STEENROD SQUARES AND HIGHER MASSEY PRODUCTS

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Recently D. Kraines [3] and P. May [5], [6] have studied a system of higher order cohomology operations which they call higher Massey products. These operations are of essential importance in studying differentials in spectral sequences and in studying the cohomology of Hopf algebras [6]. Also the recent work of P. May on 2-stage Postnikov systems depends on studying these operations.

In this paper we study the relations between these higher order Massey products and the Steenrod algebra $\mathcal{A}(2)$. More precisely, higher order Massey products can be regarded as a direct generalization of ordinary cup products, and if cup products and Steenrod squares are connected by the Cartan formula

$$\mathrm{Sq}^i(ab) = \Sigma \mathrm{Sq}^j a \cdot \mathrm{Sq}^{i-j} b,$$

our main result is a generalized Cartan formula for higher Massey products. As an application we calculate the cohomology of certain 2-stage Postnikov systems as modules over the Steenrod algebra.

Let $\langle A_1, \cdots, A_n \rangle \in H^*(X)$ be a (matric) Massey product, then our main result states that for each i, another Massey product

$$B_i = \langle \mathbf{SQ}_R{}^i A_1, \mathbf{SQ}^i A_2 \cdots \mathbf{SQ}^i A_{n-1}, \mathbf{SQ}_c{}^i A_n \rangle$$

is defined and

Theorem 0: $\mathrm{Sq}^i \langle A_1, \cdots, A_n \rangle \subset B_i$.

Here $\mathbf{SQ}^i(A)$ is the matrix

$$\begin{pmatrix} \mathrm{Sq}^0 A & 0 & 0 & 0 \\ \mathrm{Sq}^1 A & \mathrm{Sq}^0 A & 0 & 0 \\ \vdots & & & \\ \mathrm{Sq}^i A & \mathrm{Sq}^{i-1} A & \cdots & \mathrm{Sq}^0 A \end{pmatrix}$$

(if A is an $n \times m$ matrix $\mathrm{Sq}^j(A)_{r,s} = \mathrm{Sq}^j(A_{r,s})$) while $\mathbf{SQ}_R{}^i(A) = (\mathrm{Sq}^i A, \mathrm{Sq}^{i-1} A, \cdots, \mathrm{Sq}^0 A)$

$$\mathbf{SQ}_c{}^i(A) = \begin{pmatrix} \mathrm{Sq}^0 A \\ \mathrm{Sq}^1 A \\ \vdots \\ \mathrm{Sq}^i A \end{pmatrix}$$

If a, b are elements in $H*(X)$ then $\langle a, b \rangle$ is just $a \smile b$ and in this case Theorem

0 specializes to

$$\mathrm{Sq}^i\langle a, b\rangle = \left\langle (\mathrm{Sq}^i a, \cdots \mathrm{Sq}^0 a) \begin{pmatrix} \mathrm{Sq}^0 b \\ \vdots \\ \mathrm{Sq}^i b \end{pmatrix} \right\rangle$$
$$= \Sigma \mathrm{Sq}^j a \cdot \mathrm{Sq}^{i-j} b$$

as one would hope.

Actually, the proof of theorem 0 shows that the formula is valid in a much wider context than merely the cohomology of topological spaces. For example it is true for the cohomology of co-commutative Hopf algebras, as for example in $H^*(\mathfrak{A}(2)) = \mathrm{Ext}_{\mathfrak{A}(2)}{}^{**}(Z_2, Z_2)$. Our result is particularly relevant here since May has proved in [6] that ever indecomposable in $H^*(A)$ (where A is any connected, augmented algebra of finite type over a field $\mathfrak{F}$) is a matric Massey product built up successively from the elements of $H^1(A) = \mathrm{Ext}_A{}^{1,*}(\mathfrak{F}, \mathfrak{F})$. In particular if A is the Steenrod algebra $\mathfrak{A}(2)$ then $H^1(A)$ has a basis consisting of the elements denoted by Adams [1] as h_j, $i = 0, 1, \cdots$. On the other hand, it follows from Adams calculations in [1] that $\mathrm{Sq}^0(h_i) = h_{i+1}$, $\mathrm{Sq}^1(h_i) = (h_i)^2$, and $\mathrm{Sq}^i(h_j) = 0$ otherwise. Thus, in principle, we can evaluate the squaring operations in all of $H^*(\mathfrak{A}(2))$. In particular families of operations (which appear in different dimensions, but for similar reasons) turn out to be related by the iteration of Sq^0. For example there is the family $c_0, c_1, \cdots$ where $c_i = \langle (h_{i+2})^2, h_i, h_{i+1}\rangle$, and from theorem 0 we have easily $\mathrm{Sq}^0(c_i) = c_{i+1}$. A similar formula holds for the d family and the g family, and all other families known to the author.

The paper falls naturally into 2 parts. In the first four sections we review the fundamental definitions and constructions needed and apply theorem 0 to study the four fold product $\langle a, a, a, a\rangle$ which turns out to be defined whenever $a^2 = 0$. This information is then used to study certain 2-stage Postnikov systems in section 4. Then, the second part is devoted to the proof of theorem 0.

We should also mention that Lemma 2.2.2 shows that there are other types of inter-relations between Massey products and Steenrod squares, and it would be quite interesting to generalize 2.2.2 in some way to higher Massey products. In this connection we point out that there can be no interconnection for 4-fold products since $\langle a, a, a, a\rangle$ is independent in general of lower order products. However, the author believes that the Massey product $\langle a, b, c, b, a\rangle$ should contain a 3 or 4-fold matric Massey product whose entries are Steenrod squares on a, b c, but he has no idea as to what the exact formula should be.

This paper owes a debt to P. May for many helpful conversations over the material in part 2, and above all to Professor S. MacLane, without whose aid section 5 and thus the entire proof of theorem 0 would not have been possible.

1. Matric Massey Products

The definitions and main properties of higher Massey products have been given in [3] and [5]. In this section we recall the definition and reformulate it in a manner better suited to the ends we have in mind.

(1) A ∂-algebra $\mathcal{G}$ over a field $\mathfrak{F}$ is an algebra with a boundary operator ∂ and an automorphism $\epsilon: \mathcal{G} \to \mathcal{G}$ which satisfy

(i) $\epsilon^2 = 1$

(ii) $\partial(a \cdot b) = (\partial a)b + (\epsilon a) \cdot \partial b$

(iii) $\partial\epsilon + \epsilon\partial = 0$

For example, if $\mathcal{G}$ is a D.G.A. algebra then in particular it is a ∂-algebra with $\epsilon(a) = (-1)^n a$ if a has dimension n.

If $\mathcal{G}$ is a ∂-algebra then $\mathfrak{M}(\mathcal{G})$ will denote the infinite matrix ∂-algebra with entries in $\mathcal{G}$. More precisely $\mathfrak{M}(\mathcal{G})$ consists of all matrices M with a finite number of non-zero entries $M_{i,j}$ in $\mathcal{G}$. With respect to the usual matrix multiplication the algebra structure is now specified by

(i) $(\partial M)_{i,j} = \partial(M_{i,j})$

(ii) $(\epsilon M)_{i,j} = \epsilon(M_{i,j})$.

Passing to homology one has easily that $H(\mathfrak{M}(\mathcal{G})) \cong \mathfrak{M}(H(\mathcal{G}))$, the isomorphism being of algebras.

DEFINITION 1.1.1: *Let $\mathcal{G}$ be a ∂-algebra, and suppose $a_{1,2}$, $a_{2,3}$, $\cdots$ $a_{n,n+1}$ are cycles in $\mathcal{G}$ with $\{a_{i,i+1}\} = B_i$ in $H(\mathcal{G})$. Then the n-fold Massey product $\langle B_1, B_2, \cdots, B_n\rangle$ is defined if there exist elements $a_{i,j}$ with $1 \leq i < j - 1 \leq n$, $(i, j) \neq (1, n + 1)$ in $\mathcal{G}$ satisfying*

$$\partial a_{i,j} = \sum_{i<k<j} (\epsilon a_{i,k}) \cdot a_{k,j} .$$

The chain

$$\sum_j (\epsilon a_{1,j}) \cdot a_{j,n+1}$$

is a cycle which is not, in general, a boundary and it represents a class in $\langle B_1, B_2, \cdots, B_n\rangle$. The set of all such classes arising from different choices of the $a_{i,j}$ $(i < j - 1)$ is the Massey product.

It is a result of D. Kraines [3] that different choices of the $a_{i,i+1}$ give the same Massey product, so the notion is well defined in homology and depends only on the ordered set $\{B_1, \cdots, B_n\}$.

There are times when a Massey product defined in $\mathfrak{M}(H(\mathcal{G}))$ actually can be thought of as a subset of $H(\mathcal{G})$ (this is the case in theorem 0).

DEFINITION 1.1.2: *$H(\mathcal{G}) \subset \mathfrak{M}(H(\mathcal{G}))$ is the set of matrices M with $M_{1,1}$ the only possible non-zero entry. If* M_1 (M_n) *is a row (column) matrix, and the Massey product* $\langle \mathrm{M}_1, \cdots, \mathrm{M}_n\rangle$ *is defined in* $\mathfrak{M}(\mathrm{H}(\mathcal{G}))$, *then the matric Massey product* $\langle \mathrm{M}_1, \cdots, \mathrm{M}_n\rangle$ in $\mathrm{H}(\mathcal{G})$ *is defined as*

$$\mathrm{H}(\mathcal{G}) \cap \langle \mathrm{M}_1, \cdots, \mathrm{M}_n\rangle.$$

(2) We now reformulate the definition of 1.1 by means of universal models.

Let I_n be the $\mathfrak{F}$-vector space of dimension $[n(n-1)/2] - 1$ generated by the symbols $(i, j), 1 \leq i < j \leq n + 1$, $(i, j) \neq (1, n + 1)$. In turn, let F_n be the free tensor algebra $T(I_n \oplus \epsilon I_n)$ generated by two copies of I_n. (If $\mathfrak{F} = Z_2$, then only one copy of I_n is used).

We make F_n into a differential algebra by setting

$$\partial(i,j) = \sum_{i<k<j} (\epsilon(i,k)) \otimes (k,j)$$
$$\partial(\epsilon(i,j)) = -\sum_{i<k<j} (i,k) \otimes (\epsilon(k,j))$$

on generators while

$$\epsilon[(i,j)] = \epsilon(i,j),\ \epsilon[\epsilon(i,j)] = +(i,j)$$

and

$$\partial a \otimes b = (\partial a) \otimes b + \epsilon a \otimes \partial b$$
$$\epsilon(a \otimes b) = \epsilon(a) \otimes \epsilon(b)$$

(if $\mathfrak{F} = Z_2$ then $\epsilon = \text{id}$).

