A DIAGONAL MAP FOR THE COBAR CONSTRUCTION

BY BYRON DRACHMAN

Introduction

Dold and Lashof ([3]) have extended the Milnor construction of the principal universal classifying bundle to the case of an (associative) H-space X. We write this bundle as $X \to E_\infty(X) \to B_\infty(X)$. If Y is another (associative) H-space, and f is a multiplicative map from X to Y, we have a map of "bundles"

$$\begin{array}{ccc} X & \xrightarrow{\quad f \quad} & Y \\ \downarrow & & \downarrow \\ E_\infty(X) & \xrightarrow{E_\infty(f)} & E_\infty(Y) \\ \downarrow & & \downarrow \\ B_\infty(X) & \xrightarrow{B_\infty(f)} & B_\infty(Y) \end{array}$$

If f is not multiplicative, one can not define $E_\infty(f)$ and $B_\infty(f)$ in general. Sugawara [6] has therefore defined a condition on f so that $E_\infty(f)$ and $B_\infty(f)$ can be formed. It is stronger than homotopy multiplicative and weaker than multiplicative. Sugawara calls such maps strongly homotopy multiplicative. They are also called A_∞-maps (Stasheff [5]).

Clark [2] has given the algebraic analogue of this condition, so that a linear map between associative DGA algebras induce a morphism on the bar constructions.

In this paper we shall give a corresponding definition for DGA-coalgebras, and study some of their algebraic properties. In particular, we will give a homotopy condition on a map between two DGA coalgebras so that there will be an induced morphism on the cobar constructions. Using this, we will find a diagonal map for the cobar construction, so that it becomes a Hopf algebra. In a future paper we shall relate the algebraic structure to the loop space.

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I.1 The cobar construction (Adams [1])

Recall that if C is a simply connected DGA coalgebra over K, a fixed commutative ring with unit, *i.e.*, C is connected and $C_1 = 0$, then the cobar $\bar{F}(C)$ is the direct product of the D^n for all $n \geq 0$, where D^n is the n-fold tensor product of the desuspension of $\bar{C} = \text{Ker}(\epsilon)$, and where $\epsilon: C \to K$ is the augmentation. (Normally one takes the direct sum, but the free product will be more convenient). We will use infinite sum notation instead of the product notation. A typical element is therefore an infinite linear combination of elements of the form $x = [c_1 | \cdots | c_n]$, where x has bidegree $(-n, m)$, and $m = \sum_{i=1}^{n}$ degree (c_i). The differential in $\bar{F}(C)$ is defined on elements of bidegree $(-1, *)$ by

$$d[c] = [-dc] + \textstyle\sum_i (-1)^{\deg c_i'}[c_i' \mid c_i'']$$

where

$$\Delta(c) = c \otimes 1 + 1 \otimes c + \textstyle\sum_i c_i' \otimes c_i'', \quad \Delta: C \to C \otimes C$$

being the diagonal mapping of C. The differential is extended to all of $\bar{F}(C)$ by the requirement that $\bar{F}(C)$ be a DGA-algebra.

The acyclic cobar construction is $F(C) = C \otimes \bar{F}(C)$ with the contracting homotopy $s: F(C) \to F(C)$ defined by

$$s(c \otimes [c_1 \mid \cdots \mid c_n]) = \epsilon(c) \cdot c_1 \otimes [c_2 \mid \cdots \mid c_n]$$

and differential $d: F(C) \to F(C)$ defined so that

$$ds(x) + sd(x) = x - \epsilon(x) \otimes [\ \ ],$$

where $[\ \ ]$ is the unit element of $\bar{F}(C)$, and $\epsilon: F(C) \to K$ is the augmentation induced by the augmentations of C and $\bar{F}(C)$.

