A NOTE ON CERTAIN SECONDARY COHOMOLOGY OPERATIONS

BY ANTHONY HUGHES AND EMERY THOMAS*

1. The main result

Secondary cohomology operations have played an important role in recent years in homotopy theory. (See, for example, [2], [8], [12].) This note concerns a method of computation, due to Mahowald-Peterson, which applies to certain operations. Their original result is as follows. According to Adem [2], one has a relation

$$\mathrm{Sq}^2\mathrm{Sq}^n = 0,$$

which holds on integral cohomology classes of degree $\leq n + 1$. Denote by Φ a secondary operation associated with this relation [1]. Let X be a space, let SX denote the (reduced) suspension of X, and let $s: H^*(X) \approx H^*(SX)$ denote the suspension isomorphism. Now let $u \in H^{n-1}(X)$ (mod 2 coefficients). Then Φ is defined on su (since $\mathrm{Sq}^n(su) = su \smile su = 0$), and Mahowald-Peterson [8] show that Φ can be chosen so that

$$s(u \smile \mathrm{Sq}^2 u) \in \Phi(su).$$

Similar results have been obtained by Mahowald [7], with Sq^2 replaced by Sq^4 and Sq^8. These computations have played important roles in three applications: immersions of manifolds [8], Whitehead products [7], and vector fields on manifolds [12].

We prove in this note a theorem which includes as special cases the computations mentioned above.

Suppose we have a relation of the form

$$(*) \qquad \mathrm{Sq}^{2t}\mathrm{Sq}^n + \mathrm{Sq}^{n+t}\mathrm{Sq}^t + \sum\nolimits^r{}_1\, \alpha_i\beta_i = 0,$$

with the following properties.

(1.1) The relation holds on (mod 2) classes of degree $<2t + n$.

(1.2) Each operation β_i vanishes (for dimensional reasons) on classes of degree $\leq n$.

$$(1.3) \qquad 2t < n.$$

(1.4) When t is odd, Sq^t is replaced by $\delta^*\mathrm{Sq}^{t-1}$, where δ^* denotes the Bockstein coboundary operator from mod 2 coefficients to integer coefficients.

(We show in an appendix, §7, that such a relation exists.)

Let Φ be a secondary cohomology operation, defined on classes of degree $2t + n - 1$, associated with relation $(*)$. We will prove

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THEOREM 1.5. *Let* $u \in H^{n-1}(X)$ *be a class such that* $\mathrm{Sq}^t(u) = 0$. *Then* Φ *is defined on* $s^{2t}(u)$, *and* Φ *can be chosen* (*independently of* X) *so that*

$$s^{2t}(\textstyle\sum_j \mathrm{Sq}^j u \smile \mathrm{Sq}^{2t-j} u) \in \Phi(s^{2t}(u)),$$

where the summation runs from $j = 0$ *to* $j = t - 1$.

Here s^i, $i \geq 0$, denotes the iterated suspension isomorphism, defined by

$$s^0 = \text{identity}, \qquad s^i = s \circ s^{i-1}, \qquad i \geq 1.$$

Let ΩX denote the loop space and

$$\sigma: H^*(X) \to H^*(\Omega X)$$

the loop homomorphism (of degree -1). Set

$$\sigma^0 = \text{identity}, \qquad \sigma^i = \sigma \circ \sigma^{i-1}, \qquad i \geq 1.$$

Corresponding to the operation Φ defined on classes of degree $2t + n - 1$, has one the operation $\sigma^i\Phi$ defined on classes of degree $2t + n - 1 - i$, $i \geq 0$. Recalling the relation between the operators s and σ (see 6.2), one has at once

COROLLARY 1.6. *Let* Φ *be the operation, and* u *the class, given in* 1.5. *Then, for* $0 \leq i \leq 2t$,

$$s^i(\textstyle\sum_j \mathrm{Sq}^j u \smile \mathrm{Sq}^{2t-j} u) \in \sigma^{2t-i}\Phi(s^i u).$$

Remark 1. If $(*)$ is a *stable* operation (i.e., the relation holds on classes of all degrees), then the operation Φ in 1.5 can be regarded as belonging to the stable operation determined by $(*)$.

Remark 2. The proof of Theorem 1.5 follows in broad outline the method of Mahowald [7], but the details are different. We are indebted to M. Mahowald for pointing out that a somewhat more general result has been obtained by L. Kristensen (Math. Scand., 12 (1963), page 76). However, the method of Kristensen is rather different. (Operations are defined via cochains.)

Remark 3. By applying Sq^1 to relation $(*)$ one obtains a relation beginning with $\mathrm{Sq}^{2t+1}\mathrm{Sq}^n$, and $\mathrm{Sq}^1\Phi$ is then an operation associated with this relation. (One assumes now that $n + t$ is even.)

Remark 4. Hughes [17] has proved Theorem 1.5 for the case t odd without assuming (1.4).

2. The join construction

For spaces A, B we denote by $A * B$ the *join* of A and B. (See [16]). Points in the join will be written (a, t, b), with $a \in A$, $b \in B$, $0 \leq t \leq 1$. The join $A * B$ can be regarded as a proper triad (see [9]), with the two subspaces of the triad intersecting in $A \times B$ (which we regard as embedded in $A * B$ by $(a, b) \to (a, \frac{1}{2}, b)$). Thus we have a Mayer-Vietoris coboundary operator

$$\Delta: H^q(A \times B) \to H^{q+1}(A * B), \qquad q \geq 0.$$

Suppose now that A, B have integral homology of finite type; let $\bar{H}^*(\ \ )$ denote reduced cohomology with coefficients in a fixed field. One then has (see [9]):

$$\Delta: \bar{H}^*(A) \otimes \bar{H}^*(B) \approx \bar{H}^*(A * B).$$

Given classes $u \in \bar{H}^*(A)$, $v \in \bar{H}^*(B)$, we set

$$u * v = \Delta(u \otimes v) \quad \text{in} \quad \bar{H}^*(A * B). \tag{2.1}$$