F_n is clearly universal for n-fold Massey products in the following sense. Given $B_1, \cdots, B_n$ in $H(\mathfrak{S})$, the Massey product $\langle B_1, \cdots, B_n \rangle$ is defined if and only if there is a map of differential algebras

$$f: F_n \to \mathfrak{S}$$

so that $f(i, i+1)$ is a cycle representing B_i and (noting that $\sum_{1<j<n+1} (\epsilon(1,j))(j, n+1)$ represents a non-zero class U_n in $H(F_n)$) $\langle B_1, \cdots, B_n \rangle$ is $\{f_*(U_n)\} \subset H(\mathfrak{S})$ where f runs over all maps satisfying the above conditions.

(3) We now study some subcomplexes of F_n.

DEFINITION 1.3.1: *$\mathcal{C}(n)_{i,j}$ is the subcomplex of F_n generated by all chains in F_n of the form*

$$\epsilon^r[(\epsilon(i,j_1)) \otimes (\epsilon^2(j_1, j_2)) \cdots \otimes (\epsilon^r(j_{r-1}, j))]$$

with $i < j_1 < j_2 < \cdots < j_{r-1} < j$. We make $\mathcal{C}(n)_{i,j}$ into a graded complex by letting $\mathcal{C}_r(n)_{i,j}$ be the vector subspace of $\mathcal{C}(n)_{i,j}$ generated by all $r+2$-fold tensor products.

THEOREM 1.3.2: *$H_*(\mathcal{C}(n)_{i,j}) \equiv 0$ if $(i,j) \neq (1, n+1)$ while $H_*(\mathcal{C}(n)_{1,n+1})$ is generated by the cycle $\sum_{1<j<n} \epsilon(1,j) \otimes (j, n+1)$.*

Proof: Consider the dual complex Hom $(\mathcal{C}(n)_{i,j}, \mathfrak{F})$. It has a basis consisting of all $(r-1)$-tuples

$$(j_1, \cdots, j_{r-1})$$

with $1 < j_1 < \cdots < j_{r-1} < j$, or equivalently of all $(r-1)$-tuples

$$(k_1, \cdots, k_{r-1})$$

with $1 \leq k_1 < \cdots < k_{r-1} < j - i$. The dual boundary is then given by the formula

$$\delta(k_1, \cdots, k_{r-1}) = \sum_{s=1}^{r-1} (-1)^s (k_1, \cdots, \hat{k}_s, \cdots, k_{r-1}),$$

and we may define a contracting homotopy s by the formula

$$s(k_1, \cdots, k_{r-1}) = \begin{cases} (1, k_1, \cdots, k_{r-1}), & k_1 > 1 \\ 0 & k_1 = 1. \end{cases}$$

Then, if $(i, j) \neq (1, n+1)$, we easily check that

$$s\delta + \delta s = I.$$

To complete the proof note that

$$(\mathcal{C}(n+1)_{1,n+1}) \cong \mathcal{C}(n)_{1,n+1} \oplus \mathfrak{F}$$

where the extra $\mathfrak{F}$ is generated by $(1, n+1)$. This complex has trivial homology so $H_*(\mathcal{C}(n)_{1,n+1})$ is generated by $\partial(1, n+1)$. Q.E.D.

2. Steenrod operations

In this section we review the construction of the Steenrod operations, and write down the various formulas involving the $\smile_i$-products which will be of use in the sequel.

(1) Let G be a group, then $\mathfrak{F}(G)$ is the group-ring of G over $\mathfrak{F}$, i.e. $\mathfrak{F}(G)$ is the $\mathfrak{F}$-vector space with one generator for each element of G and

$$(\Sigma f_i(g_i))(\Sigma f_j(g_j)) = \Sigma(f_i \cdot f_j)(g_i \cdot g_j) \ f_i \in \mathfrak{F}, g_i \in G.$$

(From now on $\mathfrak{F} = Z_2$.)

DEFINITION 2.1: *W is the $\mathfrak{F}(Z_2)$ free acyclic complex with one generator e_i in each dimension $i \geq 0$ and*

$$\partial e_i = (1 + T)e_{i-1}$$

where T is the generator of the group $G = Z_2$.

Corresponding to the diagonal $\Delta: \mathfrak{F}(Z_2) \to \mathfrak{F}(Z_2) \otimes \mathfrak{F}(Z_2)$ given on generators by $\Delta 1 = 1 \otimes 1$, $\Delta T = T \otimes T$ there is a diagonal map

$$\Delta: W \to W \otimes W$$

given on generators by

$$\Delta e_i = \sum_{j=0}^{i} e_j \otimes T^j e_{i-j}$$

and

$$\Delta(Te_i) = \Delta(T)\Delta(e_i).$$

Now let $\mathcal{C}$ be any chain complex with a diagonal map

$$\Delta: \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$$

Set $T(c \otimes c') = c' \otimes c$, ($\mathfrak{F} = Z_2$) so $\mathcal{C} \otimes \mathcal{C}$ becomes a $\mathfrak{F}(Z_2)$ module.

Finally, suppose there is an $\mathfrak{F}(Z_2)$ equivariant map

$$F: W \otimes \mathcal{C} \to \mathcal{C} \otimes \mathcal{C}$$

satisfying

$$F(e_0 \otimes c) = \Delta c.$$

Then, following [11] there is an induced map

$$\tilde{F}: W \otimes_{\mathfrak{F}(Z_2)} \mathcal{C}^* \otimes \mathcal{C}^* \to \mathcal{C}^*$$

where $\mathcal{C}^* = \mathrm{Hom}(\mathcal{C}, Z_2)$, and we set $\mathrm{Sq}^{n-i}\{a\} = \mathrm{Sq}_i\{a\} = \{\tilde{F}(e_i \otimes a \otimes a)\}$, where a has dimension n. More generally, on the cochain level we set

$$a \smile_i b = \tilde{F}(e_i \otimes a \otimes b)$$
$$ab = a \smile_0 b = \tilde{F}(e_0 \otimes a \otimes b).$$

Note the formulas

2.1.2 $$\delta(a \smile_i b) = (\delta a) \smile_i b + a \smile_i \delta b + a \smile_{i-1} b + b \smile_{i-1} a$$

for $i \geq 1$, and in particular

2.1.3 $$\delta(a \smile_1 b) = (\delta a) \smile_1 b + a \smile_1 \delta b + ab + ba.$$

(2) In the topological category where $\mathcal{C}$ is the singular chain complex of a space X and Δ is the usual diagonal approximation we also have the "Hirsch formula"

2.2.1 $$(ab) \smile_1 c = a(b \smile_1 c) + (a \smile_1 c)b.$$

From this we obtain a curious and important corollary (to the author's best knowledge first obtained by G. Hirsch.

LEMMA 2.2.2: *Suppose $a, b \in H^*(X, Z_2)$ and $ab = 0$, then $\langle a, b, a \rangle$ is defined and contains $b \cdot \mathrm{Sq}_1(a)$.*

In particular, if $a^2 = 0$ then $\langle a, a, a \rangle$ is defined and since $a \smile \mathrm{Sq}_1(a)$ belongs to its indeterminacy it follows that $\langle a, a, a \rangle$ contains 0 and $\langle a, a, a, a \rangle$ is thus defined. In the next section we will see that if $\dim a = n$, and if Ψ_n is the stable secondary operation associated with the relation $\mathrm{Sq}^{2n-1}\mathrm{Sq}^n = 0$ then

$$\Psi_n \cap \langle a, a, a, a \rangle \neq \emptyset.$$

Indeed, in the universal example for both these operations it turns out that they intersect in a *unique* element having, therefore, the best properties of each kind.

Proof of 2.2.2: Let $\mathbf{a}$, $\mathbf{b}$ represent a, b and suppose $\delta M = \mathbf{a} \smile \mathbf{b}$, then

$$\delta(M + \mathbf{b} \smile_1 \mathbf{a}) = \mathbf{b} \cdot \mathbf{a}$$

and

$$\{M\mathbf{a} + \mathbf{a}(M + \mathbf{b} \smile_1 \mathbf{a})\} \in \langle a, b, a \rangle$$

but

$$\delta M \smile_1 \mathbf{a} = M\mathbf{a} + \mathbf{a}M + (\mathbf{a} \cdot \mathbf{b}) \smile_1 \mathbf{a} = (M\mathbf{a} + \mathbf{a}M + \mathbf{a}(\mathbf{b} \smile_1 \mathbf{a})) + (\mathbf{a} \smile_1 \mathbf{a})\mathbf{b}$$

Q.E.D.

Remark 2.2.3: For convenience write

$$\underbrace{\langle a, a, \cdots, a\rangle}_{\text{i}-\text{times}} \quad \text{as} \quad \langle a^{(i)}\rangle,$$

then it turns out that if $\langle a^{(4)}\rangle = 0$ the next possible non-zero $\langle a^{(i)}\rangle$ is $\langle a^{(8)}\rangle$, and it seems reasonable to conjecture that the only possible non-zero $\langle a^{(i)}\rangle$ are those for which i is a power of 2 (at least in the topological category).

To prove this it would be sufficient to know that there was a stable n^{th}-order cohomology operation associated to the relation

$$\mathrm{Sq}^{2^{n-1}(m-1)+1}\mathrm{Sq}^{2^{r-1}(m-1)+1} \cdots \mathrm{Sq}^{2m-1}\mathrm{Sq}^{m}$$

for each m, n.

3. The 4-fold products $\langle a^{(4)}\rangle$

The main object of this section is to prove the result (3.3.1) mentioned in 2.2, and to examine the indeterminacy of $\langle a^{(4)}\rangle$.

(1) LEMMA 3.1.1: *Let $F \xrightarrow{j} E \xrightarrow{\pi} B$ be a Serre fibration and suppose that in the Leray-Serre spectral sequence*

$\alpha \in H^{2n-1}(F, Z_2)$ transgresses to $\beta^2 \in H^{2n}(B, Z_2)$ then $g = \langle(\pi^*\beta)^{(4)}\rangle$ is defined in $H^*(E, Z_2)$ and $\alpha^2 \in j^*(g)$.