$F(C)$ is a differential C-comodule with coaction

$$\Delta_{F(C)}: F(C) \to C \otimes F(C)$$

given by

$$\Delta_{F(C)}(c \otimes z) = \Delta(c) \otimes z$$

and is a differential $\bar{F}(C)$-module with action

$$F(C) \otimes \bar{F}(C) \to F(C)$$

given by

$$(c \otimes [c_1 \mid \cdots \mid c_n]) \cdot ([b_1 \mid \cdots \mid b_m]) = c \otimes [c_1 \mid \cdots \mid c_n \mid b_1 \mid \cdots \mid b_m]$$

I.2 Some notation and formulas

We suppose that C is a DGA coalgebra over K. Let

(2) $$C^k = C \otimes \cdots (k) \cdots \otimes C,$$

be the tensor product of C with itself k times.

Then for $1 \leq i \leq k$, define $P_i: C^k \to C^k$ by

(3) $$P_i(c_1 \otimes \cdots \otimes c_k) = (-1)^{\sum_{j=1}^{i} \deg c_j}(c_1 \otimes \cdots \otimes c_k)$$

and also define

$$d_k^-: C^k \to C^k$$

by

(4) $$d_k^-(c_1 \otimes \cdots \otimes c_k) = \textstyle\sum_{i=1}^{k} (-1)^i P_{i-1}(c_1 \otimes \cdots \otimes c_{i-1} \otimes dc_i \otimes c_{i+1} \otimes \cdots \otimes c_k)$$

In particular, $d_1^-(c_1) = -dc_1$ and

$$d_2^-(c_1 \otimes c_2) = -dc_1 \otimes c_2 + (-1)^{\deg c_1} c_1 \otimes dc_2$$

Also, define $\Delta_i^k : C^k \to C^{k+1}$ by

5) $\Delta_i^k(c_1 \otimes \cdots \otimes c_k) = c_1 \otimes \cdots \otimes c_{i-1} \otimes \Delta(c_i) \otimes c_{i+1} \otimes \cdots \otimes c_k$

Define

$$i_k : C^k \to \bar{F}(C) \quad \text{by} \quad i_k(c_1 \otimes \cdots \otimes c_k) = [c_1 \mid \cdots \mid c_k]$$

Since $\bar{F}(C)$ is defined on $\bar{C}$, if any c_i has degree 0 then $i_k(c_1 \otimes \cdots \otimes c_k) = 0$.

The following formula may be verified by induction:

(6) $$di_n = i_n d_n^- + i_{n+1} \sum_{i=1}^{n} (-1)^{i+1} P_i \Delta_i^n$$

In particular, $di_1 = i_1 d_1^- + i_2 P_1 \Delta$ and

$$di_2 = i_2 d_2^- + i_3(P_1(\Delta \otimes id) - P_2(\mathrm{id} \otimes \Delta))$$

where $\mathrm{id} : C \to C$ stands for the identity mapping, as usual. The case $n = 1$ is just equation (1).

Given a DGA module M, $\mathfrak{F}^r(M)$ will stand for the submodule of M consisting of those elements of degree less than $r + 1$.

I.3 Definition of SHCM mappings

Suppose C and D are DGA coalgebras over K and $h_1 : C \to D$ is a homomorphism of DGA-modules (but not necessarily a homomorphism of coalgebras). Then to say h_1 is the initial mapping of the strongly homotopy comultiplicative (SHCM) mapping $\{h_1, h_2, \cdots, h_n, \cdots\}$ will mean that for each integer $n \geq 2, h_n$ is a K-module homomorphism of degree $n - 1$

$$h_n : C \to (D)^n$$

such that

(7) $$d_n^- h_n + h_n d = \sum_{i=1}^{n-1} (h_i \otimes h_{n-i}) P_1 \Delta + \sum_{i=1}^{n-1} (-1)^i P_i \Delta_i^{n-1} h_{n-1}$$

In particular, for $n = 2$ we have

$$d_2^- h_2 + h_2 d = (h_1 \otimes h_1) P_1 \Delta - P_1 \Delta h_1$$

which says, except that the signs are different, that the following diagram is homotopy commutative

$$\begin{array}{ccc} C & \xrightarrow{\Delta} & C \otimes C \\ \downarrow{\scriptstyle h_1} & & \downarrow{\scriptstyle h_1 \otimes h_1} \\ D & \xrightarrow{\Delta} & D \otimes D \end{array}$$

