RESULTS CONCERNING THE SCHUTZENBERGER-WALLACE THEOREM

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The principal purpose of this paper is to extend, dualize and simplify algebraic results due to J. R. Bastida and given in references [1] and [2]. The extensions are manifold in character and include what is termed the "relative" case with respect to Green's Relations (whereas Bastida treats only the "absolute" case) and, in addition, results of a non-discrete type are presented which Dr. Bastida does not consider. As to the duality, Bastida examines one-half of the possible left-right duality and in section 2 conditions are given under which the structures obtained by reversing the multiplication are topologically and algebraically the same. Simplicity is introduced because in the preliminary propositions those properties of H-slices which are truly necessary for the validity of the arguments are isolated. It is then shown that the Schutzenberger-Wallace Theorem is a consequence of these extensions of Bastida's results.

Moreover, under the assumption that S is compact, in the first section it is shown that the Dubreil semigroups $(aS)a^{(-1)}/\mathcal{E}(a)$ and $a^{(-1)}(Sa)/\mathfrak{F}(a)$ are iseomorphic. In section 2 it is then indicated that these Dubreil semigroups are extensions of the Schutzenberger groups in the sense that each Schutzenberger group is a subgroup of a Dubreil semigroup.

§1

This initial section will present introductory material including an important result which is due to P. Dubreil in its algebraic setting.

1.1. Definition. A *topological semigroup* S is a nonnull Hausdorff space together with a continuous associative multiplication. Precisely, a semigroup is such a function $m: S \times S \rightarrow S$ that

(i) S is a nonnull Hausdorff space,

(ii) m is continuous, and

(iii) m is associative; i.e., for each x, y, z in S, $m(x, m(y, z)) = m(m(x, y), z)$.

At times it will be necessary to distinguish between a semigroup and its non-topological counterpart, an *algebraic semigroup*. It is common usage to say that a semigroup is compact if S is a compact space and to say that a subset of S is closed if it is closed in a topological sense.

1.2 Definitions. The empty set will be designated by $\square$. If X and Y are subsets of S, then $X^{(-1)}Y = \{w \text{ in } S; Xw \cap Y \neq \square\}$, $X^{[-1]}Y = \{w \text{ in } S; Xw \subset Y\}$, $YX^{(-1)} = \{w \text{ in } S; wX \cap Y \neq \square\}$, and $YX^{[-1]} = \{w \text{ in } S; wX \subset Y\}$.

It is noted that if X is a singleton set, then $X^{(-1)}Y = X^{[-1]}Y$ and $YX^{(-1)} = YX^{[-1]}$.

The next result is but a fragment of a result due to A. D. Wallace, another part of which is found in [4].

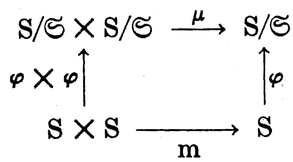
1.3 Proposition. *If Y is closed, then $X^{[-1]}Y$ is closed.*

Proof. It is easily verified that $X^{[-1]}Y = \bigcap_{a \in X} a^{(-1)}Y$ and since the function $la: S \to S$ defined by $la(s) = as$ is continuous, we have that $(la)^{-1}(Y) = a^{(-1)}Y$ is closed and, consequently, the desired result follows immediately.

1.4 Definition. Letting Y be a subset of S and Δ be the diagonal of $Y \times Y$, then an equivalence relation $\mathcal{E} \subset Y \times Y$ is a *closed congruence* on Y if and only if $\Delta\mathcal{E} \cup \mathcal{E}\Delta \subset \mathcal{E}$ and $\mathcal{E}$ is closed in $Y \times Y$ with respect to the relative topology.

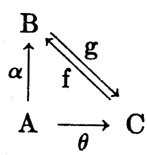
The next two results are well known and are stated here without proof. (The reader may refer to [5] and [7].)

1.5 Proposition. *If S is compact or discrete, if $\mathfrak{S}$ is a closed congruence on S and if $\varphi: S \to S/\mathfrak{S}$ is the canonical map, then there is a unique continuous function μ such that the diagram*



is analytic and thus $S/\mathfrak{S}$ is a semigroup and φ is a continuous homomorphism.