Proof: Let $\delta A = B^2$ where $\{B\} = \pi^*(\beta)$, and we can assume

$$\{j^{\#}A\} = \alpha.$$

Then

$$\begin{aligned}\delta(A^2 + B^2 \smile_1 A) &= B^2 \smile_1 B^2 = B(B \smile_1 B^2) + (B \smile_1 B^2)B \\ &= \delta\{B[B^2 \smile_2 B + B \smile_1 (B \smile_1 B)] \\ &\qquad + [B^2 \smile_2 B + B \smile_1 (B \smile_1 B)]B\}\end{aligned}$$

thus

$$c = A^2 + BL + LB$$

is a permanent cycle with $\{j^{\#}c\} = \alpha^2$, where

$$L = B \smile_1 A + B^2 \smile_2 B + B \smile_1 (B \smile_1 B).$$

On the other hand if we put

$$A_{12} = A_{23} = A_{34} = A_{45} = B$$

$$A_{13} = A_{24} = A_{35} = A$$

$$A_{14} = A_{25} = L$$

these give a defining system for $\langle(\pi^*\beta)^{(4)}\rangle$ and the lemma follows.

Now consider the 2-stage Postnikov system E_n with fiber $K(Z_2, 2n-1)$ base $K(Z_2, n)$, and k-invariant ι^2 (where ι is the fundamental class in $H^n(K(Z_2, n), Z_2)$). If γ is the fundamental class on the fiber then γ transgresses to ι^2 and it follows that $\langle \pi^*(\iota)^{(4)} \rangle$ is defined and its restriction to the fiber includes γ^2. On the other hand, since ι^2 is in the image of suspension, it follows that E_n is a stable system, and it is easy to see that γ^2 is also the restriction to the fiber of the stable secondary chohomology operation Ψ_n based on the relation

$$\mathrm{Sq}^{2n-1}\mathrm{Sq}^n = 0.$$

However, this fact by itself *does not* assure us that the intersection of the two operations is non-empty in $H^*(E_n)$ (they might differ by elements of lower filtration). We have, in fact:

THEOREM 3.1.2: $\Psi_n \cap \langle \pi^*(\iota)^{(4)} \rangle \neq \emptyset$

Here is the basic idea of the proof. If we look at a special cochain ℓ which represents Ψ_n in B_{E_n} (the classifying space for E_n) then we may evaluate the suspension homomorphism σ (see §4.2) on this cochain. Then $\{\sigma(\ell)\} \in \Psi_n$, and we can consider $\Delta = c - \sigma(\ell)$ (where c is given in the proof of 3.1.1.) It turns out that $\Delta \in \operatorname{im} \pi$, say $\Delta = \pi(a)$. Moreover a is a cocycle which represents an element in the image of the Steenrod "slash" homomorphism

$$H_2(S_4) \,/\, \iota^{(4)} \to H^{4n-2}(K(Z_2, n)).$$

(Here S_4 is the symmetric group on 4 letters.)

But this image consists of the two elements $(\mathrm{Sq}^{n-1}\iota)^2$, $(\mathrm{Sq}^{n-2}\iota)\iota^2$ and π^* of each of them is 0, thus $\{\Delta\} = 0$ and the theorem follows. Now we give the details.

Let $\{I\}$ represent the fundamental class (of dimension $n+1$) in $H^*(B_{(E_n)})$, and let $\delta B = I \smile_1 I$. Then $\Psi_n(\{I\})$ has a representative

3.1.3 $$B \smile_1 B + (\delta B) \smile_2 B + K(I)$$

where $K(I)$ is an operation depending only on I and so that $\delta K + K\delta = (I \smile_1 I) \smile_2 (I \smile_1 I)$. K may be specified more exactly as follows.

Given any $Z_2(S_4)$ free acyclic complex W_4, there is a natural $Z_2(S_4)$ equivariant map constructed by the techniques of acyclic models

$$H: W_4 \otimes \mathcal{C} \to \mathcal{C}^{(4)}$$

where $\mathcal{C}$ is the singular complex of a space or the chain complex of a simplicial complex. There is also the natural map

$$G: (W\otimes W)\otimes W \otimes \mathcal{C} \xrightarrow{1\otimes F} W\otimes W\otimes \mathcal{C}^2 \xrightarrow{1\otimes T\otimes 1} (W\otimes\mathcal{C})^{(2)} \xrightarrow{F\otimes F} \mathcal{C}^{(4)}$$

which is equivariant with respect to the Sylow 2-subgroup of S_4, $Z_2\wr Z_2$. In particular, properly understood $(W\otimes W)\otimes W$ is a $Z_2(Z_2\wr Z_2)$ free acyclic complex. Thus, there is a $Z_2(Z_2\wr Z_2) \subset Z_2(S_4)$ equivariant map

$$M: (W\otimes W)\otimes W \to W_4$$

and the diagram

3.1.4
$$\begin{array}{ccc} W_4 \otimes \mathfrak{C} & \xrightarrow{H} & \mathfrak{C}^{(4)} \\ \uparrow {\scriptstyle M \otimes (\mathrm{id})} & \nearrow {\scriptstyle G} & \\ (W \otimes W) \otimes W \otimes \mathfrak{C} & & \end{array}$$

homotopy commutes, with homotopy S. In fact, F. Adams in unpublished work has constructed a W_4, maps H, F, M so 3.1.4 actually commutes. We now assume that W_4 is the one constructed by Adams. There is also the dual diagram

3.1.5
$$\begin{array}{ccc} (W \otimes W) \otimes W \otimes_{Z_2(Z_2 \wr Z_2)} \mathfrak{C}^{*(4)} & \xrightarrow{\tilde{G}} & \mathfrak{C}^* \\ {\scriptstyle M \otimes (\mathrm{id})} \downarrow & \nearrow {\scriptstyle H} & \\ W_4 \otimes_{Z_2(S_4)} \mathfrak{C}^{*(4)} & & \end{array}$$

Finally, there must be a chain $k \in W_4 \otimes_{Z_2(S_4)} Z_2$ so

$$\partial k = M((e_1 \otimes e_1) \otimes e_2 \otimes_{Z_2(Z_2 \wr Z_2)} Z_2)$$

since

$$M_*(\{(e_1 \otimes e_1) \otimes e_2 \otimes_{Z_2(Z_2 \wr Z_2)} Z_2\}) = 0 \quad \text{in} \quad H_*(W_4 \otimes_{Z_2(S_4)} Z_2).$$

Thus

$$\begin{aligned} \delta \tilde{H}(k \otimes I \otimes I \otimes I \otimes I) &= \tilde{G}((e_1 \otimes e_1) \otimes e_2 \otimes I \otimes I \otimes I \otimes I) \\ &= (I \smile_1 I) \smile_2 (I \smile_1 I) \end{aligned}$$

since I is a cocycle, and

$$\tilde{H}(k \otimes I \otimes I \otimes I \otimes I)$$

is the desired operation $K(I)$.

Now we study the problem of suspending the various operations on I to operations on $\sigma(I)$, so as to find a representative of Ψ_n on $\sigma(I)$. The basic idea behind this part of the proof follows from ([9] p. 295 formulae 4.1, 4.2, 4.3) which we generalize as best we can to fit our more complex situation.

There is the join operation on simplices $\sigma^{i+1} = \sigma^i \vee A$ where A is a point in general position with respect to σ^i. Thus on models we have

$$H_{i+1}(W_4 \otimes \sigma^i \vee A) \to (\sigma^i \vee A)^{(4)}$$

and in W_4 there is a map s_4 of degree -3 so that

3.1.6
$$\begin{aligned} A \cap \tilde{H}_{i+1}(X \otimes (\sigma_1 \vee A)^* \otimes (\sigma_2 \vee A)^* \otimes (\sigma_3 \vee A)^* \otimes (\sigma_4 \vee A)^* \\ = H_i(s_4(X) \otimes \sigma_1^* \otimes \sigma_2^* \otimes \sigma_3^* \otimes \sigma_4^*) \end{aligned}$$

where $\sigma_1, \cdots, \sigma_4$ are subsimplices of σ^i. The map s_2 on W defined in [9] simply maps e_i to e_{i-1}, $i > 0$, $e_0 \to 0$, and the following diagram commutes,

3.1.7
$$\begin{array}{ccc} (W\otimes W)\otimes W & \xrightarrow{M} & W_4 \\ \downarrow{\scriptstyle s_2\otimes s_2\otimes s_2} & & \downarrow{\scriptstyle s_4} \\ (W\otimes W)\otimes W & \xrightarrow{M} & W_4 \end{array}$$

Remark: We freely grant that 3.1.5, 3.1.6, the definitions of s_4 and W_4 are not obvious.

Now we can complete the proof of 3.1.2. Let i_{n+1} be the fundamental class in B_{E_n} and I be a cochain representing it. Then

$$\sigma\psi_n(i)$$

has as representant (from 3.1.3)

$$\tilde{H}(s_4(k)\otimes\sigma I\otimes\sigma I\otimes\sigma I\otimes\sigma I) + (\sigma B)^2 + (\delta\sigma B) \smile_1 (\sigma B)$$

and $(\sigma I)^2 = \delta(\sigma B)$. Thus the difference between this representant of $\sigma\psi_n(i_{n+1}) \subset \psi_n(i)$ and the representant of $\langle i^{(4)}\rangle$ given in the proof of 3.1.1. is

$$S = \tilde{H}(s_4(k)\otimes\sigma I\otimes\sigma I\otimes\sigma I\otimes\sigma I) + \sigma I[((\sigma I)^2 \smile_2 \sigma I) + \sigma I \smile_1 (\sigma I \smile_1 \sigma I)$$
$$+ [(\sigma I)^2 \smile_2 \sigma I + \sigma I \smile_1 (\sigma I \smile_1 \sigma I)]\sigma I = \tilde{H}((s_4k + h)\otimes\sigma I\otimes\cdots\otimes\sigma I)],$$

where h is an appropriate chain in W_4.

We may now assume $I = (B\pi)^{\#}J$ and hence $\sigma I = \pi^{\#}(\sigma J)$ where $\{\sigma J\} = i$ the fundamental class of $K(Z_2, n)$. Thus by naturality

$$S = \pi^{\#}\tilde{H}((s_4k + h)\otimes(\sigma J\otimes\cdots\otimes\sigma J)).$$

Finally, $(s_4k + h)\otimes\sigma J\otimes\cdots\otimes\sigma J$ is a cycle and simultaneously represents an element in $H_*(S_4)$ and in $H^*(K(Z_2, n))$. In fact, the latter element is given by taking the slash product with $i^{(4)}$ of $H_*(S_4)$; see for example [8] p. 248, and the theorem now follows easily as indicated in the outline.