The motivation for the above definition is the following:

I.4. **Theorem.** If *C and D are simply connected coalgebras over K and*

$h = \{h_1, \cdots, h_n, \cdots\}$ *is a* SHCM *mapping from* C *to* D, *then* h *induces a morphism of* DGA *algebras*

$$\bar{F}(h):\bar{F}(C) \to \bar{F}(D)$$

Proof. First define $\bar{F}(h)$ for elements having one bar by

$$\bar{F}(h)[c] = i_1h_1(c) + i_2h_2(c) + \cdots + i_nh_n(c) + \cdots \tag{8}$$

and then extend $\bar{F}(h)$ to all of $\bar{F}(C)$ by the requirement that $\bar{F}(h)$ be multiplicative.

To show $\bar{F}(h)$ is a chain map, we write down $d\bar{F}(h)[c]$ and $\bar{F}(h)\, d[c]$.

$$\begin{aligned}\bar{F}(h)\, d[c] &= \bar{F}(h)(i_1 d_1^-(c) + i_2P_1\Delta(c)) \\ &= \textstyle\sum_n i_nh_n(d_1^-(c)) + F(h)(i_2P_1\Delta(c)) \\ &= \textstyle\sum_n i_nh_n(d_1^-(c)) + \sum_{n\geqq 2}\sum_{i=1}^{n-1} i_n(h_i \otimes h_{n-i})P_1\Delta(c)\end{aligned}$$

whereas

$$dF(h)[c] = d\textstyle\sum_n i_nh_n(c) = \sum_{n\geq 1}\{i_n\, d_n^-h_n(c) + i_{n+1}\sum_{i=1}^{n}(-1)^{i+1}P_i\Delta_i^n h_n(c)\}$$

To say the terms having n-bars of the two expressions are equal is to say that

$$h_n\, d_1^-(c) + \textstyle\sum_{i=1}^{n-1}(h_i \otimes h_{n-1})P_1\Delta(c) = d_n^-h_n(c) + \sum_{i=1}^{n-1}(-1)^{i+1}P_i\Delta_i^{n-1}h_{n-1}(c)$$

which is equation (7).

It will be convenient to extend $\bar{F} = \bar{F}(h)$ to a chain map

$$F = F(h):F(C) \to F(D)$$

by acyclicity as follows.

Define $F:C_0 \otimes \bar{F}(C) \to F(D)$ by

$$F(1 \otimes z) = 1 \otimes \bar{F}(z).$$

Suppose we have defined $F:\mathfrak{F}^r((C) \otimes \bar{F}(C)) \to F(D)$ so that $dF(c \otimes z) = Fd(c \otimes z)$ whenever $c \otimes z \in \mathfrak{F}^r((C) \otimes \bar{F}(C))$.

Now let $c \otimes z \in \mathfrak{F}^{r+1}((C) \otimes F(C))$. We define

$$F(c \otimes z) = s(1 \otimes \bar{F}([c]z) - F(sd(1 \otimes [c]z)).$$

Since $(c \otimes z) \in \mathfrak{F}^{r+1}((C) \otimes F(C))$, $s\,d(1 \otimes [c]z) \in \mathfrak{F}^r((C) \otimes \bar{F}(C))$ and hence the right hand side is defined. Then

$$\begin{aligned}dF(c \otimes z) &= ds\{(F(1 \otimes [c]z) - F(sd(1 \otimes [c]z))\} \\ &= ds\{F(d(c \otimes z)\} = F(d(c \otimes z)) - sd(F(d(c \otimes z))).\end{aligned}$$

But

$$d(F(d(c \otimes z))) = F(d(d(c \otimes z))) = 0$$

since

$$d(c \otimes z) \in \mathfrak{F}^r(C \otimes \bar{F}(C))$$

hence $sdF(d(c \otimes z)) = 0$.

Hence F is defined inductively on all of $C \otimes \bar{F}(C)$.

