CONSTRUCTION OF SOME NONSINGULAR BILINEAR MAPS*

By Kee Yuen Lam**

1. Introduction

Let $\mathcal{E}$ be a set of operators on a vector space V. $\mathcal{E}$ "has property (P)" if each nontrivial linear combination of operators in $\mathcal{E}$ is a monomorphism. In [2], Adams, Lax and Phillips were concerned with property (P) when V is finite dimensional. In this paper,we give a nontrivial example of such $\mathcal{E}$ on an infinite dimensional V. This example enables us to obtain certain new nonsingular bilinear maps (compare [4]), thus providing estimates on the geometrical dimension of vector bundles over real projective spaces RP^n. Specifying these estimates to the immersion problem for RP^n, we re-obtain the result of Milgram [5, Theorem 1], plus the immersion of RP^n into $R^{2n-\alpha(n)}$ when $n \equiv 0 \pmod 8$. In fact, our example $\mathcal{E}$ arises from an attempt to recast Milgram's work in a more systematic setting.

2. Notations

F denotes the space of real, complex, quaternion or Cayley numbers, with the basic units $\epsilon_0(=1)$, $\epsilon_1, \cdots, \epsilon_{d-1}$ (d = real dimension of F) as standard orthonormal basis. F^m is the vector space (over R) of m-tuples of elements of F. $F^\infty = \bigcup_m F^m$, with usual inner product and norm. If c, c' are non-negative integers satisfying $c \leq c'$, $F(c, c']$ denotes the orthogonal complement of F^c in $F^{c'}$. Operators on F^∞ will be written on the right of their arguments.

We shall have occasion to use the following arithmetic functions:

$[t]$ = the greatest integer not exceeding the real number t.

$\{t\} = t - [t]$.

$\mathrm{sgn}(a) = 1$ if $a \geq 0$; $\mathrm{sgn}(a) = -1$ if $a < 0$.

$v(b)$ = the greatest integer h such that 2^h divides b. By convention, $v(0) = \infty$.

$\alpha(n)$ = the number of 1's in the dyadic expansion of n.

(There is a relation between v and α, namely, $\sum_{1 \leq b \leq n} v(b) = n - \alpha(n)$. This can be easily proved.)

Finally, for each pair of positive integers k, $h (k > h)$, define a non-negative number $\tau(k, h)$ as follows: let $k = \sum_{j=0}^\infty \alpha_j 2^j$, $h = \sum_{j=0}^\infty \beta_j 2^j$ and $k - h = \sum_{j=0}^\infty \gamma_j 2^j$ be dyadic expansions. Then

$$\tau(k, h) = \mathrm{Card}\{j \geq 0 \mid \gamma_j = 0, \alpha_j \neq \beta_j\}.$$

Here, Card denotes cardinality. The first few values of $\tau(k, h)$ are given by

* This paper is a revision of a portion of the author's Ph.D. thesis submitted to Princeton University. The author thanks Professor Norman E. Steenrod for his advice and encouragement.

** Supported by the National Science Foundation, Grant No. GP-6724.

$\tau(3, 1) = \tau(3, 2) = 0$, $\tau(2, 1) = \tau(4, 1) = \tau(4, 2) = 1$, and $\tau(4, 3) = 2$. Note, in particular, that if $k + h + 1$ is a power of 2, then $\tau(k, h) = \alpha(h)$.

3. The operators S_b and T_a on F^∞.

(3.1) For each nonnegative integer b, S_b is defined as the monomorphism

$$(x_1, x_2, x_3, \cdots)S_b = (\underbrace{0, \cdots, 0}_{b}, x_1, x_2, x_3, \cdots).$$

In particular, S_0 is the identity operator I.

(3.2) For $-d < a < 0$ the operator T_a is defined simply by

$$(x_1, x_2, x_3, \cdots)T_a = (x_1\epsilon_{-a}, x_2\epsilon_{-a}, x_3\epsilon_{-a}, \cdots).$$

For $a \geq 0$, first put $m = 2^a$ and decompose F^∞ into the direct sum $\oplus_{k\geq 0}F(km, (k+1)m]$. Then define T_a to be the operator which maps each summand $F(km, (k+1)m]$ to itself in the following way:

$$(\underbrace{0, \cdots, 0}_{km}, x_{km+1}, \cdots, x_{km+m}, 0, \cdots)T_a$$

$$= (-1)^k(\underbrace{0, \cdots, 0}_{km}, x_{km+m}, \cdots, x_{km+1}, 0, \cdots).$$

Obviously, each $T_a (a > -d)$ is an isometry, $T_a^2 = \operatorname{sgn}(a)I$, and the adjoint of T_a is $T_a^* = \operatorname{sgn}(a)T_a$.