(2) Lemma 3.2.1: *Let $\mathcal{B} = \langle A_1, A_2, A_3, A_4\rangle$ be a 4-fold matric Massey product then the indeterminacy of $\mathcal{B}$ is contained in the set of all possible 3-fold Massey products of the form*

2.3.2
$$\left\langle (A_1, B)\begin{pmatrix} A_2 & C \\ 0 & A_3\end{pmatrix}\begin{pmatrix} D \\ A_4\end{pmatrix}\right\rangle$$

Proof: Let A_{ij}, A_{ij}' be two different defining systems for $\mathcal{B}$. Without loss of generality we may assume $A_{i,i+1} = A_{i,i+1}'$. Then

$$\mathcal{B} = A_{12}A_{25} + A_{13}A_{35} + A_{14}A_{45}$$

$$\mathcal{B}' = A_{12}A_{25}' + A_{13}'A_{35}' + A_{14}'A_{45}$$

and

2.3.3
$$\mathcal{B} + \mathcal{B}' = A_{12}(A_{25} + A_{25}') + A_{13}(A_{35} + A_{35}') + (A_{13} + A_{13}')A_{35}' + (A_{14} + A_{14}')A_{45}$$

$$\delta(A_{25} + A_{25}') = A_{23}(A_{35} + A_{35}') + (A_{24} + A_{24}')A_{45},$$

$$\delta(A_{14} + A_{14}') = A_{12}(A_{24} + A_{24}') + (A_{13} + A_{13}')A_{34}$$

thus, if $B = A_{13} + A_{13}'$, $C = A_{24} + A_{24}'$ and $D = A_{35} + A_{35}'$, a defining system for 2.3.2 is

$$(A_{12}, B) = \bar{A}_{12}$$

$$(A_{13}, A_{14} + A_{14}') = \bar{A}_{13}$$

$$\begin{pmatrix} A_{23} & C \\ 0 & A_{34} \end{pmatrix} = \bar{A}_{23}$$

$$\begin{pmatrix} D \\ A_{45} \end{pmatrix} = \bar{A}_{34}$$

$$\begin{pmatrix} A_{25} + A_{25}' \\ A_{35}' \end{pmatrix} = \bar{A}_{24}$$

and the lemma is proved.

Lemma 3.2.4: a) *Let $a, b, c, d \in H^*((X), Z_2)$ and suppose $a^2 = 0$ so then* a)

3.2.5
$$\left\langle (a, b) \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \right\rangle$$

is defined and contains 0,

b)

3.2.6
$$\left\langle (b, a, c, d) \begin{pmatrix} a & 0 & d & 0 \\ b & a & c & d \\ 0 & 0 & a & 0 \\ 0 & 0 & b & a \end{pmatrix} \begin{pmatrix} d \\ c \\ a \\ b \end{pmatrix} \right\rangle$$

is also defined and also contains 0.

Proof: a) Let $\delta M = a^2$, then a defining system for 3.2.5 is

$$(M + a \smile_1 a, a \smile_1 b), \begin{pmatrix} M + a \smile_1 a & b \smile_1 a \\ 0 & M + a \smile_1 a \end{pmatrix}$$

$$\begin{pmatrix} b \smile_1 a \\ M + a \smile_1 a \end{pmatrix}$$

$$(M \smile_1 a, M \smile_1 b + a(a \smile_2 b) + (a \smile_2 b)a + (a \smile_1 a) \smile_1 b)$$

$$\begin{pmatrix} M \smile_1 a \\ M \smile_1 b + a(a \smile_2 b) + (a \smile_2 b)a + a \smile_1 a) \smile_1 b \end{pmatrix}$$

and 3.2.5 is represented by

3.2.7 $\quad a^2(a \smile_2 b) + (a^2 \smile_1 a) \smile_1 b + (a^2 \smile_1 b) \smile_1 a + (a \smile_2 b)a^2$

after subtracting

$$\delta\{(M \smile_1 b) \smile_1 a + ((b \smile_2 a)a) \smile_1 a + a \smile_1 (a(b \smile_2 a))\}$$

from the represent of 3.2.5 obtained directly from the defining system above.

To complete the proof of 3.2.5 we must show 3.2.7 is cohomologous to 0. In fact, I assert that there is a 3-variable operation θ, so

$$(\delta\theta+\theta\delta)(A\otimes B\otimes C) = A(B \smile_2 C) + (A \smile_1 B) \smile_1 C + (A \smile_1 C) \smile_1 B + (B \smile_2 C)A.$$

In order to prove this, it suffices to observe that the right hand side of the above equation is a cochain map of degree -2, hence it is homotopic to 0. Now substituting a^2 for A, a for B, b for C, $\delta(A\otimes B\otimes C) = 0$, so $\delta\theta(a^2, a, b) = (3.2.7)$ and the proof of part a is complete.

b) For a defining system we use

$$(b \smile_1 a, M, a \smile_1 d + b \smile_1 c, d \smile_1 a)\begin{pmatrix} d \smile_1 a \\ b \smile_1 d + a \smile_1 c \\ M \\ b \smile_1 a \end{pmatrix}$$

and the resultant representant of the Massey product is seen to be the coboundary of

$$M \smile_1 c + (b \smile_1 d) \smile_1 a.$$

This completes the proof of the lemma. As a corollary we have, if $a^2 = 0$, then

$$\left\langle (a\ b) \begin{pmatrix} a & b \\ 0 & a \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} \right\rangle$$

is defined and contains 0.

(3) From section 4.1 it follows that in $H^*(E_n)$ the indeterminacy of $\langle i^{(4)} \rangle$ has the form

$$\left\langle (i\ b) \begin{pmatrix} i & b \\ 0 & i \end{pmatrix} \begin{pmatrix} b \\ i \end{pmatrix} \right\rangle$$

where b runs over all elements in $H^*(E_n)$ of dimension $n - 1$. Thus from 3.2.1 it follows that indet $(\langle i^4 \rangle)$ consists of decomposables. Again, in the next section it will become clear that there is no *primitive* decomposable among them. Hence, if $\langle i^{(4)} \rangle$ contains any primitives, it must contain a unique one. On the other hand Ψ_n (or more precisely $\sigma\Psi_{n+1}$) contains nothing but primitives since it is a stable operation. Thus from 3.1.2 we have

THEOREM 3.3.1: $\Psi_n \cap \langle i^{(4)} \rangle = \theta$ *consists of a unique element in* $H^*(E_n, Z_2)$.

(4) From our main theorem

3.4.1 $$\mathrm{Sq}^1 \langle i^{(4)} \rangle \subset \left\langle (i \ \mathrm{Sq}^1 i) \begin{pmatrix} i & \mathrm{Sq}^1 i \\ 0 & i \end{pmatrix} \begin{pmatrix} i & \mathrm{Sq}^1 i \\ 0 & i \end{pmatrix} \begin{pmatrix} \mathrm{Sq}^1 i \\ i \end{pmatrix} \right\rangle$$

which by lemma 3.2.4 (a) contains 0. Again it will follow from the results of the next section that the indeterminacy of the right hand side of 3.4.1 runs over Massey products having the same form as 3.2.6 with $a = \mathrm{Sq}^1 i$, $b = i$. Thus again it will appear that the only primitive in the right hand side of 3.4.1 is 0. But $\mathrm{Sq}^1(\theta)$ is primitive since θ is and it follows that we have

Theorem 3.4.2: $\mathrm{Sq}^1(\theta) = 0$.

4. The cohomology of E_n

The cohomology ring of E_n was first determined by Kristensen in [4]. Here we outline an alternative method for evaluating it, which, in view of 3.1.1, allows us to evaluate Massey products. We finish this section by evaluating the action of the Steenrod algebra $\mathcal{A}(2)$ in $H^*(E_2)$ and $H^*(E_3)$. Actually, our techniques together with those of Kristensen seem to allow a virtually complete determination of the $\mathcal{A}(2)$ action in $H^*(E_n)$ for all n, but we defer the full exposition of these results.

(1) There is the evident fibering

$$\begin{array}{ccc} K(Z_2, 2n-1) & \xrightarrow{j} & E_n \\ & & \downarrow{\scriptstyle \pi} \\ & & K(Z_2, n) \end{array}$$

for E_n and in the resultant Leray-Serre spectral sequence the $\mathcal{E}_2$ terms is

$$H^*(K(Z_2, n), Z_2) \otimes H^*(K(Z_2, 2n-1), Z_2).$$

The generator of the left hand part is i, that for the other is γ. We now describe all differentials, and compute the $\mathcal{E}_\infty$ term of the spectral sequence.

If I is an m-tuple of integers $(i_1, \cdots, i_m)$ with $i_1 \geq 2i_2 \geq \cdots \geq 2^{m-1} i_m$ we say *I can be halved* if each i_j in I is even and we write $\frac{1}{2}I = (\frac{1}{2}i_1, \cdots, \frac{1}{2}i_m)$. Set $|I| = \sum_{j=1}^m i_j$, then we have

Lemma 4.1.1: $d_{|I|+2n}(\mathrm{Sq}^I(\gamma)) = (\mathrm{Sq}^{1/2 I} i)^2$ *if I can be halved, otherwise*

$$d_{|I|+2n}(\mathrm{Sq}^I \gamma) = 0.$$

Moreover as I runs over all permissible sequences of excess less than $2n - 1$, *these generate all differentials in the spectral sequence.*

Proof: $H^*(K(Z_2, 2n-1), Z_2) = P(\mathrm{Sq}^I \gamma)$ where I runs over all admissible sequences of excess less than $2n - 1$. Moreover every generator transgresses to the base, and since $d_{2n}\gamma = i^2$, the first assertion of 4.1.1 follows easily. The second statement is proved by a simple induction, assuming $\mathcal{E}_r = \mathcal{E}_r^{*0} \otimes \mathcal{E}_r^{0*}$ and showing that $\mathcal{E}_{r+1}$ also decomposes in this way.

Thus it is easy to see that

4.1.2 $$\mathcal{E}_\infty^{*0} \cong \Lambda\{\mathrm{Sq}^J(\iota)\}$$

the exterior algebra on generators $\mathrm{Sq}^J(\iota)$ where J runs over all admissible sequences of excess less than n. Also

4.1.3 $$\mathcal{E}_\infty^{0*} = P(\{\mathrm{Sq}^J\gamma\}, \{(\mathrm{Sq}^K\gamma)^2\})$$

a polynomial algebra on generators $\{\mathrm{Sq}^J\gamma\}$ where J has excess less than $2n - 1$ and can not be halved $(j^*\{\mathrm{Sq}^J\gamma\} = \mathrm{Sq}^J\gamma)$ and also on generators $\{(\mathrm{Sq}^K\gamma)^2\}$ where K has excess less than $2n - 1$ and can be halved $(j^*\{(\mathrm{Sq}^K\gamma)^2\} = (\mathrm{Sq}^K\gamma)^2)$.