I.5 Homotopies of SHCM maps

We will define homotopies of SHCM mappings so that if two SHCM maps are homotopic in this special sense, then the induced mappings on the cobar constructions are homotopic in the usual sense.

DEFINITION. Let C and D be DGA coalgebras over K and let $f = \{f_1, f_2, \cdots, f_n, \cdots\}$ and $k = \{k_1, k_2, \cdots, k_n, \cdots\}$ be two SHCM maps from C to D. Then f and k are (SHCM) homotopic if for each positive integer n there is a K-module homorphism of degree n

$$g_n : C \to D^n$$

such that

$$g_n d + d_n^- g + \sum_{i=1}^{n-1} (-1)^{i+1} P_i \Delta_i^{n-1} g_{n-1} = f_n - k_n \tag{9}$$

In this case we will refer to $g = \{g_1, \cdots, g_n, \cdots\}$ as the homotopy between f and k.

As we mentioned, the reason for this definition is the following.

THEOREM. *Let C and D be two simply connected* DGA *coalgebras over K. Let f and k be two* SHCM *maps from C to D and let g be a homotopy between f and k. Then the induced morphisms*

$$\bar{F}(f) : \bar{F}(C) \to \bar{F}(D)$$

and

$$\bar{F}(k) : \bar{F}(C) \to \bar{F}(D)$$

are homotopic.

Proof. We will first define a homotopy for elements having one bar and then will extend to all of $\bar{F}(C)$ using the acyclicity of $F(D) = D \otimes \bar{F}(D)$.

First define

$$H : \mathfrak{F}^0(\bar{F}(C)) \to \bar{F}(D)$$

by

$$\bar{H}_1([\ \ ]) = [\ \ ]$$

and

$$\bar{H} : \mathfrak{F}^1(\bar{F}(C)) \to \bar{F}(D)$$

by

(10) $$H([c]) = \sum_n i_n g_n(c)$$

when $[c] \in \mathfrak{F}^1\bar{F}(C)$, *i.e.* $\deg c = 2 \cdot (\mathfrak{F}^1(\bar{F}(C)) = \mathfrak{F}^1(\bar{F}^1(C))$. We show that in this case

(11) $$d\bar{H}([c]) + \bar{H}d[c] = \bar{F}(f([c] - \bar{F}(k)[c].$$

$d\bar{H}([c]) = d\sum_n i_n g_n(c) = \sum_n \{i_n d_n^- g_n(c) + i_{n+1} \sum_{i=1}^n P_i\Delta_i^n g_n(c)\}$ by (6) and $\bar{H}(d[c]) = \bar{H}([-dc]) = \sum_n i_n g_n(d_1^-(c))$ since $\deg c = 2$ (forcing c to be primitive).

Then to say $d\bar{H}([c]) + \bar{H}(d[c]) = \bar{F}(f([c] - \bar{F}(k)[c]$ is to say

$$\sum_n i_n d_n g_n(c) + i_{n+1} \sum_{i=1}^n P_i\Delta_i^n g_n(c) + i_n g_n(d_1^-(c)) = \sum_n i_n(f_n(c) - k_n(c))$$

which follows from (and is the motivation for) the definition (9). Then define $H: \mathfrak{F}^1(F(C)) \to F(D)$ by $H(c \otimes z) = c \otimes \bar{H}(z)$ if $(c \otimes z) \in \mathfrak{F}^1(F(C))$. (Hence c has degree 0.) Then

$$dH(c \otimes z) + H\,d(c \otimes z) = F(f)(c \otimes z) - F(k)(c \otimes z).$$

Now suppose that we have extended to

$$H: \mathfrak{F}^r(F(C)) \to F(D)$$

such that

$$dH(c \otimes z) + H\,d(c \otimes z) = F(f)(c \otimes z) - F(g)(c \otimes z)$$

whenever

$$(c \otimes z) \in \mathfrak{F}^r(F(C))$$

extend H to

$$H: \mathfrak{F}^{r+1}(F(C)) \to F(D)$$

by

(12) $$H(c \otimes z) = s\{F(f)(c \otimes z) - F(k)(c \otimes z) - Hd(c \otimes z)\}$$