(3.3) In addition to the operators S_b and T_a let P_n be the projection of F^∞ to F^n. These operators in general do not commute, but we do have the following special commutation rules:

(i) $S_{b'}S_b = S_{b'+b} (= S_bS_{b'})$.

(ii) $T_aS_b = \begin{cases} S_bT_a & \text{if } v(b) > a \\ -S_bT_a & \text{if } v(b) = a \end{cases}$

(iii) $T_aT_{a'} = -\operatorname{sgn}(a)\operatorname{sgn}(a')T_{a'}T_a$ for $a \neq a'$

(iv) $T_aP_n = P_nT_a$ if $v(n) \geq a$.

All these rules can be verified from definitions and known properties of F. We also record

(3.4)

LEMMA. *Let $a \geq 0$ and $m = 2^a$. Then for each $b \geq 0$, the operator $S_bT_aP_m$ is self-adjoint.*

Proof: If $b \geq m$, $S_bT_aP_m = S_bP_mT_a = 0$ is self-adjoint. If $b < m$ let $x = (x_1, x_2, \cdots)$, $y = (y_1, y_2, \cdots)$ be arbitrary vectors in F^∞. By direct computation,

$$\langle xS_bT_aP_m, y\rangle = \langle x_1, y_{m-b}\rangle + \cdots + \langle x_{m-b}, y_1\rangle$$

$$= \langle x, yS_bT_aP_m\rangle,$$

which proves the lemma.

4. The Main Theorem.

(4.1)

THEOREM. *The set* $\mathcal{E}$ *of operators on* F^∞ *defined by*

$$\mathcal{E} = \{T_a S_b \mid -d < a < v(b)\}$$

has property (P).

Remark. $\mathcal{E}$ consists precisely of those products $T_a S_b$ for which the commutation rule $T_a S_b = S_b T_a$ holds, (see (ii) of (3.3)).

Proof. For each $r \geq 0$ and *nonzero* $x \in F^\infty$, let $\mathcal{E}^r = \{T_a S_b \mid -d < a < \min, (v(b), r + 1)\}$ and $\mathcal{E}^r(x) = \{xA \mid A \in \mathcal{E}^r\}$. Also write $n = 2^r$ and $V_k^r = F(kn\ (k+1)n]$. Corresponding to the direct sum decomposition

$$F^\infty = V_0^r \oplus V_1^r \oplus V_2^r \oplus V_3^r \oplus \cdots$$

partition $\mathcal{E}^r(x)$ into the disjoint union

$$\mathcal{E}^r(x) = E_0^r \cup E_1^r \cup E_2^r \cup E_3^r \cup \cdots$$

in which $E_k^r = \mathcal{E}^r(x) \cap [(F^{kn})^\perp - (F^{(k+1)n})^\perp]$. The following facts are clear from definition:

(1_r) If $j > k$, each vector in E_j^r projects to zero in V_k^r.

(2_r) Vectors in E_k^r never project to zero in V_k^r.

Now, $\mathcal{E}$ has property (P) iff each set of vectors $\mathcal{E}^r(x)$ is linearly independent. This is a consequence of (1_r) and the following assertion which strengthens (2_r):

(3_r) The projection $\bar{E}_k^r$ of E_k^r in V_k^r is a linearly independent set.

We shall prove (3_r) by induction on r.

Represent x as yS_p where y has nonzero first component y_1. First consider $r = 0$. One checks directly that $\bar{E}_k^0$ is empty if $k < p$, and that if $k \geq p$, $\bar{E}_k^0$ consists of the vectors

$$(\underbrace{0, \cdots, 0}_{k}, y_1 \epsilon_{-a}, 0, \cdots), \ -d < a < 0,$$

for $k - p$ odd, and consists of the additional vector

$$(\underbrace{0, \cdots, 0}_{k}, (-1)^p y_1, 0, \cdots)$$

for $k - p$ even. In any case $\bar{E}_k^0$ is linearly independent.