Finally, we remark that E_n is a loop space since the k-invariant ι^2 is in the image of suspension. Thus the Serre spectral sequence is a spectral sequence of Hopf-algebras and we have easily

LEMMA 4.1.4: $\mathcal{E}_\infty^{**}$ *is primitively generated as a Hopf algebra with generators*

$$\{\mathrm{Sq}^J\gamma\}, \quad \{(\mathrm{Sq}^K\gamma)^2\}, \quad \mathrm{Sq}^J(\iota)$$

as described above.

Note in particular that since $\mathcal{E}_\infty^{**}$ is the graded Hopf-algebra associated to $H^*(E_n)$ it follows that the only possible primitives in $H^*(E_n)$ are either in the image of π^* (and more exactly in π^* Prim $(H^*(K(Z_2, n), Z_2))$ or restrict under j^* to primitives on the fiber.

Finally, as a consequence of 3.1.1, 3.1.2 we may replace $\{(\mathrm{Sq}^K\gamma)^2\}$ by a primitive in $\langle(\mathrm{Sq}^K\iota)^{(4)}\rangle$ and we have

THEOREM 4.1.5: $H^*(E_n) = \Lambda(\pi^*(\mathrm{Sq}^J\iota)) \otimes P(\{\mathrm{Sq}^K\gamma\}, \langle(\mathrm{Sq}^{\bar{J}}\iota)^{(4)}\rangle)$ *where* $J, \bar{J}$ *run over all permissible sequences of excess less than* n *and* K *runs over all permissible sequences of excess less than* $2n - 1$ *which cannot be halved. Moreover, at least the generators* $\pi^*(\mathrm{Sq}^J\iota)$, $\langle(\mathrm{Sq}^J\iota)^{(4)}\rangle$ *can be taken as primitive.*

Now we are able to complete the proofs of 3.3.1 and 3.4.2: there are no decomposable primitives in $H^*(E_n)$ in dimensions $4n - 2$, $4n - 1$ since if there were we would also have such decomposable primitives in $\mathcal{E}_\infty$, but there are none.

(2) The suspension homomorphism

$$\sigma: H^*(X) \rightarrow H^{*-1}(\Omega X)$$

is defined as the composition sA^* where s is the suspension isomorphism

$$s: H^*(\Sigma\Omega X \xrightarrow{\cong} H^{*-1}(\Omega X)$$

while A is the adjoint map $\Sigma\Omega X \rightarrow X$. σ is clearly a natural transformation and we have the commutative diagram

4.2 1 $$\begin{array}{ccc} H^*(E_n) & \xrightarrow{\sigma} & H^*(\Omega E_n) \\ \uparrow{\scriptstyle \pi^*} & & \uparrow{\scriptstyle (\Omega\pi)^*} \\ H^*(K(Z_2, n)) & \xrightarrow{\sigma} & H^*K(Z_2, n - 1). \end{array}$$

σ on the bottom row is known from Cartan's calculations of the cohomology of the $K(\pi, n)$'s [2] or from [7] (§4) and the fact that $\mathcal{A}(2)$ commutes with σ. It is also well known that $\sigma(\alpha) = 0$ if α is decomposable (this is even true at the cochain level!).

LEMMA 4.2.2: *Let* $\alpha \in H^*(K(Z_2, n), Z_2)$ *and suppose* $\sigma(\alpha) = 0$ *in* 4.2.1, *then* $\pi^*(\alpha) = 0$, *if* α *is primitive.*

Proof: From Cartan's calculations it follows that $\alpha \in \operatorname{Ker} \sigma$ if and only if α is decomposable and all indecomposables are primitive. On the other hand, Ker π^* is precisely generated by the decomposable primitives.

THEOREM 4.2.3: $\mathrm{Sq}^{4j}\langle i^{(4)}\rangle = \langle(\mathrm{Sq}^j i)^{(4)}\rangle$.

Proof: $\mathrm{Sq}^{4j}\langle i^{(4)}\rangle$ and $\langle(\mathrm{Sq}^j i)^{(4)}\rangle$ are both primitive and both restrict to the same element on the fiber, hence from the results of 4.1

$$\mathrm{Sq}^{4j}\langle i^{(4)}\rangle = \langle(\mathrm{Sq}^J i)^{(4)}\rangle + T$$

where T is $\pi^*(\alpha)$ with α primitive. On the other hand,

$$\sigma(\mathrm{Sq}^{4j}\langle i^{(4)}\rangle) = \mathrm{Sq}^{4j}(\sigma\langle i^{(4)}\rangle) = 0,$$

since $\langle i^{(4)}\rangle$ is decomposable on the cochain level but this implies, since

$$\sigma\langle(\mathrm{Sq}^J i)^{(4)}\rangle = 0$$

also, that

$$\sigma(T) = \sigma(\pi^*\alpha) = (\Omega\pi)^*(\sigma(\alpha)) = 0.$$

But $(\Omega\pi)^*$ is a monomorphism so $\sigma(\alpha) = 0$ implies $\pi^*(\alpha) = T = 0$ by 4.2.2, and the result follows.

THEOREM 4.2.4: *Let* $n + j$ *be odd, then in* $H^*(E_n)$ *there is a primitive indecomposable* w_j *which restricts to* $\mathrm{Sq}^{2j+1}(\gamma)$ *on the fiber and*

$$\mathrm{Sq}^{4j+2}\langle i^{(4)}\rangle = w_j^2$$

(actually any primitive which restricts to $\mathrm{Sq}^{2j+1}(\gamma)$ has this property).

Proof: w_j represents the stable secondary operation on the relation $\mathrm{Sq}^1(\mathrm{Sq}^{n+j})$ evaluated on the class $\mathrm{Sq}^j i$, and for this reason it may be taken as a primitive. As before both w_j^2, $\mathrm{Sq}^{4j+2}\langle i^{(4)}\rangle$ restrict to the same element on the fiber and we have

$$\mathrm{Sq}^{4j+2}\langle i^{(4)}\rangle = (w_j)^2 + \pi^*(\alpha)$$

where α is primitive and the theorem follows as before.

Similarly we have

THEOREM 4.2.5: *Let* $n + j$ *be even then any element* w_j *which restricts to* $\mathrm{Sq}^{2j+1}\gamma$ *on the fiber satisfies*

$$\mathrm{Sq}^{4j+2}\langle i^{(4)}\rangle = w_j^2$$

Proof: $\varphi(w_j) = w_j \otimes 1 + \Sigma a' \otimes a'' + 1 \otimes w_j$ with either a' or a'' having the form $\pi^*(\alpha)\beta$ where $\alpha \neq 1$, since

$$(j^* \otimes j^*)\varphi(w_j) = \varphi j^*(w_j) = \mathrm{Sq}^{2j+1}\gamma \otimes 1 + 1 \otimes \mathrm{Sq}^{2j+1}\gamma$$

Thus $\varphi(w_j^2) = (\varphi w_j)^2 = w_j^2 \otimes 1 + 1 \otimes w_j^2$ since $(\pi^*\alpha)^2 = 0$ for any $\alpha \neq 1$ in $H^*(K(Z_2, n))$. Now, the proof goes as before.

Thus we have

COROLLARY 4.2.6: $\mathrm{Sq}^{2j+1}\langle i^{(4)} \rangle = 0$ *for all* $j \geq 0$.

Hence, we have completely determined the action of the Steenrod algebra on $\langle i^{(4)} \rangle$.

(3) We now compute the structure of $H^*(E_2, Z_2)$ over $\mathfrak{A}(2)$. From 4.1 we have

LEMMA 4.3.1: $H^*(E_2, Z_2) = \Lambda(\mathrm{Sq}^{J_\tau} i) \otimes P(\mathrm{Sq}^R(\lambda), \mathrm{Sq}^S\langle i^{(4)} \rangle)$ *where*

$$J_\tau = \mathrm{Sq}^{2^\tau}\mathrm{Sq}^{2^{\tau-1}} \cdots \mathrm{Sq}^2\mathrm{Sq}^1$$

R is any permissible monomial of excess ≤ 3 starting with 2 or 3, and S is any permissible monomial of excess less than 6 which can be halved and does not start with 2. Moreover $j^*(\lambda) = \mathrm{Sq}^1\gamma$.

From [4] we have

LEMMA 4.3.2: $\mathrm{Sq}^3\lambda = \mathrm{Sq}^2\mathrm{Sq}^1 i + \mathrm{Sq}^1 i \cdot i$

$$\varphi(\lambda) = \lambda \otimes 1 + \pi^*(i) \otimes \pi^*(i) + 1 \otimes \lambda$$

where φ is the comultiplication in $H^*(E_2)$.

But this together with 4.2.4, 4.2.5 gives the complete structure of $H^*(E_2, Z_2)$ over $\mathfrak{A}(2)$.

(4) In $H^*(E_3)$ we have

LEMMA 4.3.1: $H^*(E_3, Z_2) = \Lambda(\mathrm{Sq}^I\pi^*(i)) \otimes P(\mathrm{Sq}^R\lambda, \mathrm{Sq}^S\tau, \mathrm{Sq}^T\langle i^{(4)} \rangle)$ *where I runs over all permissible sequences of excess less than 3, R runs over all permissible sequences of excess 5 or less which start with a number greater than or equal to 2, S runs over all permissible sequences of excess 7 or less which start with 6 or 7, or S is*

$$(2^{r_1}[2^{r_2}\{2^{r_3} + 2\} + 1], \cdots, 2^{r_2-1}[2^{r_3} + 2], \cdots, 2^{r_3} + 2, 2^{r_3-1}, \cdots, 4)$$

$r_3 > 2$ and T runs over all admissible monomials which can be halved, have excess less than 8 and do not begin with 2, 6 (τ restricts to $\mathrm{Sq}^3\gamma$, λ restricts to $\mathrm{Sq}^1\gamma$).