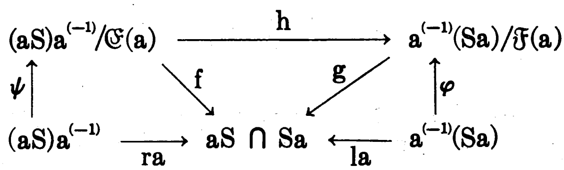
1.6 Theorem (Sierpinski). *If $\alpha: A \to B$ and $\theta: A \to C$ are onto functions with the property that $\alpha(a_1) = \alpha(a_2)$ if and only if $\theta(a_1) = \theta(a_2)$, then in the diagram there exist mutually inverse functions f and g such that $g\alpha = \theta$ and $f\theta = \alpha$. Furthermore, if A, B and C are semigroups and α, θ are morphisms, then f, g are isomorphisms; if, in addition, A is compact, B, C are Hausdorff and α, θ are continuous, then f, g are iseomorphisms.*



1.7 Definitions. Throughout this study, and in particular the next theorem, we will make frequent use of the following functions: If $a \in S$ and $B \subset S$ we will define $ra: B \to S$ by $ra(b) = ba$ and $la: B \to S$ by $la(b) = ab$. It is noted that the image of la is $aS \cap Sa$ if $B = a^{(-1)}(Sa)$, because if x is in $aS \cap Sa$, say $x = as = s'a$, then there exists an element in $a^{(-1)}(Sa)$, namely s, such that $x = as$ and, consequently, la maps $a^{(-1)}(Sa)$ onto $aS \cap Sa$. Moreover, $s' \in (aS)a^{(-1)}$ and $x = s'a$ so that ra maps $(aS)a^{(-1)}$ onto $aS \cap Sa$.

1.8 Theorem (Dubreil). *If S is compact or discrete, $a \in S$ and if we define $\mathfrak{E}(a) = \{(x, y); x, y \in (aS)a^{(-1)}$ and $xa = ya\}$ and $\mathfrak{F}(a) = \{(u, v); u, v \in$*

$a^{(-1)}(Sa)$ and $au = av\}$, then $\mathfrak{E}(a)$ and $\mathfrak{F}(a)$ are congruences on the semigroups $(aS)a^{(-1)}$ and $a^{(-1)}(Sa)$, respectively. If ψ and φ are the appropriate natural homomorphisms in the diagram, then f, g and h are such homeomorphisms that $f\psi = ra$, $g\varphi = la$ and $g^{-1}f = h$; moreover, h is an isomorphism.



Proof. The sets $a^{(-1)}(Sa)$ and $(aS)a^{(-1)}$ are nonempty because $a \in a^{(-1)}(Sa) \cap (aS)a^{(-1)}$. The fact that $a^{(-1)}(Sa)$ is closed follows from (1.3), since S is compact and Hausdorff, and it is immediate that $a^{(-1)}(Sa)$ is an algebraic semigroup; in a similar fashion $(aS)a^{(-1)}$ is a closed semigroup. It is clear that $\mathfrak{F}(a)$ is an equivalence and a right congruence on $a^{(-1)}(Sa)$. If u, x, $y \in a^{(-1)}(Sa)$ and $ax = ay$, then for some $s \in S$, $aux = sax = say = auy$ and, consequently, $\mathfrak{F}(a)$ is also a left congruence. To see that $\mathfrak{F}(a)$ is closed we merely note that $\mathfrak{F}(a) = (a^{(-1)}(Sa) \times a^{(-1)}(Sa)) \cap [(la) \times (la)]^{-1}(\Delta)$, where Δ is the diagonal of $S \times S$, and that $(la) \times (la)$ is continuous. Therefore, in view of (1.5), $a^{(-1)}(Sa)/\mathfrak{F}(a)$ is a semigroup. Clearly, $\varphi(x) = \varphi(y)$ if and only if $la(x) = la(y)$. Since la maps $a^{(-1)}(Sa)$ onto $aS \cap Sa$, in view of Sierpinski's result we see that such a homeomorphism g exists.

Arguments which are dual to the preceding ones yield the other half of the diagram and, clearly, $h = g^{-1}f$ is a homeomorphism.

If c is in $(aS)a^{(-1)}$, it is easy to see that $f(\psi(c)) = ca$ since $f\psi = ra$ and hence that $g^{-1}f(\psi(c)) = g^{-1}(ca) = g^{-1}(an) = \varphi(n)$ for some $n \in a^{(-1)}(Sa)$, the last equality holding due to the analyticity of the right-hand side of the diagram. Now, if b, c are in $(aS)a^{(-1)}$ and m, n are elements of $a^{(-1)}(Sa)$ such that $ba = am$ and $ca = an$, it follows that $bca = ban = amn$ and therefore $g^{-1}f(\psi(b)\psi(c)) = g^{-1}f(\psi(bc)) = g^{-1}(bca) = g^{-1}(amn) = \varphi(mn) = \varphi(m)\varphi(n) = g^{-1}f(\psi(b)) \cdot g^{-1}f(\psi(c))$ and, consequently, h is an iseomorphism.

§2

The principal purpose of this section is to extend, dualize and simplify results given by J. R. Bastida in [1] and [2]. A, B and T will denote subsets of a semigroup S, c will be an element of S and $D = c^{(-1)}A \cap B$.