Let A be an H-space with multiplication $m: A \times A \to A$. By the Hopf construction we obtain a map $\mu: A * A \to SA$, given by

$$\mu(a, t, a') = (m(a, a'), t). \tag{2.2}$$

Regarding SA as a proper triad (the upper and lower cones intersect in A, embedded in SA by $a \mapsto (a, \frac{1}{2})$), we see that μ is a triad map. The Mayer-Vietoris coboundary for the triad SA is simply the suspension s. Thus we have the following commutative diagram:

$$\begin{array}{ccc} H^*(A \times A) & \xrightarrow{\Delta} & H^*(A * A) \\ \uparrow m^* & & \uparrow \mu^* \\ H^*(A) & \xrightarrow[s]{} & H^*(SA) . \end{array} \tag{2.3}$$

Recall that a class $u \in \bar{H}^*(A)$ is called primitive if

$$m^*u = u \otimes 1 + 1 \otimes u.$$

If u and v are both primitive, then

$$m^*(uv) = u \otimes v \pm v \otimes u + uv \otimes 1 + 1 \otimes uv.$$

But $\Delta(uv \otimes 1) = 0 = \Delta(1 \otimes uv)$, and so by 2.3 we have:

Proposition 2.4 *Let u and v be primitive classes in $\bar{H}^*(A)$* (mod 2 *coefficients*). *Then*

$$\mu^* s(uv) = u * v + v * u.$$

In the next section we apply this to $A = \Omega X$.

3. The fiber of a map

Let B and C be spaces (with basepoint $*$) and $f: B \to C$ a basepoint preserving map. Define PC to be the space of paths λ on C such that $\lambda(1) = *$. Let E_f be the subspace of $B \times PC$ consisting of pairs (b, λ) such that $f(b) = \lambda(0)$, and let $p: E_f \to B$ denote by projection $(b, \lambda) \mapsto b$. If we regard f as a fiber map (e.g., see [5]), its fiber is E_f with p playing the role of fiber inclusion.

Given a space X define a canonical map $c: S\Omega X \to X$ by

$$c(\omega, t) = \omega(t),$$

where $\omega \in \Omega X$, $0 \leq t \leq 1$. We now state the result of Barcus-Meyer [3] on the fiber of c, E_c. By the above definition E_c is the space of pairs $([\omega, t], \lambda)$, such that

$$\omega(t) = \lambda(0),$$

where $\omega \in \Omega X$, $\lambda \in PX$, $0 \leq t \leq 1$.

We wish to define a homotopy equivalence $\Omega X * \Omega X \to E_c$. For this we use the following notation: given a path μ on X and numbers $0 \leq a \leq b \leq 1$, define a new path $\mu(a, b)$ by setting

$$\mu(a, b)(t) = \mu((b - a)t + a), \qquad 0 \leq t \leq 1.$$

Thus $\mu(a, b)$ is a path on X from $\mu(a)$ to $\mu(b)$. Now define

$$k:\Omega X * \Omega X \to E_c$$

by

$$k(\mu, t, \nu) = \begin{cases} ([\mu\nu, t], \mu(2t, 1)), & 0 \leq t \leq \frac{1}{2} \\ ([\mu\nu, t], \nu(0, 2t-1)^{-1}), & \frac{1}{2} \leq t \leq 1. \end{cases}$$

Here μ, $\nu \in \Omega X$, $\mu\nu$ denotes the usual product of paths, and $\nu(0, 2t - 1)^{-1}$ denotes the inverse path. We leave it to the reader to check that k is well-defined.

THEOREM 3.1 (Barcus-Meyer). *k is a homotopy equivalence.*

Proof. Define $h:E_c \to \Omega X * \Omega X$ by

$$h([\omega, t], \lambda) = (\omega(0, t)\lambda, t, \lambda^{-1}\omega(t, 1)).$$

The fact that h and k are homotopy inverses can now be checked by using the formulae on pages 904–905 of [3].

We set

$$\mu = p \circ k:\Omega X * \Omega X \to S\Omega X.$$

In other words,

$$\mu(\alpha, t, \beta) = (\alpha\beta, t),$$

$\alpha, \beta \in \Omega X$. By Theorem 3.1, up to homotopy μ can be regarded as a fiber inclusion.

On the other hand, if we let $m: \Omega X \times \Omega X \to \Omega X$ denote the multiplication in the H-space ΩX (i.e., $m(\alpha,\beta) = \alpha\beta$), then μ is simply the map defined by 2.2. Let $u \in \tilde{H}^*(\Omega X)$ be a primitive class (mod 2 coefficients). Then $\mathrm{Sq}^i(u)$ is also primitive, $i \geq 0$, and so by 2.4 we obtain:

COROLLARY 3.2. *Let $u \in \tilde{H}^*(\Omega X)$ be a primitive class. Then,* for $i, j \geq 0$,

$$\mu^* s(\mathrm{Sq}^i u \smile \mathrm{Sq}^j u) = \mathrm{Sq}^i u * \mathrm{Sq}^j u + \mathrm{Sq}^j u * \mathrm{Sq}^i u.$$

4. Secondary operations

Recall the relation given in §1:

$$(*) \qquad \mathrm{Sq}^{2t}\mathrm{Sq}^n + \mathrm{Sq}^{n+t}\mathrm{Sq}^t + \textstyle\sum_i \alpha_i\beta_i = 0.$$

By hypothesis 1.2 we obtain a relation

$$(**) \qquad \mathrm{Sq}^{2t}\mathrm{Sq}^{n} + \mathrm{Sq}^{n+t}\mathrm{Sq}^{t} = 0,$$

which holds on mod 2 classes of degree $\leq n$.

