SOME HOMOTOPY EQUIVALENT MANIFOLDS

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1. Introduction

It is well known that some homotopy equivalent manifolds are not even homeomorphic. Such examples of low dimensional smooth manifolds are lens spaces. Examples of high dimensional simply-connected, non-smooth manifolds are given in [4, Theorem 4]. However, it is not known whether high dimensional smooth, simply-connected, homotopy equivalent manifolds are homeomorphic. In this note we shall establish two theorems on cobordant homotopy equivalent manifolds with a highly-connected trace. In some cases the homotopy equivalent manifolds are only different from a connected-sum of a homotopy sphere. The first theorem which is a generalization of [9] is to deal with the non-simply-connected manifolds. The second theorem deals with two non-vanishing relative homology of the trace. These results are outgrowths of the study of Novikov's paper [2] and I. Tamura's paper [6].

2. Preliminaries

Manifolds considered here are to be compact, oriented, connected, and differentiable. Let G, π be respectively homotopy group of middle dimension and fundamental group of a manifold N^{2k}. Let $S(G)$ be the set of families of generators of G, and $S(\pi)$ be the set of subset of π. We define $[S(G), S(\pi)]$ to be certain class of functions from $S(G)$ to $S(\pi)$ as follows: Let $[S_1^k, S_2^k, \cdots, S_r^k]$ be a collection of immersed spheres in N^{2k} representing a set of generators of G. For dimensional reason, we may assume the singularities are isolated double points. On each sphere we connect the singularities by a tree, a contractible 1-complex. If a point P is a self-intersection of the sphere, we allow two branches of the tree to meet at the point. The totality of these trees forms a 1-complex K in N. The homotopy class of each component of K defines an element in the fundamental group of N, up to an inner automorphism of π. A function in $[S(G), S(\pi)]$ is trivial if for some choices of immersed spheres representing a set of generator of G, the resulting K gives only the identity element in π.

3. Theorems

THEOREM 1. *Let M_1^{2k-1} and M_2^{2k-1}, $(k \geqq 3)$, be two homotopy equivalent manifolds with fundamental group π, satisfying the following hypotheses:*

i) *There is a manifold N^{2k} with boundary $\partial N = M_2 \cup -M_1$ and i: $M_2 \to N$, the inclusion is $(k-1)$-connected, i.e., $\pi_q(N, M_2) = 0$, for $q = 1, 2, \cdots, (k-1)$.*

ii) *There is a continuous map g: $N \to M_2$ such that $g \mid M_2$ is the identity and $g \mid M_1$ is the homotopy equivalence.*

Then $\pi_k(N, M_2)$ can be realized in N by a handle body if and only if $[S\pi_k(N, M_2), S(\pi)]$ contains a trivial function. If this holds, then M_1 is h-cobordant to $M_2 \# \Sigma$, where $\#$ is the connected-sum and Σ is a homotopy sphere.

Note that if $\pi = 0, Z_2, Z_3, Z_4, Z_6$ or Z (cyclic groups), the Whitehead torsions are trivial, in turn, h-cobordant manifolds are diffeomorphic.

THEOREM 2. *Let M_1^{2k} and M_2^{2k} be two 2-connected homotopy equivalent manifolds with $k > 4$, satisfying the following hypotheses:*

i) *There is a 2-connected manifold N^{2k+1} with boundary $\partial N = M_2 \cup -M_1$ and the relative homology groups $H_q(N, M_1; Z) = Z + Z + \cdots + Z$ for $q = k, k+1$, and isomorphic to 0 for other q.*

ii) *The same as in theorem 1.*

Then M_1 is diffeomorphic to $M_2 \# \Sigma$.

COROLLARY. *If $k \equiv 3, 5, 6, 7$ Mod 8, the conclusion of theorem 2 is that M_1 is diffeomorphic to M_2.*