2.1 Definition. If S is a semigroup and A and T are subsets of S, then one defines $L(A, T) = A \cup TA$, $R(A, T) = A \cup AT$ and $H(A, T) = R(A, T) \cap L(A, T)$. When the context clearly indicates which subset T is under consideration, then reference to T is usually omitted, that is, we write $L(A, T) = L(A)$, etc. Moreover, for $T \subset S$, one defines the Relative Green (equivalence) Relations, $\mathfrak{L} = \{(x, y); L(x) = L(y)\}$, $\mathfrak{R} = \{(x, y); R(x) = R(y)\}$ and $\mathfrak{H} = \mathfrak{L} \cap \mathfrak{R}$. For $x \in S$, we will let $H_x(T)$ denote the $\mathfrak{H}(T)$-class (or slice) containing x; here again reference to T is omitted if the context is clear.

2.2 Proposition. *If $A \subset L(a)$ for all $a \in A$ and $B \subset L(b)$ for all $b \in B$ and if D is nonempty, then $A \subset Sb$ for all $b \in B$.*

Proof. If we let $b \in B$ and $d \in D$ and if $b = d$, then $A \subset L(cd) = L(cb) \subset Sb$; if $b \neq d$, then $d \in Tb$ and hence $A \subset L(cd) \subset Sb$.

2.3 Proposition. *If X and Y are subsets of S such that $\square \neq X \subset Y$ and if $Y^{(-1)}Y = Y^{[-1]}Y$, then $X^{[-1]}X \subset Y^{[-1]}Y$.*

Proof. The hypothesis that X is nonempty is needed to ensure that $X^{[-1]}X \subset X^{(-1)}X$ and so $X \subset Y$ implies that $X^{[-1]}X \subset X^{(-1)}X \subset Y^{(-1)}Y = Y^{[-1]}Y$.

2.4 Proposition. *If $y \in S$ such that $y^{(-1)}y \subset D^{[-1]}B \cap A^{[-1]}A$, then $y^{(-1)}y \subset D^{[-1]}D$; moreover, if also $y \in yS$, then $D^{[-1]}D$ is nonempty.*

Proof. If $x \in y^{(-1)}y$, then $Dx \subset B$ and $c(Dx) = (cD)x \subset Ax \subset A$ so that $Dx \subset c^{(-1)}A$ and, consequently, $Dx \subset D$, i.e., $x \in D^{[-1]}D$.
The second half of the result follows because $y \in yS$ if and only if $y^{(-1)}y \neq \square$.

2.5 Corollary. *If $A \subset L(a)$ for all $a \in A$, if $\square \neq B \subset L(b)$ for all $b \in B$ and if $B^{(-1)}B = B^{[-1]}B$, then $b^{(-1)}b \subset D^{[-1]}D$ for all $b \in B$; moreover, if, in addition, $b \in bS$, then $D^{[-1]}D$ is nonempty.*

Proof. We will satisfy the hypothesis of the first part of (2.4): Since $B^{(-1)}B = B^{[-1]}B$, it follows that $b^{(-1)}b \subset b^{(-1)}B \subset B^{(-1)}B = B^{[-1]}B \subset D^{[-1]}B$ and if D is nonempty we have $A \subset Sb$ by (2.2) so that for $a \in A$, $x \in b^{(-1)}b$ we obtain $ax = (sb)x = s(bx) = sb = a$, for some $s \in S$, and, consequently, $b^{(-1)}b \subset A^{[-1]}A$. It is noted that the conclusion also follows if D is empty for then $D^{[-1]}D = S$.

2.6 Proposition. *If $a \in A \subset R(a)$ and if $A^{(-1)}a$ is nonempty, then $a \in aS$.*

Proof. If $x \in A^{(-1)}a$ we have $a = a'x$ for some $a' \in A$ so that if $a = a'$ the result is immediate and if $a \neq a'$, then $a' = at$ for some $t \in T$ since $A \subset R(a)$ and hence $a = a'x = (at)x = a(tx)$.
It is noted that if A is nonempty the statement $A^{(-1)}a \neq \square$ is implied by the condition $\square \neq a^{(-1)}A \subset \{s \in S; A \subset As\}$, for then $\square \neq \{s \in S; a \in As\} = A^{(-1)}a$.

2.7 Definition. For any $A \subset S$ and $y \in S$, let us define $\mathfrak{S}(A, y) = \{(u, v); u, v \in A^{[-1]}A \text{ and } yu = yv\}$ and $\mathfrak{M}(A, y) = \{(u, v); u, v \in AA^{[-1]} \text{ and } uy = vy\}$.