Let Φ denote a secondary operation associated with $(*)$, defined on classes of degree $n + 2t - 1$. Let Ψ denote an operation associated with $(**)$, defined on classes of degree n. Theorem 1.5 is an immediate consequence of the following result.

THEOREM 4.1. *Let X be a space. The operations Φ and Ψ can be chosen, independently of X, to have the following two properties.*

(a) *Let $u \in H^{n-1}(X)$ be a class such that $\mathrm{Sq}^t u = 0$. Then Ψ is defined on su and*

$$s(\textstyle\sum \mathrm{Sq}^i u \smile \mathrm{Sq}^{2t-i} u) \in \Psi(su),$$

where the summation runs from 0 to $t - 1$.

$$\text{(b)} \qquad \{\Psi(su)\} = \sigma^{2t-1}\Phi(su),$$

where the brackets indicate that the indeterminacy is that of the right hand side.

The rest of the paper is devoted to the proof of 4.1. In this section we construct the universal examples for the operations, considering first the case t even, in relation $(*)$.

We adopt the following notation. For each integer $s \geq 1$, let $K(s) = K(Z_2, s)$, with fundamental class $\iota_s \in H^s(K(s))$. (All cohomology will now be with mod 2 coefficients.) Let

$$C(s) = K(s + \deg \beta_1) \times \cdots \times K(s + \deg \beta_r),$$

where $\beta_1, \cdots, \beta_r$ are the operations occurring in relation $(*)$. Let $p(s): V(s) \to K(s)$ denote the principal fibration with classifying map

$$(\mathrm{Sq}^t \iota_s, \beta_1 \iota_s, \cdots, \beta_r \iota_s): K(s) \to K(s + t) \times C(s).$$

Define $\delta_s = p(s)^* \iota_s \in H^s(V(s))$, and let $a(s): W(s) \to V(s)$ denote the principal fibration with $\mathrm{Sq}^n \delta_s$ as classifying map. If $s \leq 2t + n - 1$, the space $W(s)$ is the space of the universal example for operations associated with $(*)$. (The composite map $p(s) \circ a(s): W(s) \to K(s)$ is a fiber map with $\Omega K(s + t) \times \Omega C(s) \times \Omega K(n + s)$ as fiber.) Notice that we can take $W(s - 1) = \Omega W(s)$.

Similarly, let $q(s): Y(s) \to K(s)$ denote the principal fibration with $\mathrm{Sq}^t \iota_s$ as classifying map. Set $\epsilon_s = q(s)^* \iota_s$, and let $b(s): Z(s) \to Y(s)$ be the fibration with $\mathrm{Sq}^n \epsilon_s$ as classifying map. If $s \leq n$, the space $Z(s)$ is the space of the universal example for operations associated with $(**)$. Again, we can take

$$Y(s - 1) = \Omega Y(s), \qquad Z(s - 1) = \Omega Z(s).$$

Suppose now that t is odd. The universal examples for the operations are then constructed in the same way, except that $K(s + t)$ is now replaced by $K(Z, s + t)$ and $\mathrm{Sq}^t \iota_s$ by $\delta^* \mathrm{Sq}^{t-1} \iota_s$. We leave the changes to the reader to carry out. (In what follows we will use the notation for the case t even, and refer to the case t odd only if there is a difference in the way the proof is carried out.)

Consider now the following commutative diagram, with the notation explained below:

$$\begin{array}{ccccccc}
Z(n) & \xrightarrow{b(n)} & Y(n) & \xrightarrow{q(n)} & K(n) & \xrightarrow{\mathrm{Sq}^t\iota_n} & K(n+t) \\
\downarrow{\scriptstyle g} & & \downarrow{\scriptstyle f} & & \| & & \downarrow{\scriptstyle i} \\
W(n) & \xrightarrow[a(n)]{} & V(n) & \xrightarrow[p(n)]{} & K(n) & \xrightarrow[(\mathrm{Sq}^t\iota_n,\, \beta\iota_n)]{} & K(n+t) \times C(n).
\end{array}$$

Here i is the natural inclusion (choosing a basepoint $*$ in $C(n)$). Set $\beta = (\beta_1, \cdots, \beta_r)$. By hypothesis $\beta\iota_n = 0$ and so we can take $\beta\iota_n$ to be the constant map $K(n) \to * \in C(n)$. Thus the right hand square commutes, and so the map f is naturally defined (cf. [13]). Now $f^*\delta(n) = \epsilon(n)$ and so $f^*(\mathrm{Sq}^n\delta(n)) = \mathrm{Sq}^n\epsilon(n)$. Thus the map g is again the natural map defined for mappings between principal fibrations.

The following two facts are immediate consequences of the definition of secondary operations [1].

Lemma 4.2. (a) *Let* $\bar{\phi} \in H^{2n+4t-2}(W(n+2t-1))$ *be a universal example for the operation associated with* $(*)$, *defined on classes of degree* $n+2t-1$. *Then*

$$\phi = \sigma^{2t-1}\bar{\phi} \in H^{2n+2t-1}(W(n))$$

is a universal example for the operation associated with $(*)$, *defined on classes of degree* n.

(b) *Let* ϕ *be given as in* (a). *Then*

$$\psi = g^*\phi \in H^{2n+2t-1}(Z(n))$$

is a universal example for the operation associated with $(**)$.

(In (a) we identify $W(n)$ with $\Omega^{2t-1}W(n+2t-1)$.)