Then

$$\begin{aligned} dH(c \otimes z) &= ds(F(f)(c \otimes z) - F(k)(c \otimes z) - Hd(c \otimes z)) \\ &= F(f)(c \otimes z) - F(k)(c \otimes z) - Hd(c \otimes z) \\ &\quad -sd(F(f)(c \otimes z) - F(k)(c \otimes z) - Hd(c \otimes z)). \end{aligned}$$

We show the last three terms give 0. By inductive hypothesis

$$dH(d(c \otimes z)) + Hd(d(c \otimes z)) = F(f)\,d(c \otimes z) - F(k)\,d(c \otimes z),$$

hence

$$\begin{aligned} &d(F(f)(c \otimes z) - dF(k)(c \otimes z) - dHd(c \otimes z) \\ &\qquad = F(f)(d(c \otimes z)) - F(k)\,d(c \otimes z) - dHd(c \otimes z) = 0 \end{aligned}$$

and so we have

$$dH(c \otimes z) + Hd(c \otimes z) = F(f)(c \otimes z) - F(k)(c \otimes z).$$

Note that this requirement forced (12), hence H is an extension. This defines H and $\bar{H}$ inductively for all r.

I.6 The map $F(C) \to A$ for a principal construction $A \to M \to C$

Let F be an associative H space and X a countable CW-complex.

Let $F \to E \to X$ be a principal quasifibering (Dold and Lashof [3]). Then there is a strongly homotopy multiplicative map from ΩX to F which may be used to classify the bundle (Drachman [4]). The algebraic analogue of a principal fibering (or quasi-fibering) will be a principal construction $A \to M \to C$. (One thinks of C as the chains of X, $M = C \otimes A$ with some twisted differential, and A as the chains of F.) $\bar{F}(C)$, the cobar construction of C, is a model for the chains of ΩX. We will show that there is a DGA-morphism from $\bar{F}(C)$ to A for a given principal construction $A \to M \to C$. This theorem has other applications but in particular we may use it to define an Eilenberg-Zilber equivalence from $\bar{F}(C \otimes C)$ to $\bar{F}(C) \otimes \bar{F}(C)$.

I.6.1. *Definition.* $A \to M \to C$ is a principal construction if A is a DGA-algebra over K (a fixed ring with unit $\neq 0$), C is a connected DGA coalgebra over K, $M = C \otimes_K A$, M is a differential A module, and a differential C-comodule. (Hence M will have a twisted differential.) For example, $\bar{F}(C) \to F(C) \to C$ is a principal construction when C is a simply connected DGA coalgebra.

I.6.2. *Definition.* $A \to M \to C$ is a special principal construction if $A \to M \to C$ is a principal construction and if in addition A is connected, C is simply connected, and there is a contracting homotopy $s: M \to M$ such that

(1) s raises total degree by $+1$, $s(1) = 0$, $s((c \otimes x)\cdot a) = (s(c \otimes x))\cdot a$ if $c \otimes x \in M$, $a \in A$, and $\deg(x) > 0$

(2) $s\,dx + dsx = x - \epsilon(x) \otimes 1_A$

(3) If $z \in A$, $\deg(z) \neq 0$, and $s(1 \otimes z) = c \otimes x \in M$, then $\deg c \neq 0$ (hence $\deg x = \deg z + 1 - \deg c < \deg z$ since (1) holds and $C_1 = 0$).

In the above, $\deg z$ is the degree of z, $\epsilon: M \to K$ is the augmentation on M induced by the augmentations of C and A, and 1_A is the unit of A.

These are the essential properties of the cobar construction. $\bar{F}(C) \to F(C) \to C$.