2.8 Proposition. *If $A^{[-1]}A \neq \square$, then $\mathfrak{S}(A, y)$ is a congruence on $A^{[-1]}A$ if and only if $yu = yv$ implies that $y'u = y'v$ for all $y' \in y(A^{[-1]}A)$.*

Proof. For brevity and clarity, let $\mathfrak{S} = \mathfrak{S}(A, y)$ in this proof. If $\mathfrak{S}$ is a congruence on $A^{[-1]}A$, then $(\Delta\mathfrak{S} \cup \mathfrak{S}\Delta) \subset \mathfrak{S}$ where Δ is the diagonal of $A^{[-1]}A \times A^{[-1]}A$ and thus, letting $w \in A^{[-1]}A$ and $(u, v) \in \mathfrak{S}$, we have $ywu = ywv$ so that $y'u = y'v$ for all y' in $y(A^{[-1]}A)$.

Conversely, if the condition holds, namely, $yu = yv$ implies that $y'u = y'v$ for all y' in $y(A^{[-1]}A)$ and if $w \in A^{[-1]}A$, $(u, v) \in \mathfrak{S}$, then the condition implies that $(yw)u = (yw)v$ because yw is in $y(A^{[-1]}A)$ and therefore $\mathfrak{S}$ is a left congruence on $A^{[-1]}A$. It is evident that $\mathfrak{S}$ is a right congruence.

2.9 Corollary. *If b is in $B \cap bS$, if $A \subset L(x)$ for all x in A, if $\Box \neq B \subset L(x)$ for all x in B, if $B^{(-1)}B = B^{[-1]}B$ and if D is nonempty, then $\mathfrak{S}(D, b)$ is a congruence on $D^{[-1]}D$ and $\mathfrak{S}(D, b) = \mathfrak{S}(D, b')$ for $b' \in B$.*

Proof. In view of (2.5) the hypothesis implies that $D^{[-1]}D$ is nonempty so that in order to prove that $\mathfrak{S}(D, b)$ is a congruence on $D^{[-1]}D$ it suffices to show that the condition of (2.8) is satisfied. In the case that B is a singleton set, say $B = \{b\}$, we have from (2.3) that $b(D^{[-1]}D) \subset b(B^{[-1]}B) \subset B = \{b\}$ so that the condition is trivially satisfied. If card $B > 1$, then it follows that b is in Sb and thus that $B \subset Sb$. Letting $(u, v) \in \mathfrak{S}(D, b)$ and $b' \in b(D^{[-1]}D)$ we have $b' \in b(b^{(-1)}B) \subset B \subset Sb$ since $D^{[-1]}D \subset b^{(-1)}B$ so that if $b' = sb$ we obtain $b'u = (sb)u = s(bu) = s(bv) = (sb)v = b'v$ and therefore the condition of (2.8) holds.

Momentarily fixing distinct elements b and b' in B and letting (u, v) be an element of $\mathfrak{S}(D, b)$, we use the fact that $B \subset L(b)$ to obtain $b'u = (tb)u = t(bu) = t(bv) = (tb)v = b'v$, where $t \in T$; so that (u, v) is also in $\mathfrak{S}(D, b')$. Clearly, in a similar fashion we have $\mathfrak{S}(D, b') \subset \mathfrak{S}(D, b)$.

2.10 Proposition. $A^{[-1]}A \subset (xA)^{[-1]}(xA)$ *for all $x \in S$.*

Proof. If $y \in A^{[-1]}A$, then $(xA)y = x(Ay) \subset xA$ so that y is in $(xA)^{[-1]}(xA)$.

2.11 Proposition. *For any elements x, y and $z \in S$, if $x \in Sy$, then $\mathfrak{S}(A, y) \subset \mathfrak{S}(zA, x)$.*

Proof. If $(u, v) \in \mathfrak{S}(A, y)$, then in view of (2.10) it remains only to verify that $xu = xv$: Letting $x = sy$ we have $s(yu) = (sy)u = xu$ and in a similar manner $s(yv) = xv$ so that $xu = xv$.

2.12 Corollary. *If $A \subset L(a)$ for all a in A, if $B \subset L(b)$ for all $b \in B$ and if D is nonempty, then $\mathfrak{S}(D, b) \subset \mathfrak{S}(cD, a)$ where $a \in A$ and $b \in B$.*

Proof. The hypothesis is that of (2.2) so that $A \subset Sb$ and therefore in view of (2.11) the conclusion is evident.

2.13 Proposition. *If A, B and D are nonempty sets such that $A \subset L(a)$ for all $a \in A$ and $B \subset L(b)$ for all $b \in B$, if $A^{(-1)}A = A^{[-1]}A$ and $B^{(-1)}B = B^{[-1]}B$ and if $b \in bS$ for some $b \in B$, then $\mathfrak{S}(cD, a)$ is a congruence on $(cD)^{[-1]}(cD)$ for any $a \in A$.*

Proof. In view of (2.5) we have $D^{[-1]}D \neq \Box$ and so $(cD)^{[-1]}(cD)$ is nonempty since $D^{[-1]}D \subset (cD)^{[-1]}(cD)$. We will now verify that the condition of (2.8) is satisfied: In the case that $A = \{a\}$ it follows from (2.3) that $a[(cD)^{[-1]}(cD)] = \{a\}$ and as a result the condition is trivially fulfilled.