5. *Proof of 4.1*. We continue with the notation of the preceding section. Consider the following diagram:

$$\begin{array}{ccccc}
\Omega Y(n) * \Omega Y(n) & \xrightarrow{k} & K(2n-1) & & \\
\downarrow{\scriptstyle \mu} & & \downarrow{\scriptstyle j} & & \\
S\Omega Y(n) & \xrightarrow{h} & Z(n) & & \\
\downarrow{\scriptstyle c} & & \downarrow{\scriptstyle b(n)} & & \\
Y(n) & = & Y(n) & \xrightarrow[\mathrm{Sq}^n\epsilon_n]{} & K(2n).
\end{array} \tag{5.1}$$

Now $\mathrm{Sq}^n\epsilon_n = \epsilon_n \smile \epsilon_n$, and so $c^*(\mathrm{Sq}^n\epsilon_n) = 0$. Thus c lifts to a map h as shown, such that $b(n) \circ h \simeq c$. By §3, $\Omega Y(n) * \Omega Y(n)$ has the homotopy type of the fiber of c, with μ as fiber inclusion, so there is a map k as shown with, $j \circ k \simeq h \circ \mu$ (j denotes the fiber inclusion).

By construction $Y(n)$ is $(n - 1)$-connected and $\pi_n(Y(n)) = Z_2$. Thus $\Omega Y(n) * \Omega Y(n)$ is $(2n - 2)$-connected (see [16]) and

$$\pi_{2n-1}(\Omega Y(n) * \Omega Y(n)) = Z_2 .$$

Clearly $\epsilon_{n-1} * \epsilon_{n-1}$ is the fundamental class. (Recall that $Y(n - 1) = \Omega Y(n)$ and $\epsilon_{n-1} = \sigma\epsilon_n$.)

LEMMA 5.2. $k^*\iota_{2n-1} = \epsilon_{n-1} * \epsilon_{n-1}$.

Proof. By construction of the fibration $b(n)$, ι_{2n-1} transgresses to $\mathrm{Sq}^n\epsilon_n$, as then does $k^*\iota_{2n-1}$ by naturality. But[1] $\mathrm{Sq}^n\epsilon_n \neq 0$ and so $k^*\iota_{2n-1} \neq 0$, which implies $k^*\iota_{2n-1} = \epsilon_{n-1} * \epsilon_{n-1}$.

Now let

$$\tilde{\phi} \in H^{2n+4t-2}(W(n + 2t - 1)),$$
$$\phi = \sigma^{2t-1}\tilde{\phi} \in H^{n+2t-1}(W(n)),$$
$$\psi = g^*\phi \in H^{n+2t-1}(Z(n)),$$

be classes as given in 4.2. By 4.2(b) ψ represents the operation Ψ associated with relation (**), and so by definition,

$$j^*\psi = \mathrm{Sq}^{2t}\iota_{2n-1} \in H^{2n+2t-1}(K(2n - 1)).$$

Thus by 5.1 and 5.2,

$$\mu^*h^*\psi = k^*\mathrm{Sq}^{2t}\iota_{2n-1} = \mathrm{Sq}^{2t}(\epsilon_{n-1} * \epsilon_{n-1}). \tag{5.3}$$

Now by 2.1, $\epsilon_{n-1} * \epsilon_{n-1} = \Delta(\epsilon_{n-1} \otimes \epsilon_{n-1})$. Clearly Δ commutes with the Sq^i's, and so by the Cartan formula,

$$\mathrm{Sq}^{2t}(\epsilon_{n-1} * \epsilon_{n-1}) = \textstyle\sum_j \mathrm{Sq}^j\epsilon_{n-1} * \mathrm{Sq}^{2t-j}\epsilon_{n-1} + \mathrm{Sq}^{2t-j}\epsilon_{n-1} * \mathrm{Sq}^j\epsilon_{n-1} , \tag{5.4}$$

where the summation runs from 0 to $t - 1$. (Recall that $\mathrm{Sq}^t\epsilon_{n-1} = 0$).

On the other hand by 3.2, for $j \geq 0$,

$$\mu^*s(\mathrm{Sq}^j\epsilon_{n-1} \smile \mathrm{Sq}^{2t-j}\epsilon_{n-1}) = \mathrm{Sq}^j\epsilon_{n-1} * \mathrm{Sq}^{2t-j}\epsilon_{n-1} + \mathrm{Sq}^{2t-j}\epsilon_{n-1} * \mathrm{Sq}^j\epsilon_{n-1} . \tag{5.5}$$

Set

$$\omega = \textstyle\sum_j \mathrm{Sq}^j\epsilon_{n-1} \smile \mathrm{Sq}^{2t-j}\epsilon_{n-1} ,$$

in $H^{2n+2t-2}(\Omega Y(n))$, where the summation runs from 0 to $t - 1$. By 5.3–5.5 we then have

$$\mu^*(h^*\psi - s\omega) = 0. \tag{5.6}$$

Now the Serre exact sequence [10] for the fibration c holds through dimension

[1] See Appendix II, §8.

$3n - 2$. But

$$2n + 2t - 1 \leq 3n - 2,$$

since $2t < n$ by 1.3. Thus by exactness (recalling that μ is homotopic to the fiber inclusion) we obtain by 5.6: there is a class $v \in H^{2n+2t-1}(Y(n))$ such that

$$h^*\psi = s\omega + c^*v. \tag{5.7}$$

PROPOSITION 5.8. *There are classes* $d \in H^{2n+2t-1}Y(n)$ *and* $e \in H^{2n+4t-2}(K(n + 2t - 1))$ *such that*

a) d *is decomposable*
b) $v = d + q(n)^*\sigma^{2t-1}(e)$.

We prove 5.8 in the following section. Using it we now prove Theorem 4.1.

Proof of Theorem 4.1. Define

$$\phi' = \phi - (p(n) \circ a(n))^*e \in H^{2n+4t-2}(W(n + 2t - 1)),$$

and take Φ in 4.1 to be the operation given by the class ϕ'. Set

$$\psi' = g^*\sigma^{2t-1}\phi' \in H^{n+2t-1}(Z(n)),$$

and take Ψ in 4.1 to be the operation given by the class ψ'. With this choice of Φ, Ψ, 4.1(b) is now satisfied.