I.6.3. *Theorem. Let $A \to M \to C$ be a principal construction and let $A' \to M \to C$ be a special principal construction. Then there is a unique morphism of C-comodules and A-modules*

$$f: M' \to M \quad \textit{inducing} \quad \bar{f}: A' \to A \quad \textit{such that}$$

(i) *f is a chain map*

(ii) *$f(c \otimes 1_{A'}) = c \otimes 1_A$ if $c \in C$ (and hence)*

(iii) *$f(c \otimes z) = c \otimes \bar{f}(z)$ if $c \in C$ and $z \in A'$.*

Proof. Let $\mathfrak{F}^m(A')$ be the submodule of A' consisting of all elements of degree less that $m+1$. Let $\mathfrak{F}^m(M') = C \otimes \mathfrak{F}^m(A')$. We shall construct, by induction on m, $f\colon \mathfrak{F}^m(M') \to M$. If $m = 0$ then f is defined uniquely by $f(c \otimes 1_{A'}) = c \otimes 1_A$.

Now let $r > 0$ and suppose inductively that f is extended to $f\colon \mathfrak{F}^r(M') \to M$ so that f is a morphism of C-comodules inducing $\bar{f}\colon \mathfrak{F}^r(A') \to A$, $(f(c \otimes z) = c \otimes \bar{f}(z)$ whenever $c \otimes z \in \mathfrak{F}^r(A'))$ such that $df(y) = fd(y)$ whenever $y \in \mathfrak{F}^r(M')$ and also $dy \in \mathfrak{F}^r(M')$.

Now let $z \in \mathfrak{F}^{r+1}(A')$. Suppose $\deg z > 0$. By condition (3) of I.6.2, $s(1 \otimes z)$ and $sd(1 \otimes z)$ are in $\mathfrak{F}^r(M')$. We define then

$$f(1 \otimes z) = f(sd(1 \otimes z)) + df(s(1 \otimes z))$$

since the terms on the right hand side are already defined. Note that the requirement that f be a chain map and the equation $sd(1 \otimes z) + ds(1 \otimes z) = 1 \otimes z$ forces this definition. In other words, the proof yields uniqueness as well as existence.

We want now to show

$$\Delta_M f(1 \otimes z) = (1_C \otimes f)\Delta_{M'}(1 \otimes z). \tag{13}$$

where $\Delta_{M'}$, $\Delta_{M'}$ are the comodule actions. But

$$\Delta_M f(1 \otimes z) = \Delta_M fsd(1 \otimes z) + \Delta_M\, dfs(1 \otimes z)$$

and by inductive hypothesis,

$$\Delta_M fsd(1 \otimes z) = (1_C \otimes f)\Delta_{M'} sd(1 \otimes z).$$

Hence one must show

$$(1_C \otimes f)\Delta_{M'}(ds(1 \otimes z)) = \Delta_M\, df(s(1 \otimes z)).$$

We leave this to the reader. Once this is done, we will have induced

$$\bar{f}\colon \mathfrak{F}^{r+1}(A') \to A$$

and we extend to all of $\mathfrak{F}^{r+1}(M')$ by defining

$$f(c \otimes z) = c \otimes \bar{f}(z)$$

whenever $c \otimes z \in \mathfrak{F}^{r+1}(M')$.

(iii) forces this definition, and we have the commutative diagram

$$\begin{array}{ccc} \mathfrak{F}^{r+1}(M') & \xrightarrow{\quad f \quad} & M \\ \downarrow & & \downarrow{\scriptstyle \Delta_M} \\ C \otimes \mathfrak{F}^{r+1}(M') & \xrightarrow[1_C \otimes f]{} & C \otimes M \end{array}$$

and thus an inductive definition of the C-comodule morphism f. After that, it will remain to show that $\bar{f}$ is multiplicative.

We use induction on m. If $m = 0$ then of course $\bar{f}: \mathfrak{F}^0(A') \to A$ preserves products.