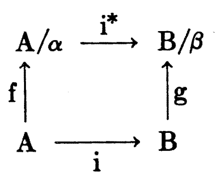
If card $A > 1$, then we obtain $a \in Sa$ and, therefore, $A \subset Sa$. Letting $(u, v) \in$

$\mathfrak{S}(cD, a)$ and $a' \in a[(cD)^{[-1]}(cD)]$ we have a' in $a(A^{[-1]}A) \subset Sa$ since $(cD)^{[-1]}(cD) \subset A^{[-1]}A$ by (2.3). Consequently, if $a' = sa$, then we see that $a'u = sau = sav = a'v$ and thus the condition of (2.8) holds.

The next result is well known and it is due to B. J. Pettis; consequently, its proof is omitted.

2.14 Proposition. *If a compact semigroup is algebraically a group, then it is a topological group.*

2.15 Proposition (Induced Homomorphism Theorem). *If α and β are congruences on semigroups A and B, respectively, such that $A \subset B$ and $\alpha \subset \beta$, then in the diagram, where f and g are the appropriate canonical maps and i is the inclusion map, there exists a homomorphism i^* such that the diagram is analytic.*



Proof. This proposition follows from an evident extension of the version of the Induced Homomorphism Theorem given in [3] and its corollary there.

2.16 Lemma. *If $A^{(-1)}A \subset A^{[-1]}A$ and if $B^{(-1)}B \subset B^{[-1]}B$, then $D^{(-1)}D \subset D^{[-1]}D$ and, consequently, $D^{[-1]}D = d^{(-1)}D$ for all d in D provided that D is nonempty.*

Proof. First of all, it is clear that $D^{(-1)}D \subset B^{(-1)}B \subset B^{[-1]}B$. Since we may assume that $D^{(-1)}D$ is nonempty, say $t \in D^{(-1)}D$, then, using the fact that $cD \subset A$, we have that $(At \cap A) \supset (cDt \cap cD) \supset c(Dt \cap D) \neq \square$. Again noting that $cD \subset A$, it follows that $cDt \subset At \subset A$ so that $Dt \subset c^{(-1)}A$. As a result, $Dt \subset c^{(-1)}A \cap B = D$ and so $t \in D^{[-1]}D$.

The second conclusion follows from the fact that $D^{[-1]}D = \bigcap \{d^{(-1)}D; d \in D\} \subset d^{(-1)}D \subset \bigcup \{d^{(-1)}D; d \in D\} = D^{(-1)}D$.

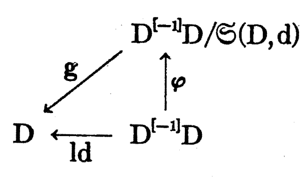
2.17 Theorem. (a) *Let S be compact or discrete and A and B be nonempty sets satisfying these three conditions:*

(i) *$A \subset L(a)$ for all a in A,*

(ii) *$B \subset L(b) \cap R(b)$ for all $b \in B$,*

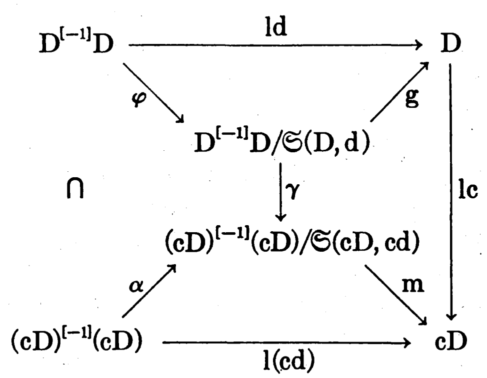
(iii) *$A^{(-1)}A = A^{[-1]}A$ and $B^{(-1)}B = B^{[-1]}B$.*

If card $B > 1$ and if D is both nonempty and closed, then this diagram is analytic:



where $D^{[-1]}D/\mathfrak{S}(D, d)$, *for* d *in* D, *is a topological group,* φ *is the canonical map and* g *is a homeomorphism.*

(b) *If, in addition,* card $A > 1$, *the preceding analytic diagram may be extended:*



where $(cD)^{[-1]}(cD)/\mathfrak{S}(cD, cd)$ *is a topological group,* α *is a canonical map,* m *is a homeomorphism and* γ *is a continuous epimorphism.*

Proof. We will consider only the case where S is compact since the situation where S is discrete follows in a similar manner with the topological results omitted.

