# THE MOD $p$ HOMOTOPY TYPE OF $BSO$ AND $F/PL$

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## 1. Introduction

Let $p$ be an odd prime. In this paper we give some characterizations of the mod $p$ homotopy type of $BSO$. As one consequence, we show that $BSO$ is decomposable into a product of $\frac{p-1}{2}$ indecomposable spaces (mod $p$). As another consequence, we show that $F/PL$ is of the same mod $p$ homotopy type as $BSO$. This was proved by Sullivan [9] by different methods in his work on the Hauptvermutung, but we believe our proof gives a different insight into the problem. We wish to thank D. Sullivan for many helpful discussions during the period when these theorems were proved.

## 2. Statements of Results

Let $p$ be an odd prime. Let $C_p$ be the class of groups of finite order prime to $p$. A map, $f: X \to Y$, in the category of simply connected $CW$-complexes, is called a mod $p$ homotopy equivalence if $f_{\#}: \pi_*(X) \to \pi_*(Y)$ is a $C_p$-isomorphism, or, equivalently, if $f^*: H^*(Y) \to H^*(X)$ is an isomorphism.[1] The equivalence relation generated by this notion is called "being of the same mod $p$ homotopy type" and is written $X \sim_p Y$.

We now give two theorems, each of which characterizes the mod $p$ homotopy type of $BSO$.

Theorem 2.1. *Let $X$ be a space such that*

1) $$\pi_i(X) = \begin{Bmatrix} 0 & i \not\equiv 0(4) \\ Z & i \equiv 0(4) \end{Bmatrix} \bmod C_p$$

*and*

2) $H^{4t+1}(X; Z) \in C_p,\ t \geq 1.$ *Then* $X \sim_p BSO$.

Let $X^{(i)}$ be the $i^{\text{th}}$ part of the Postnikov septem for $X$. That is, $\exists\, p^{(i)}: X \to X^{(i)}$ such that $p_{\#}^{(i)}: \pi_j(X) \to \pi_j(X^{(i)})$ is an isomorphism if $j \leq i$ and $\pi_j(X^{(i)}) = 0$, $j > i$. Let $X_{(i)} = \text{fibre } (X \to X^{(i-1)})$. Our second characterization involves the $k$-invariants of $X_{(i)}$.

Theorem 2.2. *Let $X$ be a space such that*

1) $$\pi_i(X) = \begin{Bmatrix} 0 & i \not\equiv 0(4) \\ Z & i \equiv 0(4) \end{Bmatrix} \bmod C_p$$

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[1] All cohomology groups have $Z_p$ coefficients unless otherwise stated.

*and*

2) *the first possible p-torsion k-invariant of* $X_{(4t)}$, *namely that in* $H_{(4t)}{}^{4t+2p-1}(X^{(4t+2p-6)}; Z) \approx Z_p \pmod{C_p}$ *is non-zero,* $t \geq 1$.
*Then* $X \sim_p BSO$.

A space $X$ is called decomposable mod $p$ if $X \sim_p X_1 \times X_2$ with $X_1 \not\sim_p pt$. Our next theorem shows that $BSO$ is decomposable mod $p$ if $p > 3$ and gives the decomposition into indecomposable factors.

THEOREM 2.3.[2] *There exists an indecomposable* (mod $p$) *space* $Y_p$ *such that*

$$BSO \sim_p \prod_{i=0}^{(p-1)/2-1} \Omega^{4i} Y_p$$

*and*

$$BU \sim_p \prod_{i=0}^{p-2} \Omega^{2i} Y_p .$$

*Furthermore, these* mod $p$ *homotopy equivalences preserve the* mod $p$ *H-space structure.*

$$\text{Recall that } \pi_i(F/PL) = \begin{cases} 0 & i \equiv 1(2) \\ Z_2 & i \equiv 2(4) \\ Z & i \equiv 0(4). \end{cases}$$

Sullivan [9] has shown that $F/PL$ is of the same mod 2 homotopy type as a product of Eilenberg-MacLane spaces except for a non-zero $k^5(F/PL) \in H^5(Z_2, 2; Z) = Z_4$ which is twice the generator. He has also shown that $F/PL \sim_p BSO$ if $p$ is odd. Clearly $F/PL$ satisfies hypothesis 1) of theorem 2.2. In section 5, we show it also satisfies hypothesis 2) and obtain the following corollary.