4. Proofs of theorems

Proof of theorem 1. By the hypothesis 2, the homotopy and the homology exact sequences of pairs (N, M_1) and (N, M_2) split (see [9]). By the Whitney imbedding theorem we can realize $\pi_k(N, M_2)$, which is regarded as a subgroup of $\pi_k(N)$, by immersed spheres with only isolated double points as singularities. On each sphere we connect these singularities by a tree. The totality of these trees is a 1-complex in N. It is clear that these spheres have no singularities in the complement of a neighborhood of K. Since $[S\pi_k(N, M_2), S(\pi)]$ contains a trivial function, K is contractable in N. By the lemma 2.7 of [5], there exists a $2k$-cell E containing K. For dimensional reason, there is no danger of E meeting the non-singular parts of the spheres. E is only a regular neighborhood of a cone of K. E serves as the $2k$-cell of the handle body, and the non-singular parts of immersed spheres (the complements of neighborhoods of the trees) together with their small normal neighborhoods in N as handles form a handle body. The boundary of this handle body is a homotopy sphere by a lemma of Wall [p. 169, 8]. The lemma says that the boundary of a $(k-1)$-connected $2k$-handle body is a homotopy sphere if the intersection pairing is non-singular. Since the handle body is simply-connected, we can lift the handle body to the universal covering space $\tilde{N}$ of N. Remove the interior of the handle body and a path connecting it to $\tilde{M}_2$. We also remove the corresponding parts in N. Now M_1 and $M_2 \# \Sigma$ are h-cobordant because their universal covering spaces are homotopy equivalent by a theorem of Whitehead.

Proof of theorem 2. By the homology hypothesis there is a Morse function defined on N with indices compatible with the homology [6]. Let us assume temporarily that $H_k(N, M_1; Z) = Z$, only one copy of infinite cyclic group. Then the Morse function implies that N is obtained from thickening M_1 with two handles attached on one side of it. That is, $N = M_1 \times [0, 1] \cup_{\varphi} [D^k \times D^{k+1}] \cup_{\eta} [D^{k+1} \times D^k]$, where $\varphi: \partial D^k \times D^{k+1} \to M_1 \times \{1\}$ a disjoint differentiable imbedding which attaches a handle to $M_1 \times \{1\}$, and $\eta: \partial D^{k+1} \times D^k \to M_1 \times [0, 1] \cup_{\varphi} [D^k \times D^{k+1}]$ likewise. Since φ does not kill any homotopy group $\pi_{k-1}(M_1)$, φ is null-homotopic. η cannot be onto the first handle otherwise it contradicts to the fact that $H_k(N, M_1; Z)$ is free. η is also null-homotopic because it does not kill any $\pi_k(M_1)$. By the Haefliger's theorem [1, theorem d'existence] we can find a differentiable k-disk in $M_1 \times [0, 1]$ bounding $\varphi(\partial D^k \times \{0\})$ in N. This handle together with a neighborhood of the disk form a D^{k+1}-bundle over a k-sphere. Since η is null-homotopic, we may assume $\eta(\partial D^{k+1} \times \{0\})$ covered by a $(k + 1)$-disk $\{0\} \times D^{k+1}$ of $D^k \times D^{k+1}$ which is the normal neighborhood of the previous k-disk. This is a plumbing manifold. Its boundary is a homotopy sphere Σ by Milnor [2]. Remove the interior of this plumbing manifold and a path connecting it to M_2. Examining the Mayer-Vietoris sequence, we see that M_1 and $M_2 \# \Sigma$ are h-cobordant, therefore diffeomorphic. If $H_k(N, M_1) = H_{k+1}(N, M_1)$ is isomorphic to more than one copy of Z, we simply pair them by duality and remove them in the same manner.

Proof of the corollary. Take two copies of D^k-bundle H_η over a $(k + 1)$-sphere and identify their boundaries. That is to form a double. Now we remove a disk from one of the $(k + 1)$-spheres and remove also the fibres over the disk. Then the resulting manifold is $D^k \times D^{k+1} \cup H_\eta$ which has a standard sphere as boundary. On the other hand, if $k \equiv 3, 5, 6, 7 \text{ Mod } 8$, any D^{k+1}-bundle H_φ over k-sphere is trivial by Bott's stable homotopy theorem. The plumbing manifold $H_\varphi \# H_\eta = D^k \times D^{k+1} \cup H_\eta$. This argument is due to I. Tamura in [7, lemma 2].

Added in proof. Our theorem 2 overlaps a theorem of R. DeSapio (*Almost diffeomorphisms of manifolds*, Pacific J. Math. **26** (1968) p. 56). However, our corollary to the theorem is still outstanding. An example of two 8-dimensional homotopy equivalent, but not homeomorphic, smooth manifolds was discovered by R. Lashof and M. Rothenberg (*On the Hauptvermutung, Triangulation of Manifolds, and h-cobordism.* Bull. Amer. Math. Soc. **72** (1966) 1040–43).

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