Suppose inductively that (3_{r-1}) is true. To prove (3_r), let $'E_k^r$ be the set of vectors in E_k^r not belonging to $\mathcal{E}^{r-1}(x)$. Since

$$V_k^r = V_{2k}^{r-1} \oplus V_{2k+1}^{r-1},$$

$$E_k^r = E_{2k}^{r-1} \cup E_{2k+1}^{r-1} \cup {}'E_k^r,$$

the linear independence of $\bar{E}_k^r$ will follow from (2_r), (1_{r-1}) and (3_{r-1}) via some elementary argument in linear algebra once we established the following key lemma:

(4.2)

LEMMA. *Suppose z, w are distinct vectors in E_k^r such that one of them (say z) belongs to $'E_k^r$. Then their projections in V_k^r are mutually orthogonal.*

Proof. Write $z = xT_rS_c$, $w = xT_aS_b$, $(a \leq r)$, with $x = yS_p$ as before. By definition of E_k^r, z and w lie in $(F^{kn})^\perp$ but not in $(F^{(k+1)n})^\perp$. These being invariant subspaces of T_r and T_a, the same must be true for the vectors $zT_r = yS_{p+c}$ and $wT_a = \text{sgn}(a)yS_{p+b}$. From this it follows that $z, w \in E_k^r$ only if the following conditions hold among a, b and c:

$$(4.3)\qquad \begin{aligned} &c \geq 0, \quad v(c) > r, \quad kn < p + c + 1 \leq (k+1)n; \\ &b \geqq 0, \quad v(b) > a, \quad kn < p + b + 1 \leq (k+1)n. \end{aligned}$$

In particular, $z \neq w$ implies $a < r$.

Now extend the definition of S_b for *negative* b by

$$(x_1, x_2, x_3, \cdots)S_b = (x_{1-b}, x_{2-b}, x_{3-b}, \cdots).$$

These operators still satisfy (ii) of (3.3), and the formula $S_{b'}S_b = S_{b'+b}$ remains valid as long as $b' \geq 0$. Moreover, S_b and S_{-b} are adjoint to each other, and the projection of F^∞ to V_k^r is given by $S_{-kn}P_nS_{kn}$.

Projecting z to V_k^r gives the vector

$$\bar{z} = (yS_pT_rS_c)S_{-kn}P_nS_{kn} = (-1)^k yS_{p+c-kn}T_rP_nS_{kn}.$$

Similarly, $\bar{w} = yS_{p+b-kn}T_aP_nS_{kn}$. Since S_{kn} preserves inner product, and P_n is self-adjoint idempotent,

$$\begin{aligned} \langle \bar{z}, \bar{w} \rangle &= (-1)^k \langle yS_{p+c-kn}T_rP_n, yS_{p+b-kn}T_aP_n \rangle \\ &= (-1)^k \langle yA, yB \rangle, \end{aligned}$$

where $A = S_{p+c-kn}T_rP_n$ is self-adjoint by (3.4) and $B = S_{p+b-kn}T_a$. Using (4.3) and the various commutation rules listed in (3.3), we have

$$\begin{aligned} BA^* = BA &= S_{p+b-kn}T_aS_{p+c-kn}T_rP_n \\ &= S_{p+b+c-kn}T_aS_{p-kn}T_rP_n \\ &= S_{p+c-kn}T_aS_{p+b-kn}T_rP_n \\ &= S_{p+c-kn}T_aP_nT_rS_{-(p+b-kn)} \qquad \text{by (3.4)} \\ &= -\text{sgn}(a)\, S_{p+c-kn}P_nT_rT_aS_{-(p+b-kn)} \\ &= -AB^*, \end{aligned}$$

so $\langle \bar{z}, \bar{w} \rangle = 0$ and lemma (4.2) is established. This ends the proof of Theorem (4.1).

Remark. Let F^∞ be completed to the Hilbert space $\mathcal{H}$. All the operators in $\mathcal{E}$ extend to $\mathcal{H}$, and our proof actually shows that $\mathcal{E}$, as a set of operators on $\mathcal{H}$, has property (P).

5. Nonsingular bilinear maps.

(5.1) LEMMA. *Suppose $h < k$ and let $\mathcal{E}(h+1, k)$ be the set of operators in $\mathcal{E}$ which map F^{h+1} into F^k. Then*

$$\text{Card } \mathcal{E}(h+1, k) = d(k-h) + \tau(k, h),$$

where $\tau(k, h)$ is as defined in §2.