Since card $B > 1$ and $B \subset R(b)$ for all b in B, it follows that $d \in dS$ for each $d \in D$ and so, by (2.5), $D^{[-1]}D$ is nonempty. Using (1.3), $D^{[-1]}D$ is closed and compact because D is closed and S is compact. It is immediate from (2.9) that $\mathfrak{S}(D, d)$ is a congruence on $D^{[-1]}D$ for any $d \in D$ and, moreover, $\mathfrak{S}(D, d)$ is closed because it is easily verified that $\mathfrak{S}(D, d) = (D^{[-1]}D \times D^{[-1]}D) \cap (ld \times ld)^{-1}(\Delta)$ and so $D^{[-1]}D/\mathfrak{S}(D, d)$ is a compact topological semigroup by (1.5). With (2.14) in mind we proceed to show that $D^{[-1]}D/\mathfrak{S}(D, d)$ is algebraically a group:

We may select an element q of $d^{(-1)}d$ because $d^{(-1)}d$ is nonempty if and only if d is in dS. Then, since $D \subset L(d)$ we have $dx \in L(d)$ for $x \in D^{[-1]}D$ and so it is easily verified that $dx = dxq$. Since $D^{[-1]}D$ is a semigroup we have $(x, xq) \in \mathfrak{S}(D, d)$ and thus $\varphi(x) = \varphi(x)\varphi(q)$ so that $\varphi(q)$ is a right unit for $\varphi(x)$. If $d = dx$ we have $d = dxx$ and if $d \neq dx$, then $d = dxt$ for some $t \in T$ because $B \subset R(b)$ for all $b \in B$ so that in either case there is an element x' such that $d = dxx'$ and, consequently, $dq = dxx'$. In view of (2.16), x' is in $D^{[-1]}D$ and therefore $\varphi(q) = \varphi(x)\varphi(x')$ indicating that $\varphi(x)$ has a right inverse.

In order to show that $D \overset{T}{=} D^{[-1]}D/\mathfrak{S}(D, d)$ we will make use of Dubreil's result, that is, (1.8): Card $B > 1$ implies that $B \subset L(d) \subset Sd$ and thus using (2.3) we have $D^{[-1]}D \subset d^{(-1)}B \subset d^{(-1)}(Sd)$. The fact that $D^{(-1)}D \subset D^{[-1]}D$ implies that the restriction of ld to $D^{[-1]}D$ has as its image D, for if $d' \in D$ and $d' = d$ we know that $d^{(-1)}d$ is nonempty and if $d' \neq d$, $d' = dt$ for some $t \in T$ so that in either case $d' = dt'$ and thus $t' \in D^{(-1)}D \subset D^{[-1]}D$; the other inclusion is clear because $D^{[-1]}D \subset d^{(-1)}D$ implies that $d(D^{[-1]}D) \subset D$. Therefore,

$\varphi(D^{[-1]}D) \overset{T}{=} D$ because $g\varphi = ld$ where g is the appropriate homeomorphism of (1.8).

To prove the second part of the theorem we begin by proving the existence of a continuous epimorphism γ. Since $D^{[-1]}D \subset (cD)^{[-1]}(cD)$ and $\mathfrak{S}(D, d) \subset \mathfrak{S}(cD, cd)$, the Induced Homomorphism Theorem gives us the existence of a function γ such that $\gamma\varphi = \alpha i$ where i is the inclusion map. φ is closed because it is continuous, $D^{[-1]}D$ is compact and $D^{[-1]}D/\mathfrak{S}(D, d)$ is Hausdorff from a result in [6] and thus since it is also true that φ is an onto function and $\alpha i = \gamma\varphi$ is continuous, we have that γ is continuous from another result in [6]. It is clear that γ is a homomorphism and so it remains to verify that it is an onto function: If Y is an element of $(cD)^{[-1]}(cD)/\mathfrak{S}(cD, cd)$ and y is in Y, then $cdy = cd'$ for some $d' \in D$. If $d = d'$ and q is in $d^{(-1)}d$, then $cdy = cd = cdq$ and it follows that (y, q) is in $\mathfrak{S}(cD, cd)$ so that $\gamma(\varphi(q)) = Y$. If $d \neq d'$, then for some $t \in T$ we see that $d' = dt$ because $B \subset R(d)$ and we note that $t \in D^{(-1)}D = D^{[-1]}D$. Then, since $cdy = cdt$, it is true that (y, t) is in $\mathfrak{S}(cD, cd)$ and it follows that $\gamma(\varphi(t)) = Y$. We conclude, therefore, that γ is an onto function.

It is next noted that $(cD)^{[-1]}(cD)$ is nonempty because $D^{[-1]}D \subset (cD)^{[-1]}(cD)$ and that in an analogous manner to the proof of the first part of the theorem it is easy to verify that $(cD)^{[-1]}(cD)$ is closed and compact, that $\mathfrak{S}(cD, cd)$ is a closed congruence on $(cD)^{[-1]}(cD)$ and, hence, that $(cD)^{[-1]}(cD)/\mathfrak{S}(cD, cd)$ is a topological semigroup. Then, since $D^{[-1]}D/\mathfrak{S}(D, d)$ is a group and γ is an epimorphism, it follows that $(cD)^{[-1]}(cD)/\mathfrak{S}(cD, cd)$ is a topological group.