Now suppose we have proved that

$$f(z_1 \cdot z_2) = f(z_1) \cdot f(z_2) \quad \text{whenever} \quad z_1 \cdot z_2 \in \mathfrak{F}^m(A').$$

Suppose that $z_1 \cdot z_2 \mathfrak{F}^{m+1}(A')$. If z_1 or z_2 has degree 0, then clearly $\bar{f}(z_1 \cdot z_2) = \bar{f}(z_1) \cdot \bar{f}(z_2)$, so we may assume both have degree greater than 0.

$$f(z_1 \cdot z_2) = f(1 \otimes z_1 z_2) = f(sd(1 \otimes z_1 z_2)) + df(s(1 \otimes z_1 z_2))$$

whereas

$$\begin{aligned}\bar{f}(z_1) \cdot \bar{f}(z_2) &= f(1 \otimes z_1) \cdot \bar{f}(z_2) = f(sd(1 \otimes z_1)) \cdot \bar{f}(z_2) + df(s(1 \otimes z_1)) \cdot f(z_2) \\ &= f((s\,d(1 \otimes z_1)) \cdot z_2) + d(f(s(1 \otimes z_1)) \cdot \bar{f}(z_2)) \\ &\quad - (-1)^{\deg f(s(1 \otimes z_1))} f(s(1 \otimes z_1)) \cdot d\bar{f}(z_2) \\ &= f((s\,d(1 \otimes z_1)) \cdot z_2) + df(s(1 \otimes z_1) \cdot z_2) + (-1)^{\deg z_1} f(s(1 \otimes z_1)) \\ &\qquad \cdot d\bar{f}(z_2) = f(s)((d(1 \otimes z_1)) \cdot z_2) + (-1)^{\deg z_1} s(1 \otimes z_1) \cdot d(z_2) \\ &\quad + df(x(1 \otimes z_1) \cdot z_2 \\ &= f(sd(1 \otimes z_1 z_2)) + df(s(1 \otimes z_1 z_2)).\end{aligned}$$

Hence we have $\bar{f}$ multiplicative by induction.

I.7 The equivalence of $\bar{F}(C \otimes C')$ and $\bar{F}(C) \otimes \bar{F}(C')$

We shall show how to apply the previous theorem to get an equivalence between $\bar{F}(C \otimes C')$ and $\bar{F}(C) \otimes \bar{F}(C')$. We suppose C and C' are simply connected DGA coalgebras over K with coproducts $\Delta: C \to C \otimes C$, $\Delta': C' \to C' \otimes C'$. Then $C \otimes C'$ is a simply connected DGA coalgebra with coproduct

$$\Delta_2 = (1 \otimes T \otimes 1)(\Delta \otimes \Delta')$$

(T is the standard twisting map), and differential d_2

$$d_2(x \otimes y) = dx \otimes y + (-1^{\deg x}\, x \otimes dy.$$

The augmentation $\epsilon_2: C \otimes C' \to K$ is given by $\epsilon_2(x \otimes y) = \epsilon(x) \cdot \epsilon'(y)$. Hence $A \to M \to C \otimes C'$ is a special principal construction where $M = F(C \otimes C') = (C \otimes C') \otimes \bar{F}(C \otimes C')$ and $A = \bar{F}(C \otimes C')$. This special construction has the contracting homotopy s given by

$$\begin{aligned} s(c \otimes c' \otimes [c_1 \otimes c_1'| \cdots |c_n \otimes c_n']) \\ = \epsilon_2(c \otimes c')(c_1 \otimes c_1') \otimes [c_1 \otimes c_2'| \cdots |c_n \otimes c_n']\end{aligned}$$

Now let $A' = \bar{F}(C) \otimes \bar{F}(C')$. Then A' is a DGA algebra with products $(x \otimes y) \cdot (z \otimes w) = (-1)^{\deg y \cdot \deg z}(x \cdot z) \otimes (y \cdot w)$. Let $M' = (C \otimes C') \otimes A'$. We give M' a structure so that $A' \to M' \to C \otimes C'$ is a principal construction. In other words, $M' = C \otimes C' \otimes A'$ is identified with $C \otimes F(C) \otimes C' \otimes F(C') =$