Proof. $\mathcal{E}(h+1, k)$ obviously contains the operator $T_a S_b$ if $-d < a < 0$ and $h + 1 + b \leq k$. There are $(d-1)(k-h)$ such operators. The remaining operators have the form $T_a S_b$ $(= S_b T_a)$, where $0 \leq a < v(b)$ and a, b satisfy the additional restriction that T_a must map $F(b, h+1+b]$ into F^k. This restriction amounts to $g \leq k$, where $g = 2^a([h/2^a]+1) + b$ is the smallest multiple of 2^a not exceeded by $h + 1 + b$. If we set $b = 2^{a+1}(s-1)$, the restriction can be written as

$$s \leq \frac{k-h}{2^{a+1}} + \frac{1}{2}\left\{\frac{h}{2^a}\right\} + \frac{1}{2}. \tag{5.2}$$

It follows that the total number of remaining operators is

$$\begin{aligned}\text{Card } \{T_a S_b \mid a, b \geq 0; b = 2^{a+1}(s-1) \quad &\text{for some integer } s \text{ satisfying } (5.2)\} \\ &= \textstyle\sum_{a \geq 0} \sigma_a(k, h),\end{aligned}$$

where $\sigma_a(k, h)$ denotes the integral part of the right hand side of (5.2). If $k = \sum_{j \geq 0} \alpha_j 2^j$, $h = \sum_{j \geq 0} \beta_j 2^j$ and $k - h = \sum_{j \geq 0} \gamma_j 2^j$ are dyadic expansions, then by direct inspection

$$\sigma_a(k, h) = \gamma_a + \textstyle\sum_{j > a} \gamma_j \alpha^{j-a-1}, \tag{5.3}$$

unless when we run into the situation that

$$\gamma_a = 0 \quad \text{and} \quad \textstyle\sum_{0 \leq j < a} (\beta_j + \gamma_j) 2^j \geq 2^a, \tag{5.4}$$

in which case $\sigma_a(k, h)$ is 1 plus the value given in (5.3). But the inequality in (5.4) holds iff there is a carry at the a-th digital place when $k - h$ is added to h in dyadic arithmetic. Thus (5.4) is equivalent to the condition that $\gamma_a = 0$ and $\alpha_a \neq \beta_a$, which by definition occurs $\tau(k, h)$ times as a runs from 0 to ∞. Consequently

$$\text{Card } \mathcal{E}(h+1, k) = (d-1)(k-h) + \textstyle\sum_{a \geq 0} (\gamma_a + \sum_{j > a} \gamma_j 2^{j-a-1}) + \tau(k, h).$$

One can now verify easily that the summation term on the right hand side actually yields the value $k - h$, so that Card $\mathcal{E}(h+1, k) = d(k-h) + \tau(k, h)$, as is to be proved.

(5.5) THEOREM. *For any* $k > h \geq 0$ *there exists a nonsingular bilinear map*

$$R^{d(h+1)} \times R^{d(k-h)+\tau(k,h)} \to R^{dk}$$

for $d = 1, 2, 4$ *or* 8.

Proof. Let $A_1, A_2, A_3, \cdots$ be an enumeration of the operators in $\mathcal{E}$. By Theorem 4.1, the R-bilinear map $\phi: F^{\infty} \times R^{\infty} \to F^{\infty}$ given by $\phi(x, e_i) = xA_i$ is nonsingular, i.e. $\phi(x, u) = 0$ iff $x = 0$ or $u = 0$. (Here $\{e_i\}_{i \geq 1}$ is the standard basis of R^{∞}.) By the previous lemma, there is a subspace V in R^{∞} of dimension $d(k - h) + \tau(k, h)$ such that ϕ restricts to a map $F^{h+1} \times V \to F^k$. The theorem follows.

As an example, putting $k = 5$, $h = 2$ and $d = 8$ in (5.5), we get a nonsingular bilinear map $R^{24} \times R^{25} \to R^{40}$. This is better than the map $h: R^{24} \times R^{24} \to R^{40}$ obtained by regarding R^{24} and R^{40} as the spaces of 3-tuples and 5-tuples of Caley numbers respectively, and using the formula

$$h((x_0, x_1, x_2), (y_0, y_1, y_2)) = (z_0, z_1, z_2, z_3, z_4)$$

with $z_k = \sum_{i+j=k} x_i y_j$.

6. Topological applications

Let $k\xi_n$ denote the k-fold whitney sum of the Hopf bundle ξ_n over RP^n, and $gd(k\xi_n)$ its geometrical dimension. It is known [4, Proposition 3] that the existence of a nonsingular bilinear map $R^{n+1} \times R^r \to R^k$ implies $gd(k\xi_n) \leq k - r$. Applying (5.5), one may conclude

(6.1) PROPOSITION. *If* $n = dh + d - 1$, *then* $gd\,(dk\xi_n) \leq dh - \tau(k, h)$, where $d = 1, 2, 4$ or 8.

In some special cases, the estimate on $gd(k\xi_n)$ obtained in this proposition can be combined with some results of Gitler [3, Theorem 2.1] to yield the exact value of the geometrical dimension in question.