As in the proof of the first part of the theorem, we will use Dubreil's result in order to obtain $cD \overset{T}{=} (cD)^{[-1]}(cD)/\mathfrak{S}(cD, cd)$: Card $A > 1$ implies that $A \subset L(cd) \subset Scd$ and thus using (2.3) we find that $(cD)^{[-1]}(cD) \subset (cd)^{(-1)}A \subset (cd)^{(-1)}(Scd)$. If $d' \in D$ and if $d' = d$, then $(cd)^{(-1)}(cd)$ is nonempty since $d^{(-1)}d$ is nonempty and if $d' \neq d$, then $d' = dt$ for some $t \in T$ and hence $cd' = cdt$ so that in either case $cd' = cdt'$ and t' is in $D^{(-1)}D \subset D^{[-1]}D \subset (cD)^{[-1]}(cD)$; the other inclusion is clear because $(cD)^{[-1]}(cD) \subset (cd)^{(-1)}(cD)$ implies that $cd[(cD)^{[-1]}(cD)] \subset cD$. Therefore, $\varphi[(cD)^{[-1]}(cD)]$ is homeomorphic to cD since $m\varphi = l(cd)$ where m is the appropriate homeomorphism, namely, g, of (1.8).

2.18 Proposition. *Under the hypotheses of* (2.17), *if T is a subsemigroup then B, D and cD are contained in $\mathfrak{H}$-slices.*

Proof. $T^2 \subset T$ implies that $\mathfrak{H} = \{(x, y) \in S \times S; H(x) = H(y)\}$ and so for $b \in B$, $H_b = \{x \in S; H(x) = H(b)\}$. It then follows that, since $B \subset R(b) \cap L(b) = H(b)$ for $b \in B$, we have $B \subset H_b$. Clearly, $D \subset H_b$ because $D \subset B$.

Next we notice that $cD \subset A \subset L(a)$ for $a \in A$ and, in particular, $cD \subset L(cd)$ for $d \in D$. Also, we find that $cD \subset cB \subset cR(b) = R(cb)$ for $b \in B$ so that $cD \subset R(cd)$ for $d \in D$. Consequently, $cD \subset L(cd) \cap R(cd) = H(cd)$ for $d \in D$ and it follows easily that $cD \subset H_{cd}$.

2.19 Theorem. *Suppose S is compact or discrete and let us define* $K = Hw^{(-1)} \cap$

J where $w \in S$ and where H and J are nonempty sets in S satisfying these three conditions:

(i) *$H \subset R(h)$ for all $h \in H$,*

(ii) *$J \subset R(x) \cap L(x)$ for all $x \in J$,*

(iii) *$HH^{(-1)} = HH^{[-1]}$ and $JJ^{(-1)} = JJ^{[-1]}$.*

If card $J > 1$ and if K is both nonempty and closed, then this diagram is analytic:

$$\begin{array}{ccc} KK^{[-1]}/\mathfrak{M}(K,k) & & \\ \psi \uparrow & \searrow^{f} & \\ KK^{[-1]} & \xrightarrow[rk]{} & K \end{array}$$

where $KK^{[-1]}/\mathfrak{M}(K, k)$, $k \in K$, is a topological group, ψ is the canonical map and f is a homeomorphism.

Proof. All the results preceding (2.17) may be easily "dualized" so that this theorem may be proved in a manner analogous to the proof of (2.17).

2.20 Theorem. *If the hypotheses of part (a) of (2.17) and (2.19) hold and if d is in $D \cap K$, then we naturally speak of the results of (2.19) as being the "mirror image" of the results in part (a) of (2.17) in view of this analytic extension of Dubreil's diagram,*

$$\begin{array}{ccccccccc} KK^{[-1]}/\mathfrak{M} & \subset & (dS)d^{(-1)}/\mathfrak{E} & & \xrightarrow{\quad h \quad} & & d^{(-1)}(Sd)/\mathfrak{F} & \supset & D^{[-1]}D/\mathfrak{S} \\ \psi' \uparrow & & \psi \uparrow & \searrow^{f} & & \swarrow^{g} & \uparrow \varphi & & \uparrow \varphi' \\ KK^{[-1]} & \subset & (dS)d^{(-1)} & \xrightarrow[rd]{} & dS \cap Sd & \xleftarrow[ld]{} & d^{(-1)}(Sd) & \supset & D^{[-1]}D \end{array}$$

where for brevity $\mathfrak{M} = \mathfrak{M}(K, d)$, $\mathfrak{F} = \mathfrak{F}(d)$, $\mathfrak{E} = \mathfrak{E}(d)$ and $\mathfrak{S} = \mathfrak{S}(D, d)$ and ψ' is the restriction of ψ to $KK^{[-1]}/\mathfrak{M}$ and similarly for φ'. Moreover, if $D = K$, then in the diagram the restriction of h to $KK^{[-1]}/\mathfrak{M}$ is an iseomorphism with image $D^{[-1]}D/\mathfrak{S}$.