Again from [4] we have

LEMMA 4.3.2: (a) $\varphi(\tau) = \tau \otimes 1 + \pi^*(\mathrm{Sq}^1 i) \otimes \pi^*(\mathrm{Sq}^1 i) + 1 \otimes \tau$
(b) $\varphi(\lambda) = \lambda \otimes 1 + 1 \otimes \lambda$
(c) $\mathrm{Sq}^1(\lambda) = \pi^*(\mathrm{Sq}^3\mathrm{Sq}^1 i)$
(d) $\mathrm{Sq}^1(\tau) = 0$

Next I claim we have the relation

4.3.3 $$\mathrm{Sq}^2(\tau) = \langle i^{(4)} \rangle + \mathrm{Sq}^4(\lambda) + (\mathrm{Sq}^1 i) \cdot \mathrm{Sq}^2\mathrm{Sq}^1 i$$

Proof: Both sides restrict to the same thing on the fiber, moreover φ of both sides are equal by 4.3.2. Hence they differ by a primitive T from the base, but, as before, by looking at σ of both sides we see that T must equal 0.

Similarly

$$\mathrm{Sq}^3\tau = \mathrm{Sq}^5(\lambda) + \mathrm{Sq}^1 i\cdot \mathrm{Sq}^3\mathrm{Sq}^1 i$$

and from the relation $\mathrm{Sq}^2\mathrm{Sq}^3 = \mathrm{Sq}^5 + \mathrm{Sq}^4\,\mathrm{Sq}^1$ we have

$$\mathrm{Sq}^5\tau = \mathrm{Sq}^6\mathrm{Sq}^3\mathrm{Sq}^1 i + \mathrm{Sq}^2\mathrm{Sq}^1 i\cdot\mathrm{Sq}^3\mathrm{Sq}^1 i = \mathrm{Sq}^1(\mathrm{Sq}^4\tau),$$

and this completes the analysis of $H^*(E_3)$.

5. The set of higher associating homotopies

(1) The proof of Theorem 0 proceeds by constructing representants A for $\mathrm{Sq}^i\langle B_1, \cdots, B_n\rangle$, B for $\langle \mathbf{SQ}_R{}^i(B_1),\ \mathbf{SQ}^i(B_2), \cdots,\ \mathbf{SQ}^i(B_{n-1}),\ \mathbf{SQ}_c{}^i(B_n)\rangle$ and a cochain Q so $\delta Q = A + B$. The existence of Q in turn depends on certain cochain operations which we now construct.

We assume given two categories $\mathcal{C}$, $\mathcal{C}'$ whose objects are non-negatively graded chain complexes over Z_2, both of which are representable by acyclic models. Further, for each object $C \in \mathcal{C}$, $C' \in \mathcal{C}'$ there are natural transformations

$$\Delta: C \to C\otimes C$$
$$\Delta': C' \to C'\otimes C'$$

both coassociative. Moreover we assume given a functor

$$H_0: \mathcal{C} \to \mathcal{C}'$$

carrying models to models satisfying

$$\Delta' H_0 = (H_0\otimes H_0)\Delta$$

when restricted to the 0-dimensional parts C_0, C_0' of the objects. Of course $H_0\otimes H_0\Delta$ may not equal $\Delta' H_0$ in higher dimensions, but the two functors will certainly be homotopic. More exactly define

$$\Delta_{i,k}: \underbrace{C\otimes \cdots \otimes C}_{i \text{ times}} \to \underbrace{C\otimes \cdots \otimes C}_{i+1 \text{ times}}$$

as $1\otimes \cdots \otimes \overbrace{1\otimes\Delta\otimes 1}^{k^{\text{th}} \text{ position}}\otimes \cdots \otimes 1$ and set

$$\sigma_i = \Sigma(-1)^k\Delta_{i,k}$$

Similarly $\sigma_i' = \Sigma(-1)^k\Delta_{i,k}'$.

From the fact that Δ, Δ' are coassociative it follows that the iteration satisfies

5.1.1 $$\sigma_{i+1}\sigma_i = \sigma_{i+1}'\sigma_i' = 0$$

5.1.2
$$\sigma_i = \sigma_j \otimes \underbrace{1\otimes \cdots \otimes 1}_{i-j} + (-1)^j \underbrace{1\otimes \cdots \otimes 1}_{j} \otimes \sigma_{i-j}$$

and we have

THEOREM 5.1.3: *There exist functors H_k of degree k, $H_k: C \to (C')^{k+1}$ and*

$$\partial H_k \pm H_k \partial = \sigma_k' H_{k-1} + \left(\textstyle\sum_{r=0}^{k-1} (-1)^{k-r} H_{k-r-1} \otimes H_r\right)\Delta$$

Proof: If H_k is determined on models it extends by naturality and representability to all of $\mathcal{C}$. We can assume H_k is defined and 0 (for $k \geq 1$) on the 0-skeletons. Now suppose $H_0, \cdots, H_{k-1}$ are defined on all of $\mathcal{C}$. In order to construct H_k it suffices to show

$$M = \sigma_k' H_{k-1} + \left(\textstyle\sum_{r=0}^{k-1} H_{k-r-1} \otimes H_r\right)\Delta$$

is a chain map. But

$$\begin{aligned} M\partial \pm \partial M = \sigma'&\left(\textstyle\sum_{r=0}^{k-2} (-1)^{k+r-1} H_{k-r-2} \otimes H_r\right)\Delta \\ &+ \textstyle\sum_0^{k-1} (-1)^{k+r} (\sigma_{k-r-1} \otimes 1) H_{k-r-2} \otimes H_r \\ &+ \textstyle\sum_0^{k-1} (-1)^{k+r-1} (1 \otimes \sigma_r) H_{k-r-2} \otimes H_r \\ &+ \textstyle\sum (-1)^{j+1} H_{k-r-j-2} \otimes H_j \otimes H_r \\ &+ \textstyle\sum (-1)^{r+s} H_{k-r-1} \otimes H_{r-s-1} \otimes H_s = 0 \end{aligned}$$

from 5.1.2 and the theorem follows.

(2) Here are explicit formulae in the first few cases:

$$\begin{aligned} \partial H_1 + H_1 \partial &= \Delta' H_0 - H_0 \otimes H_0 \Delta \\ \partial H_2 - H_2 \partial &= (\Delta' \otimes 1 - 1 \otimes \Delta') H_1 + (H_1 \otimes H_0 - H_0 \otimes H_1)\Delta \\ \partial H_3 + H_3 \partial &= (\Delta' \otimes 1 \otimes 1 - 1 \otimes \Delta' \otimes 1 + 1 \otimes 1 \otimes \Delta') H_2 \\ &\quad - (H_2 \otimes H_0 - H_1 \otimes H_1 + H_0 \otimes H_2)\Delta \end{aligned}$$

(3) We now suppose we are given a category $\mathcal{C}^0$ of non-negatively graded chain complexes with acyclic models, then the objects of $\mathcal{C}$ are the tensor products

$$W \otimes C$$

where C is an object in $\mathcal{C}^0$, and the objects in $\mathcal{C}'$ are of the form $C \otimes C$. H_0 is the map F of section 2 which gives rise to the $\smile_i$ operations. Δ in $\mathcal{C}$ is $(1 \otimes T \otimes 1)\Delta \otimes \Delta$ and Δ in C' is $(1 \otimes T \otimes 1)\Delta \otimes \Delta$.

Finally, we are interested in the dual complexes

$$\mathrm{Hom}_{Z_2}(C, Z_2) = C^*$$

$$\mathrm{Hom}_{Z_2}(C \otimes C, Z_2) = (C \otimes C)^*$$

and the dual maps $\tilde{H}_i : W \otimes (C \otimes C)^{*(i+1)} \to C^*$. Here we have

$$(\delta\tilde{H}_i + \tilde{H}_i\delta)(e_n \otimes \bar{x}_1 \otimes \cdots \otimes \bar{x}_{i+1})$$

4.3.1
$$= \tilde{H}_{i-1}(e_n \otimes \textstyle\sum_j \bar{x}_1 \otimes \cdots \otimes \bar{x}_j \bar{x}_{j+1} \otimes \cdots \otimes \bar{x}_{i+1})$$
$$+ \textstyle\sum_{r,j} \tilde{H}_{i-j-1}(e_r \otimes \bar{x}_1 \otimes \cdots \otimes \bar{x}_{i-j}) \smile \tilde{H}_j(T^r e_{n-r} \otimes \bar{x}_{i-j+1} \otimes \cdots \otimes \bar{x}_{i+1})$$

(4) Here are two examples

4.4.1
$$\delta\tilde{H}_1(e_n \otimes \bar{x}_1 \otimes \bar{x}_2) + \tilde{H}_1(\delta(e_n \otimes \bar{x}_1 \otimes \bar{x}_2))$$
$$= \tilde{H}_0(e_n \otimes \bar{x}_1 \cdot \bar{x}_2) + \textstyle\sum_r H_0(e_r \otimes \bar{x}_1) \smile H_0(T^r e_{n-r} \otimes \bar{x}_2)$$

and this is exactly the formula used by Steenrod in [10] to prove the Cartan formula for the Sq^i.

$\tilde{H}_2$ is somewhat more myterious however.

$$(\delta\tilde{H}_2 + \tilde{H}_2\delta)(e_n \otimes \bar{x}_1 \otimes \bar{x}_2 \otimes \bar{x}_3)$$

4.4.2
$$= \tilde{H}_1(e_n \otimes (\bar{x}_1\bar{x}_2 \otimes \bar{x}_3 + \bar{x}_1 \otimes \bar{x}_2\bar{x}_3))$$
$$+ \textstyle\sum_r \tilde{H}_1(e_r \otimes \bar{x}_1 \otimes \bar{x}_2) \smile \tilde{H}_0(T^r e_{n-r} \otimes \bar{x}_3)$$
$$+ \textstyle\sum_r \tilde{H}_0(e_r \otimes \bar{x}_1) \smile \tilde{H}_1(T^r e_{n-r} \otimes \bar{x}_2 \otimes \bar{x}_3).$$

From consideration of the first sum in this last example it becomes clear why we call the $\tilde{H}_i$ higher associating homotopies.

(5) We extend the $\tilde{H}_i$ to the matric algebras $\mathfrak{M}(\mathcal{C}^*)$, $\mathfrak{M}(\mathcal{C}'^*)$ by setting

$$(\tilde{H}_0[e_k \otimes M \otimes_{Z_2} N])_{i,j} = \tilde{H}_0(e_k \otimes M_{i,j} \otimes N_{i,j})$$

where M, N are matrices in $\mathfrak{M}(\mathcal{C}^*)$, and more generally we put

$$\tilde{H}_k(e_n \otimes (M_1 \otimes N_1) \otimes \cdots \otimes (M_{i+1} \otimes N_{i+j}))_{r,t}$$

4.5.1
$$= \textstyle\sum_{s_1 s_2 \ldots s_i} \tilde{H}_k[e_n \otimes (M_1(r, s_1) \otimes N_1(r, s_1))$$
$$\otimes (M_2(s_1, s_2) \otimes N_2(s_1, s_2)) \otimes \cdots \otimes (M_{i+j}(s_i, t) \otimes N_{i+1}(s_i, t))].$$

We may now easily check that 4.3.1 holds, except that we replace ordinary cup products by matric multiplication.