Notice that by the diagram in §4, $p(n) \circ a(n) \circ g = q(n) \circ b(n)$, and so

$$\psi' = \psi - (q(n) \circ b(n))^*\sigma^{2t-1}(e).$$

Therefore, by 5.7, 5.8, and the commutativity of 5.1,

$$h^*\psi' = s\omega + c^*v - c^*q(n)^*\sigma^{2t-1}(e) = s\omega + c^*d.$$

But d is decomposable by hypothesis, and so $c^*d = 0$. Thus

$$h^*\psi' = s\omega.$$

Since $c^*\epsilon_n = s\sigma\epsilon_{n-1} = s\epsilon_{n-1}$, (see 6.2), this means, by 5.1, that

$$s\omega \in \Psi(s\epsilon_{n-1}).$$

Now $SY(n - 1)$ ($\equiv S\Omega Y(n)$) is the universal example for the class u given in 4.1, and so from the definition of ω we see that Ψ satisfies 4.1(a), which completes the proof.

6. *Proof of 5.8.* Consider the fibration $p(n)$:

$$K(n + t - 1) \xrightarrow{i} Y(n) \xrightarrow{p(n)} K(n),$$

where i denotes the fiber inclusion. Let $v \in H^{2n+2t-1}Y(n))$ be the class given in 5.7. As a first step towards proving 5.8 we have

LEMMA 6.1. $i^*v = 0$.

We begin the proof by giving a specific choice for the map h which occurs in diagram 5.1. By definition $Z(n - 1)$ is the total space of the fibration with $\mathrm{Sq}^n \epsilon_{n-1}$ as classifying map. But $\mathrm{Sq}^n \epsilon_{n-1} = 0$, and so up to homotopy type we have

$$Z(n - 1) = Y(n - 1) \times K(2n - 2).$$

Let $d: Y(n - 1) \to Z(n - 1)$ denote the canonical inclusion, and $\pi: Z(n - 1) \to Y(n - 1)$ the projection. Notice that

$$\pi d = \text{identity, and } \pi \simeq b(n - 1) = \Omega b(n).$$

Recall that if we are given spaces A, B and a map $f: A \to \Omega B$, there is associated canonically a map

$$g: SA \to B$$

given by $g(a, t) = f(a)(t)$. Now $Z(n - 1) = \Omega Z(n)$. Let

$$h: SY(n - 1) \to Z(n)$$

denote the map corresponding to d. Since $\pi d =$ identity and since $\pi \simeq \Omega b(n - 1)$, one readily shows that

$$b(n) \circ h \simeq c,$$

the canonical map associated to $1: Y(n) = Y(n)$. In other words, in diagram 5.1 we have given a specific choice for the map h.

Let $f: A \to \Omega B$, $g: SA \to B$ be maps related as above. One then has (see [1])

$$f^* \sigma = s^{-1} g^*, \tag{6.2}$$

on cohomology.

By 5.7 we have

$$s^{-1} h^* \psi = \omega + s^{-1} c^* v,$$

and so by 6.2 and the definition of h and c we obtain

$$d^* \sigma \psi = \omega + \sigma v. \tag{6.3}$$

Consider now the following commutative diagrams:

$$\begin{array}{ccc} K(2n - 1) \times K(n + t - 1) & \xrightarrow{j} & Z(n) \\ \downarrow \pi & & \downarrow b(n) \\ K(n + t - 1) & \xrightarrow{i} & Y(n) \end{array}$$

$$\begin{array}{ccc} K(2n - 2) \times K(n - t - 2) & \xrightarrow{j'} & Y(n - 1) \times K(2n - 2) \\ \uparrow d' & & \uparrow d \\ K(n - t - 2) & \xrightarrow{i'} & Y(n - 1). \end{array}$$

Here j is the fiber inclusion for the fibration $q(n) \circ b(n): Z(n) \to K(n)$, π is the projection, $i' = \Omega i$, $j' = \Omega j$, and $d' = d \mid K(n-t-2)$. We have identified $\Omega Z(n)$ with $Y(n-1) \times K(2n-2)$, and so the spaces in the right hand diagram are all considered to be the loops of the appropriate spaces in the left hand diagram.

By definition, $\sigma\psi$ is a class in $H^{2n+2t-2}(Z(n-1))$ such that

$$j'^*\sigma\psi = \mathrm{Sq}^{2t}\iota_{2n-2} \otimes 1 + 1 \otimes \mathrm{Sq}^{n+t}\iota_{n-t-2} = \mathrm{Sq}^{2t}\iota_{2n-2} \otimes 1,$$

since $\mathrm{Sq}^{n+t}\iota_{n-t-2} = 0$. Therefore

$$i'^*d'^*\sigma\psi = d'^*j'^*\sigma\psi = 0,$$

and so by 6.3,

$$i'^*\omega + i'^*\sigma v = 0.$$

By definition $\omega = \sum_j \mathrm{Sq}^j\epsilon_{n-1} \smile \mathrm{Sq}^{2t-j}\epsilon_{n-1}$; since $i'^*\epsilon_{n-1} = 0$ and since $i'^*\sigma = \sigma i^*$, we obtain:

$$\sigma(i^*v) = 0.$$

We now must consider separately the two cases, t odd and t even. Suppose first that t is odd. Then i^*v is a class in $H^{2n+2t-1}(K(Z, n+t-1))$, (see §4), but by Serre [11] σ is a monomorphism in this dimension, and so $i^*v = 0$ as claimed. Suppose, on the other hand, that t is even. Then i^*v is a class in $H^{2n+2t-1}(K(n+t-1))$. By Serre [11], the kernel of σ in degree $2n+2t-1$ is generated by $\iota' \smile \mathrm{Sq}^1\iota'$, where $\iota' = \iota_{n+t-1}$. Therefore

$$i^*v = a(\iota' \smile \mathrm{Sq}^1\iota'), \tag{6.4}$$

where $a \in Z_2$.