Proof. The first part of this theorem follows easily from the results of (1.8), (2.17) and (2.19). In addition, from (2.19) we see that the restriction of f to $KK^{[-1]}/\mathfrak{M}$ is K and from (2.17) we find that $g^{-1}(D) = D^{[-1]}D/\mathfrak{S}$ and from Dubreil's result we recall that $h = g^{-1}f$ is an iseomorphism, so that putting these remarks together the conclusion follows because $h(KK^{[-1]}/\mathfrak{M}) = g^{-1}f(KK^{[-1]}/\mathfrak{M}) = g^{-1}(K) = g^{-1}(D) = D^{[-1]}D/\mathfrak{S}$.

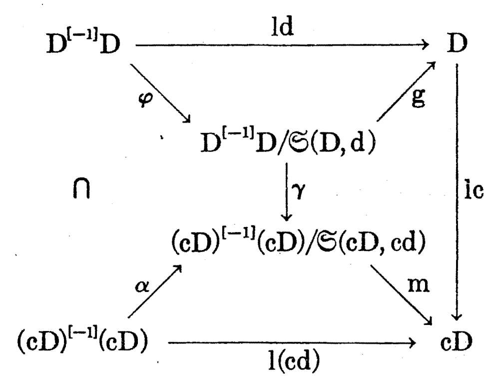
In view of its position in the diagram, an iseomorphism such as that expressed in (2.20) is known as "turning the corner."

The next theorem has been presented in its algebraic context for $T = S$ in [1] and it formed the cornerstone of that work. It is in view of this last remark that we attach the author's name to the theorem in its presentation. This result is also important to me because it served as the prime motivation of this paper. Also, subsequent to this theorem it will be shown that a well-known theorem, due originally to M. P. Schutzenberger and to A. D. Wallace in its present formulation, follows in part as a corollary.

2.21 Lemma. *$H_w^{(-1)}H_w = H_w^{[-1]}H_w$ for w in S.*

Proof. The reader may find the proof of this result in [5] where it is shown to be a consequence of Green's Lemma.

2.22 Theorem (Bastida). *Let S be compact or discrete, T be a closed subset of S and $D = c^{(-1)}H_x \cap H_y$. If card $H_x > 1$, card $H_y > 1$ and D is nonempty, then this diagram is analytic:*



where d is in D, $D^{[-1]}D/\mathfrak{S}(D, d)$ and $(cD)^{[-1]}(cD)/\mathfrak{S}(cD, cd)$ are topological groups, φ and α are canonical maps, g and m are homeomorphisms and γ is a continuous epimorphism.

Proof. We will easily verify that the hypotheses of (2.17) are fulfilled, where H_x and H_y will be A and B, respectively. In view of (2.21) and because H_x and H_y are $\mathcal{H}$-slices, we have that (i), (ii) and (iii) of (2.17) hold. If S is compact and T is closed, then H_x and H_y are closed so that $c^{(-1)}H_x$ is closed and, consequently, D is closed. Therefore, it may be seen that all the hypotheses of (2.17) are satisfied and hence this proposition now follows as an immediate corollary.

2.23 Theorem (Schutzenberger-Wallace). *If S is compact or discrete, if T is a closed subset of S and if y is an element of S such that card $H_y > 1$, then H_y is homeomorphic to the topological group, $y^{(-1)}H_y/\mathfrak{S}(H_y, y)$, and the groups $y^{(-1)}H_y/\mathfrak{S}(H_y, y)$ and $H_y y^{(-1)}/\mathfrak{M}(H_y, y)$ are iseomorphic.*

Proof. Using the dual of (2.21) we see that card $H_y > 1$ implies that $H_yH_y^{[-1]}$ is nonempty so that letting $H_y = H_x$ in (2.22) and c be an element of $H_yH_y^{[-1]}$, we have $D = c^{(-1)}H_y \cap H_y = H_y$ because $H_y \subset c^{(-1)}H_y$. The first part of this theorem now follows as a corollary to (2.22) since we have that $y^{(-1)}H_y = H_y^{[-1]}H_y$ from [5].

In a similar manner we may choose an element w in $H_y^{[-1]}H_y$ so that the set K of (2.19) and (2.20) is H_y. Therefore, by (2.20), we may turn the corner and find that $y^{(-1)}H_y/\mathfrak{S}(H_y, y)$ and $H_y y^{(-1)}/\mathfrak{M}(H_y, y)$ are iseomorphic.

* * *

I am grateful to Dr. A. D. Wallace for his comments.

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