently obtained by David Handel, using a different method.

CENTRO DE INVESTIGACIÓN DEL I P N

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