6. Theorem 0 for 3-fold products

The proof of Theorem 0 in full generality is very messy and computational. To aid in following it, we now consider the special case of 3-fold products in detail.

Let an element α of a 3-fold product in $\mathcal{C}^*$ be defined by means of a map

$$h : F_3 \to \mathcal{C}^*.$$

Now consider the cochain

$$\chi = \tilde{H}_2(e_i \otimes h(1,2) \otimes h(1,2) \otimes h(2,3) \otimes h(2,3) \otimes h(3,4) \otimes h(3,4))$$
$$+ \tilde{H}_1[(e_i \otimes h(1,3) \otimes h(1,2)(2,3)$$
$$+ e_{i-1} \otimes h(1,3) \otimes h(1,3)) \otimes h(3,4) \otimes h(3,4)]$$
$$+ \tilde{H}_1[e_i \otimes h(1,2) \otimes h(1,2) \otimes h(2,4) \otimes h(2,3)(3,4)]$$
$$+ \tilde{H}_1[e_{i-1} \otimes h(1,2) \otimes h(1,2) \otimes h(2,4) \otimes h(2,4)].$$

A short calculation gives

$$\delta\chi = \tilde{H}_0(e_{i-1} \otimes h(1,3)h(3,4) \otimes h(1,3)h(3,4) + h(1,2)(2,4)) \otimes h((1,2)(2,4))$$
$$+ \tilde{H}_0(e_i \otimes h((1,3)(3,4) + (1,2)(2,4)) \otimes h((1,2)(2,3)(3,4)))$$
$$+ \textstyle\sum_r \tilde{H}_0(e_r \otimes h(1,3) \otimes h((1,2)(2,3))$$
$$+ e_{r-1} \otimes h(1,2) \otimes h(1,3)) \smile \tilde{H}_0(e_{i-r} \otimes h(3,4) \otimes h(3,4))$$
$$+ \textstyle\sum_r \tilde{H}_1(e_r \otimes h(1,2) \otimes h(1,2) \otimes h(2,3) \otimes h(2,3))$$
$$\smile \tilde{H}_0(e_{i-r} \otimes h(3,4) \otimes h(3,4))$$
$$+ \textstyle\sum_r \tilde{H}_0(e_r \otimes h(1,2) \otimes h(1,2)) \smile [\tilde{H}_0(e_{i-r-1} \otimes h(2,4) \otimes h(2,4)$$
$$+ T^r e_{i-r} \otimes (h(2,4) \otimes h(2,3)(3,4)))]$$
$$+ \textstyle\sum_r H_0(e_r \otimes h(1.2) \otimes h(1,2))$$
$$\smile H_1(e_{i-r} \otimes h(2,3) \otimes h(2,3) \otimes h(3,4) \otimes h(3,4)).$$

The last 4 sums above represent a defining system for

$$\langle \mathrm{Sq}^{m-i+1}\, {}_R A_1,\ \mathrm{SQ}^{m-i+1} A_2,\ \mathrm{SQ}^{m-i+1}\, {}_C A_3 \rangle$$

Thus, to complete the proof for this case, it suffices to show that the first two terms on the right above represent $\mathrm{Sq}^{m-i+1}(\alpha)$. Let them be A_1, A_2 then

$$\delta\tilde{H}_0(e_i \otimes h((1,3)(3,4) + (1,2)(2,4)) \otimes h((1,3)(3,4)))$$
$$= A_1 + A_2 + \tilde{H}_0(e_{i-1} \otimes h((1,2)(2,4) + (1,3)(3,4)) \otimes h((1,2)(2,4)$$
$$+ (1,3)(3,4))$$

and this completes the proof.

Notice especially that the proof depends only on the properties of F_3, the formal properties of the maps $\tilde{H}_i$, but not on h. We exploit this fact in the remainder of the proof by suppressing h entirely. One other thing which should be pointed out is that the cochains used in χ can be bigraded by using (1) the $\tilde{H}_s$ and (2) the (e_j) involved (in fact we need only use $i - j$), and in terms of the bigrading we can specify

$$\chi = \tilde{H}_2(e_i \otimes C_0^0) + \tilde{H}_1(e_i \otimes C_0^1 + e_{i-1} \otimes C_1^1).$$

Then the remainder of the proof becomes essentially a formal manipulation, assuming certain things about $\delta(C_j^i)$ (and $\sigma^* C_j^i$). We develop these notions further in the next section.

7. Normalizing systems

(1) Let $\Gamma(n)$ be the set of ascending sequences of integers

$$\gamma = (1, \gamma_1, \cdots, \gamma_r, n)$$

with $1 \leq \gamma_1 < \gamma_2 < \cdots < \gamma_r < n$, and for $\gamma \in \Gamma(n)$ set

7.1.1
$$P_\gamma(n) = (\mathcal{C}(n)_{1,1+\gamma_1})^2 \otimes (\mathcal{C}(n)_{\gamma_1+1,\gamma_2+1})^2 \otimes \cdots \otimes (\mathcal{C}(n)_{\gamma_r+1,n+1})^2 \subset [(\mathcal{C}(n)_{1,n+1})^2]^{r+1}$$

(the notation on the right is explained in 1.3). Since $\mathcal{C}(n)_{1,n+1}$ is a graded complex (1.3.1) $P_\gamma(n)$ is also graded in the usual way. There is a second way of grading the $P_\gamma(n)$ which we now explain.

DEFINITION 7.1.2: *$\gamma \in \Gamma(n)$ has length r if*

$$\gamma = (1, \gamma_1, \cdots, \gamma_r, n).$$

The set of $\gamma \in \Gamma(n)$ of length r is denoted $\Gamma_r(n)$.

DEFINITION 7.1.3: $P_{r+1}(n) = \sum_{\gamma \in \Gamma_r(n)} P_\gamma(n)$

$$P(n) = \sum_r P_r(n).$$

We say an element $\lambda \in P_r(n)$ has bi-degree (r, k) if it has dimension k (from the grading of $\mathcal{C}(n)_{1,n+1}$), and we write

$$P(n)(r, k)$$

for the set of all elements of bi-degree (r, k).

The map σ^* is defined in §5 and we have

7.1.4
$$\sigma^*: P(n)(r, k) \to P(n)(r-1, k+4)$$

At the same time $\partial^\otimes$ is defined using the boundary for $\mathcal{C}(n)_{1,n+1}$ given in §1.2, and we have

7.1.5
$$\partial^\otimes: P(n)(r, k) \to P(n)(r, k+1)$$

7.1.6
$$\partial^\otimes \sigma^* = \sigma^* \partial^\otimes$$

7.1.7
$$(\sigma^*)^2 = (\partial^\otimes)^2 = 0$$

Note also that the minimum dimension of an element in $P(n)(r, *)$ is $-2r$.

Moreover for each γ there is only one generator of minimum dimension, namely

$$(1, \gamma_1 + 1)^2 \otimes \cdots \otimes (\gamma_r + 1, n + 1)^2,$$

which is invariant under the interchange operator

$$\mathbf{T} = \underbrace{T \otimes \cdots \otimes T}_{r \text{ times}}$$

(2) For technical reasons it is necessary to enlarge $P(n)$ somewhat. We first define $\overline{\mathcal{C}(n)_{1,n+1}}$ as $\mathcal{C}(n)_{1,n+1}$ together with the -1 dimensional generator $(1, n + 1)$ and we set

$$\partial(1, n + 1) = \textstyle\sum_{j=2}^{n} (1, j)(j, n + 1).$$

Now, if we replace $\mathcal{C}(n)_{1,n+1}$ by $\overline{\mathcal{C}(n)_{1,n+1}}$ in 7.1.1, 7.1.3 we obtain the *closed* bigraded complex $\bar{P}(n)$. Note that $\bar{P}(n)(r, s) = P(n)(r, s)$ for $r > 1$, and $H_*(\bar{P}(n)(1, *)) = 0$ while $H_*(P(n)(1, *))$ has one generator $(\Sigma(1, j)(j, n + 1))^{\otimes 2}$.

On the other hand

$$H_*(P(n)(r, *)) = H_*(\bar{P}(n)(r, *)) = 0$$

for $r > 1$.

(3) DEFINITION 7.3.1: *An open normalizing system N of length n is a system of elements $C_j^i \in P(n)$ so $0 \leq i, j < n - 1$ and*

(1) $C_j^i \in P(n)(n - i, 3i - j - 2n)$

(2) $\partial^{\otimes} C_j^i = \sigma^* C_j^{i-1} + (1 + T) C_{j-1}^i$

(3) $C_0^0 = (1, 2)^{\otimes 2} \otimes (2, 3)^{\otimes 2} \otimes \cdots \otimes (n, n + 1)^{\otimes 2}$.

The associated *closed* normalizing system $\bar{N}$ for $\bar{P}(n)$ is the open system above together with elements in $\bar{P}(n)$

$$C_0^{n-1}, \cdots, C_{n-1}^{n-1}$$

which continue to satisfy 7.3.1 (1), (2) above.

Note that $C_{i+j}^i = 0$ $j > 0$ for dimensional reasons. Similarly C_i^i is invariant under T for each i, and if $C_{n-1}^{n-1} \neq 0$ then it must equal $(1, n + 1) \otimes (1, n + 1)$, that is, if a normalizing system exists at all. Note also that if $\{C_j^i\}$ forms a normalizing system N then so does $\{\mathbf{T}(C_j^i)\}$ as well as $\{\mathbf{T}^j(C_j^i)\}$. Thus a normalizing system is not, in general, unique.