COROLLARY 2.4. *If $p$ is an odd prime, then* $F/PL \sim_p BSO$. *Furthermore, the* mod $p$ *homotopy equivalence can be chosen to be an H-map.*

## 3. Proofs of theorems 2.1 and 2.2.

Let $k^{4t+1}(BSO) \in H^{4k+1}(BSO^{(4t-1)}; Z)$ be the $k$-invariant of $BSO$. It is well-known that the odd primary part of this group is cyclic and that some power of 2 times $k^{4t+1}(BSO)$ generates the odd primary part. Form a new space $B_p$ with

$$\pi_i(B_p) = \begin{cases} 0 & i \not\equiv 0(4) \\ Z & i \equiv 0(4) \end{cases}$$

and $k^{4t+1}(B_p)$ the generator of the $p$-primary part of $H^{4t+1}(B_p{}^{(4t-1)}; Z)$. Let $X$ satisfy the conditions of theorem 2.1, we will show that there exists an $f: X \to B_p$ such that $f^*$ is an isomorphism on $H^*(\quad)$.

Consider the following diagram.

[2] This theorem was proved independently by J. F. Adams and D. W. Anderson (see [1]).

$$\begin{array}{ccccc}
 & & B_p & & \\
 & & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
K(Z,4t) & \rightarrow & B_p{}^{(4t)} & & \\
 & & \downarrow & & \\
 & & B_p{}^{(4t-4)} & \rightarrow & K(Z, 4t+1) \\
 & \nearrow & \downarrow & & \\
 & & \vdots & & \\
 & & \downarrow & & \\
X & \xrightarrow{f_1} & K(Z, 4) & &
\end{array}$$

Find $f_1 : X \rightarrow K(Z, 4)$ such that $f_1{}^* : H^4(Z, 4) \rightarrow H^4(X)$ is an isomorphism. Assume we have lifted $f_1$ to $f_{t-1} : X \rightarrow B_p{}^{(4t-4)}$. The obstruction to lifting $f_{t-1}$ to $f_t$ is $f_{t-1}{}^*(k^{4t+1}(B_p)) \in H^{4t+1}(K; Z)$. Since $H^{4t+1}(X; Z) \in C_p$ and $k^{4t+1}(B_p)$ is of order a power of $p$, this obstruction is 0 and we may lift $f_1$ to $f = f_\infty : X \rightarrow B_p$. We now wish to show $f^{(4t)} : X^{(4t)} \rightarrow B_p{}^{(4t)}$ is an isomorphism on $H^*(\quad)$ by induction on $t$. Consider the following diagram.

$$\begin{array}{ccccc}
K(Z, 4t) & \xrightarrow{g} & K(Z, 4t) & & \\
\downarrow i & & \downarrow & & \\
X^{(4t)} & \xrightarrow{f^{(4t)}} & B_p{}^{(4t)} & & \\
\downarrow & & \downarrow & & \\
X^{(4t-4)} & \xrightarrow{f^{(4t-4)}} & B_p{}^{(4t-4)} & \rightarrow & K(Z, 4t+1)
\end{array}$$

We assume $f^{(4t-4)*}$ is an isomorphism on $H^*(\quad)$. $H^{4t+1}(B_p{}^{(4t-4)}; Z) \approx Z_{p\varnothing(t)}$ (mod $C_p$) with a generator $x = k^{4t+1}(B_p)$. $f^{(4t-4)*}$ is a $C_p$-isomorphism on $H^*(\quad; Z)$, hence $H^{4t+1}(X^{(4t-4)}; Z) \approx Z_{p\varnothing(t)}$, the isomorphism being given by $f^{(4t-4)*}$. Let $sx = k^{4t+1}(X)$. Since $H^{4t}(X; Z) \in C_p$, $s \not\equiv 0(p)$. Let $\iota \in H^{4t}(Z, 4t; Z)$. Then $g^*(\iota) = a\iota$. By naturality with transgression, we have $\tau g^*(\iota) = asx = f^{(4t-4)*}\tau(\iota) = x$. Hence, if $x \neq 0$, $a \not\equiv 0(p)$ and $g^*$ is an isomorphism on $H^*(\quad)$. Thus $f^{(4t)*}$ is an isomorphism on $H^*(\quad)$ and the induction is complete. If however $x = 0$, i.e. $Z_{p\varnothing(t)} = 0$, then $B_p{}^{(4t)} \sim_p B_p{}^{(4t-4)} \times K(Z, 4t)$ and $X^{(4t)} \sim_p X^{(4t-4)} \times K(Z, 4t)$. Let $y \in H^{4t}(X^{(4t)}; Z)$ be such that $i^*(y) = r\iota$, $r \not\equiv 0(p)$. If $g^*(\iota) \equiv 0(p)$, change $f^{(4t)}$ by a map $X^{(4t)} \rightarrow K(Z, 4t)$ realizing $y$. Then the new $f^{(4t)}$ can be extended to a new $f_\infty$ by the above argument and the new $g$ is such that $g^*(\iota) \not\equiv 0(p)$. This again completes the induction step.