Consider now the extended fiber sequence given by $p(n)$ [5]:

$$K(n-1) \xrightarrow{\ \ell\ } K(n+t-1) \xrightarrow{\ i\ } Y(n) \xrightarrow{\ p(n)\ } K(n).$$

The map i can be regarded as a fiber map with ℓ as fiber inclusion, where ℓ is given by $\ell^*\iota' = \mathrm{Sq}^t\iota_{n-1}$. Hence,

$$\ell^*(\iota' \smile \mathrm{Sq}^1\iota') = \mathrm{Sq}^t\iota_{n-1} \smile \mathrm{Sq}^{t+1}\iota_{n-1} \neq 0,$$

since t is even (by 1.3)). But by exactness, $\ell^*i^*v = 0$, and so $a = 0$ in 6.4. Thus $i^*v = 0$, which proves 6.1.

To simplify matters we adopt the following notation; fix n and set

$$Y = Y(n), \qquad K = K(n), \qquad F = K(n+t-1),$$
$$\iota = \iota_n, \qquad x = \mathrm{Sq}^t\iota_n, \qquad q = q(n).$$

In order to prove 5.8 we need to refer to results from another paper [15]. There we have defined morphisms

$$\nu: H^j(Y) \cap \mathrm{Kernel}\ i^* \to H^j(Y \# F),$$
$$\tau: H^j(Y \# F) \to H^{j+1}(K)/I_j,$$

where $0 \leq j \leq 2n + 2t - 1$, and where $I_j = 0$ if $j < 2n + 2t - 1$ and $I_{2n+2t-1}$ denotes the (linear) subspace of $H^{2n+2t}(K)$ spanned by x^2.

These morphisms have the following properties.

(6.5) The following sequence is exact:

$$H^{2n+2t-1}(K) \xrightarrow{q^*} (n)\, H^{2n+2t-1}(Y) \cap \text{Kernel } i^*$$

$$\xrightarrow{\nu} H^{2n+2t-1}(Y \# F) \xrightarrow{\tau} H^{2n+2t}(K)/(x^2).$$

(6.6) Given classes u and v in $\bar{H}^*(Y)$ such that $i^*u = 0$ and $\deg u + \deg v \leq 2n + 2t - 1$, then

$$\nu(u \smile v) = u \otimes i^*v.$$

(6.7) Given classes v in $\bar{H}^*(F)$ and w in $\bar{H}^*(K)$ such that $\deg v + \deg w \leq 2n + 2t - 1$, then

$$\tau(q^*w \otimes v) = w \smile \hat{\tau}(v),$$

where $\hat{\tau}$ denotes the Serre transgression operator for the fibration

$$F \xrightarrow{i} Y \xrightarrow{q} K.$$

Using this material the proof of 5.8 follows at once from

LEMMA 6.8. *Let $w \in H^{2n+2t-1}(Y \# F)$ be a class in the kernel of τ. Then there is a decomposable class $d \in H^{2n+2t-1}(Y) \cap$ Kernel i^* such that*

$$\nu(d) = w.$$

Assuming this for the moment we prove 5.8.

Proof of 5.8. Let v be any class in $H^{2n+2t-1}(Y) \cap$ Kernel i^*. By the exactness of 6.5, $\tau\nu(v) = 0$, and so by 6.8 there is a decomposable class d_1 such that

$$\nu(d_1) = \nu(v).$$

Set $v_1 = v - d_1$. Then $\nu(v_1) = 0$, and so again by exactness there is a class $e_1 \in H^{2n+2t-1}(K)$ such that $q^*e_1 = v_1$. By Serre [11] there is a decomposable class $d_2 \in H^{2n+2t-1}(K)$ and a class $e \in H^{2n+4t-2}(K(n + 2t - 1))$ such that

$$e_1 = d_2 + \sigma^{2t-1}(e).$$

Set $d = d_1 + q^*d_2$. We then have

$$v = d + q^*\sigma^{2t-1}(e),$$

which completes the proof of 5.8. (We have proved in 6.1 that if v is the class given in 5.8, then $v \in$ Kernel i^*.)

We are left with proving 6.8. For this we need the following lemmas about vector spaces.

LEMMA 6.9. *Let U, V, and W be vector spaces and let $\alpha: V \to W$ be a linear map.*

Set $\beta = 1 \otimes \alpha : U \otimes V \to U \otimes W$. *Then*

$$\text{Kernel } \beta = U \otimes \text{Kernel } \alpha.$$

For proof see, for example, [6, 5.9.8]. We will need the following special case.

LEMMA 6.10. *Let* U, $\{V_i\}$, $\{W_i\}$, $1 \leq i \leq n$, *be vector spaces, and let* $\alpha : V_i \to W_i$ *be linear maps,* $1 \leq i \leq n$. *Set*

$$P = U \otimes (V_1 \oplus \cdots \oplus V_n), \qquad Q = U \otimes (W_1 \oplus \cdots \oplus W_n),$$

and define $\beta : P \to Q$ *by*

$$u \otimes (v_1, \cdots, v_n) \mapsto u \otimes (\alpha_1 v_1, \cdots, \alpha_n v_n).$$

Then,

$$\text{Kernel } \beta = U \otimes (\text{Kernel } \alpha_1 \oplus \cdots \oplus \text{Kernel } \alpha_n).$$

Let R be a Z_2-polynomial algebra and let M be the linear subspace spanned by a set of generators for R. Let S and T be subspaces of M such that $S \cap T = 0$. Finally, let m be any element in M, and let (m^2) denote the (linear) subspace of R generated by m^2. Define a map

$$\gamma : S \otimes T \to R/(m^2)$$

by $s \otimes t \to \{st\}$, where the brackets denote the coset of st in $R/(m^2)$.

LEMMA 6.11. *The linear map* γ *is injective.*

The proof is elementary and is left to the reader.