LEMMA 7.3.2: *For each $n \geq 2$ there exist normalizing systems N, $\bar{N}$ of length n.*

Proof: Since $C_j^i \in P(n)(n - i, 3i - j - 2n)$ and $P(n)(n - i, *)$ has trivial homology (since $i < n - 1$) it suffices to show by induction that

$$\partial^{\otimes}(\sigma^* C_j^{i-1} + (1 + \mathbf{T}) C_{j-1}^i) = 0.$$

But $\partial^{\otimes}[\sigma^* C_j^{i-1} + (1 + T)C_{j-1}^{i}] = \sigma^*(\sigma^* C_j^{i-2} + (1 + T)C_{j-1}^{i-1}) + (1+T)[\sigma^*(C_{j-1}^{i-1}) + (1 + T)C_{j-2}^{i}] = (\sigma^*(1+T) + (1+T)\sigma^*)(C_{j-1}^{i-1}) = 0.$

Thus since C_0^0 is in all cases given, we can start the induction and this shows the existence of N. $\bar{N}$ is obtained similarly since $H_*(\bar{P}(n)(1, *)) = 0$. This completes the proof.

(4) Here are two examples of closed normalizing systems. For $n = 2$

$$C_0^0 = ((1,2)\otimes(1,2))\otimes((2,3)\otimes(2,3))$$
$$C_0^1 = (1,3)\otimes(1,2)(2,3)$$
$$C_1^1 = (1,3)\otimes(1,3)$$

For $n = 3$

$$C_0^1 = ((1,3)\otimes(1,2)(2,3))\otimes((3,4)\otimes(3,4)) + ((1,2)\otimes(1,2))\otimes((2,4)\otimes(2,3)(3,4))$$
$$C_1^1 = ((1,3)\otimes(1,3))\otimes((3,4)\otimes(3,4)) + ((1,2)\otimes(1,2))\otimes((2,4)\otimes(2,4))$$
$$C_0^2 = ((1,3)(3,4) + (1,2)(2,4))\otimes(1,3)(3,4)$$
$$C_1^2 = ((1,3)(3,4) + (1,2)(2,4))\otimes(1,4)$$
$$C_2^2 = (1,4)\otimes(1,4)$$

It might be useful at this stage to compare these formulae with those in section 6.

(5) We now give a more explicit method of constructing open and closed normalizing systems.

There is a "shift" isomorphism $S_j: \mathcal{C}(r)_{1,r+1} \to \mathcal{C}(n)_{j+1,j+r+1}$ defined on generators by

$$S_j(a, b) = (a + j, b + j)$$

Lemma 7.5.1: *It is possible to choose the closed normalizing systems $\bar{N}$ so that the associated open systems are given inductively by*

7.5.2 $\quad C_j^{n-t-1}(n) = \sum_{r=1}^{n-1} \sum_{j_1+j_2=j} C_{j_1}^{r-1}(r)\otimes T^{j_1} S_r(C_{j_2}^{n-r-t}(n-r))$

Proof: Bidegrees are right and

$$C_0^0(n) = C_0^0(1)\otimes S_1 C_0^0(1)\otimes \cdots \otimes S_{n-1}C_0^0(1)$$
$$= C_0^0(1)\otimes S_1 C_0^0(n-1)$$

thus it suffices to show (2) of 7.3.1 is satisfied, but this follows by an easy induction. Finally, after obtaining the open normalizing system of length n we extend it in any way we can to obtain the associated closed system, and continue to the definition of the open system of length $n + 1$, etc.

LEMMA 7.5.3: *For any choice of $\bar{N}$ with N satisfying* 7.5.2 *we have*

$$C_{n-1}^{\ n-1} = (1, n+1)\otimes(1, n+1)$$

Proof: For $n = 2, 3$ see 7.4. Now, by induction

$$C_{n-2}^{\ n-2}(n) = (1, n)\otimes(1, n)\otimes(n, n+1)\otimes(n, n+1) + (1, 2)\otimes(1, 2)\otimes(2, n+1)\otimes(2, n+1)$$

and $\sigma C_{n-2}^{\ n-2}(n)$ can only be contained in $\partial^{\otimes}(A)$ or $\partial^{\otimes}(\mathbf{T}A)$ where

$$A = (1, n+1)\otimes((1, 2)(2, n+1) + (1, n)(n, n+1))$$

but only one of A, $\mathbf{T}(A)$ can occur in $C_{n-2}^{\ n-1}$. Thus exactly 1 must occur. However $(1 + \mathbf{T})A \neq 0$, hence $(1 + \mathbf{T})C_{n-2}^{\ n-1}$ is not zero and

$$(1, n+1)\otimes(1, n+1) = C_{n-1}^{\ n-1}.$$

(6) Let $\{C_j^{\ i}\}$, $\{C_j^{\ i\prime}\}$ be normalizing systems N, N' for $P(n)$. Then we can construct a system $\{D_j^{\ i}\}$ of homotopies linking them as follows:

DEFINITION 7.6.1: $\{D_j^{\ i}\}$ *links* N, N' *if*
(1) $D_j^{\ i} \in P(n)(n - i, 3i - j - 2n - 1)$
(2) $\partial D_j^{\ i} = C_j^{\ i} + (C_j^{\ i})' + (1 + \mathbf{T})D_{j-1}^{\ i} + \sigma^* D_j^{\ i-1}$.
Proceeding as in the proof of 7.3.2 we now have

LEMMA 7.6.2: *Given normalizing systems N, N' for $P(n)$ there is a homotopy system $\{D_j^{\ i}\}$ linking them.*
(Note that $D_i^{\ i} = 0$ for dimensional reasons).

8. The proof of Theorem 0

(1) In $W\otimes_{\mathbf{T}}P(n)$ we define chains $G(r, n, i)$ by

8.1.1 $$G(r, r, i) = \textstyle\sum_j e_{i+j}\otimes C_{n-2-j}^{\ r}$$

where $\{C_k^{\ r}\}$ forms an open normalizing system of length n. We have

8.1.2 $$\begin{aligned}\delta G(r, n, i) &= \Sigma e_{i+j-1}\otimes_{\mathbf{T}}(1 + \mathbf{T})C_{n-2-j}^{\ r} + \Sigma e_{i+j}\otimes_{\mathbf{T}}\sigma C_{n-2-j}^{\ r}\\ &= \Sigma e_{i+j}\otimes_{\mathbf{T}}\sigma^* C_{n-2-j}^{\ r-1}\\ &= 1\otimes\sigma^* G(r - 1, n, i).\end{aligned}$$

Thus if we let

$$Q(n, i) = \Sigma\tilde{H}_{n-1-r}\{G(r, n, i)\}$$

we find

$$\delta Q(n, i) = \tilde{H}_0[1\otimes\sigma^*(G(n-2, n, i))] + \textstyle\sum_j A_j \smile B_j(n, i)$$

where A_j, B_j are cochains which we will examine in 8.2. Now we have

LEMMA 8.1.4: *$H_0[1\otimes\sigma^*(G(n - 2, n, i))]$ represents a* "formal" *element in*

$$\mathrm{Sq}_i\ \langle(1, 2), (2, 3), \cdots, (n, n+1)\rangle$$

in particular $\{\sigma^*[G(n - 2, n, i)]\}$ *does not depend on the choice of normalizing system.*

Proof: Let N, N' be normalizing systems for $P(n)$ and let $\{D_j^i\}$ link them. Set

$$D = \Sigma e_{i+j} \otimes \sigma^* D_{n-2-j}{}^{n-2}$$

then

$$\delta D = \sigma^*(G(n - 2, n, i)) + \sigma^*(G'(n - 2, n, i))$$

and we have the second part of the lemma. To prove the first part we need

LEMMA 8.1.5: *Let A be a locally finite complex over Z_2, then*

$$H_*(W \otimes_T A \otimes A) \cong H_*(W \otimes_T H(A) \otimes H(A)).$$

The proof follows directly from Lemma 5.2 p. 204 of [11]; see for example the proof of 3.3 p. 237 of [8].

Hence (from 7.2) it follows that

$$\tilde{H}_i(W \otimes_T P(n)) = H_i(W \otimes_T Z_2) = Z_2$$

where the generators can be represented by the explicit chains

8.1.6 $$e_i \otimes_T [\Sigma(1, j)(j, n + 1)] \otimes [\Sigma(1, j)(j, n + 1)].$$

Now observe that the normalizing system given in 7.5.1 satisfies

$$\begin{aligned}\alpha &= 1 \otimes \sigma^*(e_i \otimes C_{n-2}{}^{n-2}) \\ &= e_i \otimes_T [(1, 2)(2, n + 1) \otimes (1, 2)(2, n + 1) \\ &\qquad + (1, n)(n, n + 1) \otimes (1, n)(n, n + 1)]\end{aligned}$$

and there is no element in $W \otimes_T P(n)$ which contains α in its boundary. Thus $\sigma^*(G(n - 2, n, i))$ must represent the nontrivial generator 8.1.6 and the lemma follows.

(2) We now examine the sum $\Sigma A_i \smile B_i$ in the right hand side 8.1.3. We now assume N satisfies 7.5.2, and indeed all the $N(m)$ are built up according to the proof of 7.5.1.

Here are the specific elements A_i B_i:

$$\textstyle\sum_{s,t,\text{etc.}} \tilde{H}_{n-1-r-s}(\Sigma e_{i+j-t} \otimes_T C_{j_1}{}^{r-t}) \smile \tilde{H}_{s-1}(T^{s+j-l} \Sigma e_t \otimes_T S_{j_1}(C_{j-j_1}{}^t)).$$

We can rewrite this as

$$\textstyle\sum_{\gamma,k,R,r} \tilde{H}_{k-1}[G(\gamma - k, \gamma, R - \gamma + 2)] \smile \tilde{H}_{n-r-k-1}[T^R S_\gamma(G(r + k - \gamma, n - \gamma, \gamma - R + i))].$$

Now setting $\lambda = r + k$, fixing γ, R, λ and summing over k we have

$$\begin{aligned} &\textstyle\sum_{\lambda,R,\gamma} \bar{Q}(\gamma, R - \gamma + 2)H_{n-\lambda-1}[T^R S_\gamma(G(\lambda - \gamma n, - \gamma, \gamma - R + i))] \\ 8.2.1 \qquad &= \textstyle\sum_{R,\gamma} \bar{Q}(\gamma, R - \gamma + 2) \smile \mathbf{T}^R\bar{Q}(n - \gamma, \gamma - R + i) \\ &= \textstyle\sum_{S,K}\bar{Q}(S, K) \smile \mathbf{T}^{K-S}\bar{Q}(n - S, i - K - 2). \end{aligned}$$

Here $\bar{Q}(S, K) = Q(S, K) + H_0[C(S - 1, S - 1)]$ is the chain obtained from the closed normalizing system, and

$$\delta\bar{Q}(S, K) = \Sigma A_j \smile B_j(S, K).$$

Hence 8.2.1 represents exactly the desired matric Massey product and the proof of Theorem 0 is now complete.

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