Let 2^N be a sufficiently large power of 2. By Adams [1, Theorem 7.4], $2^N\xi_n$ is a trivial bundle, so $\nu(RP^n) = (2^N - n - 1)\xi_n$ is a stable normal bundle for RP^n. A theorem of Hirsch (see, for example, Sanderson [6, Theorem 2.1]) asserts that RP^n immerses in $R^{n+\ell}(\ell > 0)$ if and only if $\ell \geq gd(\nu(P^n))$. Our estimates on $gd(k\xi_n)$ now give

(6.2) THEOREM. *Let* $d = 1, 2, 4$ *or* 8. *If* $n \not\equiv 1, 3, 7$ *and* $n + 1 \equiv 0 \pmod{d}$, *then* RP^n *immerses in* $R^{2n-\alpha(n)-\beta(d)}$, *where* $\beta(d) = (d - 1) - \alpha(d - 1)$.

Proof. Write $n + 1 = d(h + 1)$ and put $k = (2^N/d) - h - 1$ in (6.1) to obtain

$$\begin{aligned} gd(\nu(RP^n)) &\leq dh - \tau(k, h) \\ &= dh - \alpha(h) \\ &= n - \alpha(n) - [(d - 1) - \alpha(d - 1)]. \end{aligned}$$

All the claimed immersions follow.

7. Relation with the construction of Milgram [5].

In terms of the operators S_b and T_a on F^∞ it is possible to restate the main result of [5, §2 and Theorem 6] essentially as the following

(7.1) THEOREM (Milgram). *Let* $n = 2^r$. *Let* $A_1, A_2, \cdots, A_t$ *be an enumeration of the set*

$$\mathfrak{M} = \{T_r T_a S_{b-n} P_n \mid -d < a < v(b); 1 \le b < 2n\},$$

considered as a set of operators from F^n *to itself. Then*

(i) $\langle w, wA_j\rangle = 0$ *for each* $w \in F^n$;

(ii) *If* $w \in F^n$ *is non-zero, there are always* $dn - 1$ *independent vectors among* $\{wA_j\}_{1 \le j \le t}$;

(iii) $t = \text{Card}\, \mathfrak{M} = 2\,dn - r - (d + 1)$.

We shall sketch a proof of this theorem which avoids an unpleasant double induction used in [5]. Let $A_j = T_r T_a S_{b-n} P_n$. If $wA_j = 0$, (i) is trivially true. Otherwise let $wT_r = x \in F^n$. In the notation of (4.1), the vectors xT_{r+1}, $xT_a S_b$ belong to $'E_0^{r+1}$ and E_0^{r+1} respectively, and so project to mutually orthogonal vectors in V_0^{r+1} $(= F^{2n})$. Since xT_{r+1} has zeros at its first n components, orthogonality is not destroyed by further projecting into $F(n, 2n]$, resulting (as can be easily checked) in the image vectors $xT_r S_n$ and $xT_a S_{b-n} P_n S_n$ respectively. Since S_n preserves inner product, $\langle w, wA_j\rangle = 0$ follows.

To establish (ii) notice that if we put $wT_r = x$ as before, the set $\{wA_j S_n\}_{1 \le j \le t}$ contains as a subset the $\bar{E}_1^r$ of (4.1). Since $\bar{E}_1^r$ is linearly independent by assertion (3_r) in (4.1), it suffices to show that Card $\bar{E}_1^r = dn - 1$. This is done by a counting process, and we omit the details. Likewise, (iii) is obtained by direct counting.

Finally, Milgram observed that if F is not the Cayley numbers, each A_j is F-linear when F^n is considered as a left vector space over F. This leads to immersion results for complex and quaternionic projective spaces, as given in [5, Theorem 2].

REFERENCES

[1] J. F. ADAMS. *Vector Fields on Spheres*, Ann. of Math., **75** (1962), 603–32.

[2] J. F. ADAMS, P. LAX and R. PHILLIPS. *On Matrices whose Real Linear Combinations are Nonsingular*, Proc. Amer. Math. Soc. **16** (1965) 318-22.

[3] S. GITLER. *Projective Sitefel Manifolds—II. Applications*, Topology **7**, (1968) 47–53.

[4] K. Y. LAM. *Construction of Nonsingular Bilinear Maps*, Topology **6** (1967) 423–26.

[5] J. MILGRAM. *Immersing Projective Spaces*, Ann. of Math. **85** (1967), 473–82.

[6] B. J. SANDERSON, *Immersions and Embeddings of Projective Spaces*, Proc. Lond. Math. Soc. (3), **14** (1964), 135–53.

BROWN UNIVERSITY and
CENTRO DE INVESTIGACIÓN DEL IPN.