Since $BSO$ satisfies the conditions of theorem 2.1, we have $f$: $BSO \rightarrow B_p$. Hence $X \sim_p BSO$, and the proof of theorem 2.1 is complete.

We now turn to the proof of theorem 2.2. Let $X$ satisfy the conditions of theorem 2.2.

$$X^{(2p-2)} \sim_p K(Z, 4) \times K(Z, 8) \times \cdots \times K(Z, 2p-2) \sim_p B_p{}^{(2p-2)}$$

by dimensional reasons. Consider the following diagram.

$$\begin{array}{ccc} & & B_p \\ & & \downarrow \\ X & & \vdots \\ \downarrow & \searrow f_{t-1} & \downarrow \\ \vdots & & B_p^{(4t)} \\ \downarrow & & \downarrow \\ X^{(4t-4)} & \rightarrow & B_p^{(4t-4)} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ X^{(2p-2)} & \rightarrow & B_p^{(2p-2)} \\ \downarrow & & \downarrow \\ \vdots & & \vdots \\ \downarrow & & \downarrow \\ K(Z,4) & \rightarrow & K(Z,4) \end{array}$$

Let $f_1:X \rightarrow K(Z, 4)$ be as before. By the above comments there exists an extension $f_{(p-1)/2}:X \rightarrow B_p^{(2p-2)}$ such that $f_{(p-1)/2}^{(2p-2)*}$ is an isomorphism on $H^*(\ \ )$. Inductively, assume that there exists an extension $f_{t-1}:X \rightarrow B_p^{(4t-4)}$ such that $f_{t-1}^{(4t-4)*}$ is an isomorphism on $H^*(\ \ )$, and show that $f_t$ exists. Consider $X_{(4t-2p+2)}^{(4t-4)} \xrightarrow{j} X^{(4t-4)}$. $j^*(k^{4t+1}(X)) \neq 0 \in Z_p$ by hypothesis. Hence $k^{4t+1}(X) = sx \in Z_{p\varnothing(t)}$, where $x$ is a generator and $s \not\equiv 0(p)$, and $Z_{p\varnothing(t)} \approx H^{4t+1}(X^{(4t-4)}; Z) \approx H^{4t+1}(B_p^{(4t-4)}; Z)$ (mod $C_p$). Since $s \not\equiv 0(p)$, $H^{4t+1}(X; Z) \in C_p$ and we may extend $f_{t-1}$ to $f_t$. The proof that $f_t^{(4t)*}$ is an isomorphism is the same as in the proof of theorem 2.1 except that we know $x \neq 0$. This proves theorem 2.2.

## 4. Construction of $Y_p$

In order to prove theorem 2.3, we want to construct a space $Y_p$ such that 1)

$$\pi_i(Y_p) = \begin{cases} 0 & i \not\equiv 0(r) \\ Z & i \equiv 0(r), \end{cases}$$

where $r = 2p - 2$, and 2) the first $k$-invariant of $Y_{p(rt)}$, namely $k^{(rt+2p-1)} \in H^{r(t+1)+1}(Y_{p(rt)}^{(rt)}; Z) \approx Z_p$ (mod $C_p$) is non-zero. To do this, we use the "cobordism with singularities" theory of Sullivan [10], the main result of which, we now state.