Proof of 6.8. Recall that $H^*(K)$ is a Z_2-polynomial algebra on generators given by admissible sequences of Steenrod operators applied to the fundamental class. (See Serre [11].) In particular, the cohomology of K through dimension $2n - 1$ is spanned by generators. Define a subspace S of $\bar{H}^*(K)$ by $S = (\sum_{i=n}^{n+t-1} \cdot H^i(K)) \oplus$ (subspace of $H^{n+t}(K)$ spanned by all generators except x). Then by Serre [10],

$$(6.12) \qquad q^* : S \approx \textstyle\sum_{i=n}^{n+t} (Y).$$

Similarly, define

$T =$ (subspace of $H^{n+t}(K)$ spanned by x) $\oplus$ $(\sum_{j=n+t+1}^{n+2t} H^j(K))$. By 1.3, $n + 2t < 2n$, and so T (as well as S) is a linear subspace of $H^*(K)$ spanned by generators. Define

$$\gamma : S \otimes T \to H^*(K)/(x^2),$$

by

$$u \otimes v \to \{uv\}.$$

According to 6.11, γ is injective. Let

$$\hat{\tau}_j : H^j(F) \to H^{j+1}(K),$$

$0 \leq j \leq n + t - 2$, denote the Serre transgression. Define

$$\beta : S \otimes \left(\sum_{j=n+t-1}^{n+2t-1} H^j(F)\right) \to S \otimes T$$

by

$$u \otimes (v_0, \cdots, v_t) \mapsto u \otimes (\hat{\tau}_{n+t-1}v_0, \cdots, \hat{\tau}_{n+2t-1}v_t),$$

where $v_i \in H^{n+t-1+i}(F)$, $0 \leq i \leq t$. By Lemma 6.10,

$$\text{Kernel } \beta = S \otimes (\text{Kernel } \hat{\tau}_{n+t-1} \oplus \cdots \oplus \text{Kernel } \hat{\tau}_{n+2t-1}). \tag{6.13}$$

On the other hand, given classes $u \in S$ and $v \in H^j(F)$, such that $j + \deg u = 2n + 2t - 1$, we have by 6.7,

$$\tau(q^*u \otimes v) = \{u \smile \hat{\tau}_j(v)\} = \gamma\beta(u \otimes v).$$

Therefore, since γ is injective and q^* is an isomorphism in 6.12, it follows by 6.13 that

$$\begin{aligned} &\text{Kernel } \tau \text{ in dimension } 2n + 2t - 1 \\ &\qquad = \sum_{j=n+t-1}^{n+2t-1} H^{2n+2t-1-j}(Y) \otimes \text{Kernel } \hat{\tau}_j . \end{aligned} \tag{6.14}$$

Using 6.14 we now can prove 6.8. Suppose then that $w \in H^{2n+2t-1}(Y \# F)$ is a class in the kernel of τ. By 6.14 we can write w as a sum of terms of the form $u \otimes v$, where $u \in \bar{H}^*(Y)$, $v \in \bar{H}^*(F)$, $\hat{\tau}(v) = 0$, and $\deg u + \deg v = 2n + 2t - 1$. In particular $\deg u \leq n + t$; thus u is in Image q^* and so $i^*u = 0$. (We use here the fact that $x \neq 0$ and that $Sq^1 x \neq 0$ when t is even. Thus q^* is surjective through deg $n + t$.) Since $\deg v \leq n + 2t - 1$, it follows by Serre [10] that

$$v = i^*\bar{v}, \qquad \text{where } \bar{v} \in \bar{H}^*(Y).$$

But by 6.6,

$$u \otimes i^*\bar{v} = \nu(u \smile \bar{v}),$$

and so w can be written as a sum of terms of the form $\nu(u \smile \bar{v})$, where $i^*u = 0$. Since $u \smile \bar{v}$ is decomposable, the proof of 6.8 is complete.

7. Appendix I

Let t and n be positive integers such that $2t \leq n$. By Adem [2] we have the following relation:

$$Sq^{2t}Sq^n + Sq^{n+t}Sq^t = \sum_{i=0}^{t-1} \binom{n-1-i}{2t-2i} Sq^{2t+n-i}Sq^i.$$

The fact that there is a relation (*) satisfying (1.1)–(1.4), is then an immediate consequence of the following lemma.

LEMMA 7.1. *Let b and a be positive integers with $b > 2a$. Then*

$$Sq^b Sq^a = \sum \gamma_i \delta_i + \epsilon Sq^{b+a},$$

where $\epsilon = 0$ or 1, and where each operation δ_i is zero on classes of degree $< b - a$.

Proof: Again by Adem [2],

$$\mathrm{Sq}^b\mathrm{Sq}^a = \mathrm{Sq}^{2a}\mathrm{Sq}^{b-a} + \sum_{i=0}^{a-1} \binom{b-a-1-i}{2a-2i} \mathrm{Sq}^{b+a-i}\mathrm{Sq}^i.$$

Since Sq^{b-a} is zero on classes of degree $< b - a$, the lemma follows at once by an inductive argument on a. (To start the induction we note that with $a = 1$, $\mathrm{Sq}^b\mathrm{Sq}^1 = \mathrm{Sq}^2\mathrm{Sq}^{b-1} + \epsilon\mathrm{Sq}^{b+1}$.) We leave the details to the reader.

8. Appendix II

We stated in §5 that $\mathrm{Sq}^n\epsilon_n \neq 0$; we give the proof in this section. By considering the Serre exact sequence for the fibration

$$K(n+t-1) \xrightarrow{i} Y(n) \xrightarrow{q(n)} K(n),$$

we see that $\mathrm{Sq}^n\epsilon_n = 0$ if, and only if, there is an element α in the mod 2 Steenrod algebra such that

$$(*) \qquad \alpha\mathrm{Sq}^t = \mathrm{Sq}^n.$$

THEOREM 8.1. *Let n and t be positive integers, with $t < n$, and let α be an element of the* mod 2 Steenrod *algebra such that* $\deg \alpha = n - t$. *If*

$$\alpha\mathrm{Sq}^t = \mathrm{Sq}^n,$$

then $2t > n$.