$F(C) \otimes F(C')$ by means of the isomorphism $i:F(C) \otimes F(C') \to M'$ where

$$i(c_1 \otimes x_1) \otimes (c_2 \otimes x_2)) = (-1)^{hk}(x_1 \otimes c_2) \otimes (x_1 \otimes x_2)$$

where

$$h = \deg x_1, \qquad k = \deg x_2$$

$F(C) \otimes F(C')$ is a $F(C) \otimes F(C')$-module and $C \otimes C'$-comodule in the obvious way. Now let M' have the differentiation, module, and comodule actions obtained by identifying M' with $F(C) \otimes F(C')$. We note that $F(C) \otimes F(C')$ has by $s_2 = s \otimes 1 + \epsilon \otimes s'$ where s and s' are the contracting homotopies for $F(C)$ and $F(C')$.

Thus we see that $F(C) \otimes F(C')$ forms a special principal construction, and by identifying M' with $F(C) \otimes F(C')$, we have $A' \to M' \to C \otimes C'$ a special principal construction.

Thus by Theorem I.6.3 we have a unique map

$$h:M \to M' \quad \text{inducing} \quad \bar{h}:A \to A'$$

and

$$g:M' \to M \quad \text{inducing} \quad \bar{g}:A' \to A$$

satisfying (i), (ii), (iii).

hg and gh satisfy (i), (ii), (iii) also, and so by the uniqueness, $hg = id$ and $gh = id$, $\overline{hg} = \overline{id}$ and $\overline{gh} = id$.

Hence $\bar{F}(C) \otimes \bar{F}(C')$ and $\bar{F}(C \otimes C')$ are equivalent.

I.8 The diagonal map for the cobar construction

Suppose C and D are DGA comodules. The usual way to define a morphism between $\bar{F}(C) \to \bar{F}(D)$ is to have a comultiplicative map $h:C \to D$ inducing

$$h_*:F(C) \to F(D)$$

given by

$$[c_1| \cdots |c_n] \to [h(c_1)| \cdots |h(c_n)].$$

To say $h:C \to D$ is comultiplicative means the following diagram is commutative.

$$\begin{array}{ccc} C & \xrightarrow{\quad h \quad} & D \\ {\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \Delta_D} \\ C \otimes C & \xrightarrow{\; h \otimes h \;} & D \otimes D \end{array}$$

However, in the case that C is the chain complex of a space X, and $D = D \otimes C$, the diagonal map $\Delta:C \to C \otimes C$ is not comultiplicative, hence $\Delta_*:\bar{F}(C) \to \bar{F}(C \otimes C) \approx \bar{F}(C) \otimes \bar{F}(C)$ is not defined this way. One can, however, form a SHCM map $h = \{h_1, \cdots, h_n, \cdots\}$ from C to $C \otimes C$ where $h_1 = \Delta:C \to C \otimes C$

and in this case we have

$$\Delta : \bar{F}(C) \xrightarrow{\bar{F}(H)} F(C \otimes C) \xrightarrow{\approx} \bar{F}(C) \otimes \bar{F}(C),$$

a morphism of DGA-algebras. The maps $\{h_n\}$ are chosen by acyclic-carriers techniques, the only modification necessary is that the signs are more complicated. This construction will be carried out in detail in a future paper.

CENTRO DE INVESTIGACION DEL IPN

REFERENCES

[1] J. F. ADAMS, On the cobar construction, Colloquie de Topologie algébraique, Louvain 1956.

[2] A. CLARK, *Homotopy commutativity and the Moore spectral sequence*, Pacific J. Math., **15** (1965), 65–74.

[3] A. DOLD and R. LASHOF, *Principal quasifiberings and fibre homotopy equivalence of bundles*, Ill. J. Math., **3** (1959), 285–305.

[4] B. DRACHMAN, Thesis, Brown University, 1966.

[5] J. STASHEFF, *Homotopy associativity of H-Spaces I*, Trans. Amer. Math. Soc., **108** (1963), 293–312.

[6] M. SUGAWARA, *On the homotopy commutativity of groups and loop spaces*, Mem. Coll. Sci. Univ. Kyoto, Ser. A., **33** (1960), 257–69.