Let $\Omega_*(K)$ be a multiplicative bordism theory with coefficients $\Omega_* = \Omega_*(pt.)$ and associated with a spectrum $M$. Let $I \subset \Omega_*$ be an ideal with a sequence of generators $(c_1, c_2, \cdots)$ such that $c_{i+1}$ is not a zero divisor in $\Omega_*/(c_1, \cdots, c_i)$. Then there is a multiplicative bordism theory with singularities, $\Omega_*'(K)$, with coefficients $\Omega_*' = \Omega_*/(c_1, \cdots)$, associated with a spectrum $M'$, and a map $f:M \rightarrow M'$ inducing $\Omega_* \rightarrow \Omega_*'$.

In our application, we set $\Omega_* = \Omega_*^U = X[c_1, c_2, \cdots]$, dim $c_i = 2i$ (see [5]), and $I = (c_1, c_2, \cdots, \hat{c}_{p-1}, \cdots)$. Then $\Omega_*' = Z_2[c_{p-1}]$. Assume $M'$ is an $\Omega$-spectrum (see [11]) and define $Y_p = M_0'$, the first term of $M'$. Clearly $Y_p$ satisfies 1) above. In order to prove 2), we first note that $M'$ is periodic of period $r = 2p - 2$. Let $c: S^r \to M'$ represent $c_{p-1}$. Consider the composite $S^r \wedge M' \to M' \wedge M' \to M'$ and its adjoint $g: M' \to \Omega^r(M')$. $g_{\#}: \pi_*(M') \to \pi_*(\Omega^r(M'))$ sends $(c_{p-1})^a$ into $(c_{p-1})^{a+1}$ and hence is an isomorphism. Thus $M' \sim \Omega^r(M')$.

LEMMA 4.1. *The first k-invariant* $k^{r+1}(M') \in H^{r+1}(K(Z, 0); Z) \approx Z_p \pmod{C_p}$ *is non-zero.*

Before proving lemma 4.1, we conclude the proof of theorem 2.3. By lemma 4.1 and periodicity, the first $k$-invariants of $M_{(tr)}'$ are all non-zero. Hence $Y_p$ satisfies 2) above. Consider the product $\prod_{i=0}^{(p-1)/2-1} \Omega^{4i}(Y_p)$. It is easy to check that it satisfies the conditions of theorem 2.2 and hence $BSO \sim_p \prod_{i=0}^{(p-1)/2-1} \Omega^{4i}(Y_p)$. For $BU$, one needs a theorem analogous to theorem 2.2; we leave the details to the reader. To show that the $H$-space structures are the same mod $p$, apply the functor $\Omega^4$ to the above equation.

We now prove lemma 4.1. Consider the map $f: MU \to M'$. If lemma 4.1 is false, then there exists $u \in H^r(M')$ such that $c^*(u) \neq 0 \in H^r(S^r)$, $c: S^r \to M'$ representing $c_{p-1}$. But $c = f\bar{c}$, $\bar{c}: S^r \to M$ representing $c_{p-1}$. Hence $c^*(f^*(u)) \neq 0$. But Milnor has shown that all mod $p$ Chern numbers of $c_{p-1}$ are zero [6]. This is a contradiction.

## 5. $F/PL$

In this section, we show that $F/PL$ satisfies condition 2) of theorem 2.2 and prove corollary 2.4. In [10], Rourke gives a sketch of a proof of the fact that $\Omega^4(F/PL) \sim_p F/PL$ without using the result that $F/PL \sim_p BSO$. Hence to check condition 2) of theorem 2.2, we need only show that

$$k^{2p+3}(F/PL) \in H^{2p+3}(F/PL^{(2p+1)}; Z) \approx Zp \pmod{C_p}$$

is non-zero.

F/PL is the fibre of the map $BSPL \to BSF$. Using known results about $H^*(BSPL)$ and $H^*(BSF)$ (c.f. [12]), it is easy to check this fact for $p = 3$. The cases $p > 3$ are a bit more complicated.

LEMMA 5.1. *Assume* $p > 3$. *Then* $\pi_{2p+2}(BSO) \to \pi_{2p+2}(BSPL)$ *is an isomorphism* mod $C_p$.