By hypothesis 1.3, $2t < n$ and hence $(*)$ cannot hold, for our choice of n and t. Thus $\mathrm{Sq}^n\epsilon_n \neq 0$, as asserted in §5.

The proof of 8.1 will follow from a more technical result. For any positive integer q, consider its dyadic expansion:

$$q = 2^i + \cdots + 2^j, \qquad 0 \leq i < \cdots < j.$$

Define

$$\sigma(q) = \mathrm{i}, \qquad \lambda(q) = j.$$

(If $q = 2^k$, then $\sigma(q) = k = \lambda(q)$). We will prove

PROPOSITION 8.2. *Let n, t and α be as in* 8.1 *and suppose that* $(*)$ *holds. Then*

$$\binom{n}{t} \equiv 1 \bmod 2 \quad \textit{and} \quad \sigma(t) \geq \lambda(n-t).$$

Assuming this we see that 8.1 follows at once. For if $\binom{n}{t} \equiv 1$ and $\sigma(t) \geq \lambda(n-t)$, then in particular $2t > n$, as is easily seen by writing out the dyadic expansions for n and t.

To prove 8.2 we show first that if $(*)$ holds then $\binom{n}{t} \equiv 1 \bmod 2$; for this we use the cohomology of the stable rotation group SO. Recall that $H^*(\mathrm{SO})$ has a simple system of generators $h_1, h_2, \cdots, h_j, \cdots$ where h_j has degree j, such that

$$\mathrm{Sq}^i h_j = \binom{j}{i} h_{i+j}. \tag{8.3}$$

(See [4].) Thus

$$\mathrm{Sq}^n h_n = h_{2n} \neq 0.$$

If $(*)$ holds, then

$$h_{2n} = \alpha \mathrm{Sq}^t h_n = \binom{n}{t} \alpha h_{n+t},$$

and so $\binom{n}{t} \equiv 1 \bmod 2$ as claimed.

To show that $\sigma(t) \geq \lambda(n-t)$, we argue by assuming the opposite. That is, we assume now that

$$\binom{n}{t} \equiv 1 \bmod 2 \quad \text{and} \quad \sigma(t) < \lambda(n-t), \tag{$**$}$$

and show that this implies that $(*)$ cannot hold. Let

$$g = \sigma(t), \qquad h = \lambda(t), \qquad k = \lambda(n).$$

Thus,

$$t = 2^g + \cdots + 2^h,$$
$$n = e + 2^g + \cdots + 2^k,$$

where either $e = 0$ or $e > 0$ and $\lambda(e) < g$. Since $\sigma(t) < \lambda(n-t)$, we have $k > g$. Define Q to be the integer obtained from n by "filling" in all missing powers of two between g and k. Thus

$$Q = e + 2^g + 2^{g+1} + \cdots + 2^{k-1} + 2^k.$$

Now $\binom{Q}{n} \equiv 1 \bmod 2$, and so by 8.3,

$$\mathrm{Sq}^n h_Q = h_{n+Q}.$$

We prove

LEMMA 8.4. *Let n and t be integers satisfying $(**)$. Then for all sets of positive integers $(a, b, \cdots, d)$ such that $a + b + \cdots + d = n - t$ we have*

$$\mathrm{Sq}^a \mathrm{Sq}^b \cdots \mathrm{Sq}^d \mathrm{Sq}^t h_Q = 0.$$

Since α, in $(*)$, must be a sum of monomials of the form $\mathrm{Sq}^a \cdots \mathrm{Sq}^d$ given above, this shows that

$$\alpha \mathrm{Sq}^t h_Q = 0,$$

which means that $(*)$ cannot hold. Thus we are left with proving 8.4.

Proof of 8.4. Notice that

$$Q + 2^g = 2^{k+1} + e$$

and so

$$Q + t = 2^{k+1} + P + e$$

where $P = t - 2^g$. Let $(a, b, \cdots, d)$ be integers as in 8.4. We suppose now that

$$\mathrm{Sq}^a\mathrm{Sq}^b \cdots \mathrm{Sq}^d\mathrm{Sq}^t h_Q = h_{n+Q}\,.$$

and show that this then leads to a contradiction.

By 8.3, since $2^{k+1} > n$, we must have

$$\binom{P+e}{d} \equiv 1, \cdots, \binom{P+e+d+\cdots+b}{a} \equiv 1, \tag{8.5}$$

all mod 2. Now for any positive integer s, let

$$\alpha(s) = \text{number of ones in the dyadic expansion of } s.$$

(We set $\alpha(0) = 0$.) It is easily seen that if $\binom{s}{r} \equiv 1$, then $\alpha(r + s) \leq \alpha(s)$, and so by 8.5 we have

$$\alpha(P + e + d + \cdots + b + a) \leq (P + e). \tag{8.6}$$

But

$$\alpha(P + e) = \alpha(P) + \alpha(e), \quad \text{and} \quad \alpha(P) = \alpha(t) - 1.$$

Moreover,

$$P + e + d + \cdots + b + a = n - 2^g + e$$

and so by 8.6 we obtain

$$\alpha(n - 2^g + e) \leq \alpha(t) + \alpha(e) - 1. \tag{8.7}$$

On the other hand, $\lambda(e) < g$ and so $\lambda(2e) \leq g$. Thus,

$$\alpha(n - 2^g + e) = \alpha(n) - 1. \tag{8.8}$$

Finally, because $\lambda(n - t) > \sigma(t)$, one easily shows that

$$\alpha(t) + \alpha(e) \leq \alpha(n) - 1.$$

Thus, by 8.7 and 8.8, we have

$$\alpha(n) - 1 \leq \alpha(n) - 2,$$

which is a contradiction. This completes the proof.

BOSTON COLLEGE,
UNIVERSITY OF CALIFORNIA, BERKELEY, U.S.A.

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