*Proof.* Consider the exact sequence of Hirsch-Mazur [3], $0 \to \pi_{2p+2}(BSO) \to \pi_{2p+2}(BSPL) \to \Gamma_{2p+1} \to 0$, where $\Gamma_{2p+1}$ is in an exact sequence $0 \to \theta^{2p+1}(\partial\pi) \to \Gamma_{2p+1} \to \text{Coker } J_{2p+1} \to 0$. Coker $J_{2p+1} \in C_p$, so we need only show $\theta^{2p+1}(\partial\pi) \in C_p$. Now $\theta^{2p+1}(\partial\pi)$ is cyclic of order $\epsilon_m 2^{2m-2}(2^{2m-1} - 1)\,\text{num}\,(4B_m/m)$, where $m = (p+1)/2$ and $\epsilon_m = 1$ or $2$ [4]. We must show that $p \nmid (2^p - 1)$ and $p \nmid \text{num}\,(B_m)$. Since $2^{p-1} \equiv 1(p)$, $2^p \equiv 2(p)$, so p $\nmid (2^p - 1)$. $B_1 \not\equiv 0(p)$, by inspection. A special case of Kummer's congruences [5] (cf. p. 276 of [7]) is that $B_1/2 \pm B_{1+(p-1)/2/(p+1)} \equiv 0(p)$ if $p > 3$, hence $p \nmid \text{num}\,(B_{(p+1)/2})$ if $p > 3$.

We now conclude the proof of the fact that $F/PL$ satisfies condition 2) of theorem 2.2. We use the naturality of $k$-invariants with respect to the natural map $BSO \to BSPL$. $K^{2p+3}(BSPL) \in H^{2p+3}(BSPL^{(2p+1)}; \pi_{2p+2}(BSPL))$ and $k^{2p+3}(BSO) \in H^{2p+3}(BSO^{(2p+1)}; \pi_{2p+2}(BSO))$ both have the same image in $H^{2p+3}(BSO^{(2p+1)}; \pi_{2p+2}(BSPL))$. $k^{2p+3}(BSO) \neq 0$, hence by lemma 5.1, its image in $H^{2p+3}(BSO^{(2p+1)}; \pi_{2p+2}(BSPL))$ is $\neq 0$ also. Hence $k^{2p+3}(BSPL) \neq 0$. We now do a similar argument with the map $F/PL \to BSPL$. $k^{2p+3}(BSPL) \in H^{2p+3}\cdot(BSPL^{(2p+1}; \pi_{2p+2}(BSPL))$ and $k^{2p+3}(F/PL) \in H^{2p+3}(F/PL^{(2p+1)}; \pi_{2p+2}(F/PL))$ have the same image in $H^{2p+3}(F/PL^{(2p+1)}; \pi_{2p+2}(BSPL))$. Since $k^{2p+3}(BSPL) = \lambda\beta P^1(\iota_4)$, $\lambda \not\equiv 0\,(p)$, by the above, its image in $H^{2p+3}(F/PL^{(2p+1)}; \pi_{2p+2}(BSPL))$ is also $\lambda\beta P^1(\iota_4) \neq 0$ because $\iota_4 \to \iota_4$ if $p > 3$. Hence $k^{2p+3}(F/PL) \neq 0$ also, and condition 2) of theorem 2.2 is satisfied.

A proof similar to that of theorem 2.2 proves the following theorem.

THEOREM 5.2. *Let $X$ be a space such that*

1) $$\pi_i(X) = \begin{cases} 0 & i \not\equiv 2(4), i = 2 \\ Z & i \equiv 2(4), i > 2 \end{cases} \bmod C_p$$

*and*

2) *the first possible $p$-torsion $k$-invariant of $X_{(4t+2)}$ is non-zero, $t \geq 1$. Then* $X \sim_p \Omega^{-2}(BSO) = \Omega^2(BSO_{(8)})$.

Boardman and Vogt [2] have shown that there is a space $B^2(F/PL)$ such that $\Omega^2 B^2(F/PL) = F/PL$ as $H$-spaces. Clearly $B^2(F/PL)$ satisfies the conditions of theorem 5.2. Hence $F/PL = \Omega^2 B^2(F/PL) \sim_p \Omega^2\Omega^2(BSO_{(8)}) = \Omega^4(BSO_{(8)}) =$ BSO, where all maps are $H$-maps. This concludes the proof of corollary 2.4.

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