EXISTENCE OF SOLUTIONS OF RETARDED FUNCTIONAL EQUATIONS WITH NON CONTINUOUS INITIAL CONDITIONS

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1. Introduction

Let r and a be positive numbers and let Γ be a set of functions defined on $[-r, a]$ into R^n. Let $\Omega \subseteq \Gamma$ and let Λ be a set of functions from $[-r, 0]$ into R^n such that for every $t \in [0, a]$ and every $y \in \Omega$ the function $y_t(\theta) = y(t + \theta)$ with $\theta \in [-r, 0]$ is an element of Λ. Let now f be a function from $\Lambda \times [0, a]$ into R^n. We say that a function $x \in \Gamma$ is a solution of the functional differential equation

$$\frac{dx}{dt} = f(x_t, t)$$

with initial condition $x_0 \in \Lambda$, if $x(t) = x_0(t)$ for all $t \in [-r, 0]$ and

$$x(t) = x_0(0) + \int_0^t f(x_s, s)\, ds$$

for all $t \in [0, a]$.

Existence theorems for such equations have been given in [3] for f lipschitzian in its first argument and integrable respect to t. In [2] this result is obtained for continuous f and in [5] existence is proved under Caratheodory type of conditions. In all the above mentioned papers, with the exception of [5], the initial condition is assumed to be continuous. In the book [6] the existence of solutions for piecewise continuous initial conditions is mentioned but no adequate reference is given. The basic object of the present work is to give an existence theorem for these equations under general conditions on f and almost no restrictions on the initial conditions.

2. Existence Theorem

Using the same notation as in [4] let $\psi(\theta)$ be a nonnegative continuous function defined on some interval $[0, \alpha]$ with $\psi(0) = 0$ and such that the series

$$\sum_{j=1}^{\infty} 2^j \psi\left(\frac{\theta}{2^j}\right)$$

is uniformly convergent in $[0, \alpha]$. Under such conditions if we define

$$\Psi(\theta) = \sum_{j=1}^{\infty} \frac{2^j}{\theta} \psi\left(\frac{\theta}{2^j}\right),$$

then $\Psi(0) = 0$ and $\Psi(\theta) \to 0$ with $\theta \to 0^+$. By ω_1, ω_2 and ω_3 we shall always denote continuous and increasing functions, ω_1 and ω_2 defined on $[0, \alpha]$ and ω_3 defined on $[0, 2\omega_1(\alpha)]$, all vanishing at zero and bounded below by linear functions with positive slopes. These functions are assumed to be such that if we define

$$\psi(\theta) = \omega_3[2\omega_1(\theta)]\omega_2(\theta),$$

and define $\Psi(\theta)$ as before in terms of $\psi(\theta)$, these functions satisfy the conditions mentioned before.

With the symbol $\mathfrak{F}(M, \omega_1, \omega_2, \omega_3, \sigma)$, where M represents a subset of $X \times R^1$ for some banach space X and $\sigma \in (0, \alpha]$, we mean all functions F continuous on M, range in X and such that

$$\| F(x, t_1) - F(x, t_2) \| \leq \omega_1(| t_1 - t_2 |)$$

and

$$\| F(x_2, t_2) - F(x_2, t_1) - F(x_1, t_2) + F(x_1, t_1) \| \leq \omega_2(| t_1 - t_2 |)\omega_3(\| x_1 - x_2 \|)$$

for all admissible values of the arguments and $| t_2 - t_1 | \leq \sigma$. Here $\| \quad \|$ is the norm in X.

Remark 2.1. It is straightforward to prove that if $F(x, t) \in \mathfrak{F}$ where the variable x ranges over some set of functions with modulus of continuity $2\omega_1$, the conditions of Theorem 2.1 in [4] are satisfied.

Let y_0 be a function from $[-r, a]$ into R^n such that $y_0(\tau) = y_0(0)$ for $\tau \in [0, a]$ (y_0 need not be continuous or anything special in $[-r, 0]$). Let $\Gamma = \Gamma(r, a, y_0)$ be the set of functions x from $[-r, a]$ into R^n such that $x(\tau) = y_0(\tau)$ for $\tau \in [-r, 0)$ and are continuous in $[0, a]$. We can consider Γ as a banach space defining the linear operations by

$$(\alpha x + \beta y)(\tau) = \begin{cases} y_0(\tau), & \tau \in [-r, 0) \\ \alpha x(\tau) + \beta y(\tau), & \tau \in [0, a] \end{cases}$$

for all reals α, β and $x, y \in \Gamma$; and defining the norm in Γ by

$$\| x \|_\Gamma = \sup_{0 \leq t \leq a} | x(\tau) |.$$

Let Ω be a convex subset of Γ that contains y_0; (T_1, T_2) will be a real interval and F a function from $\Omega \times (T_1, T_2)$ into Γ. We are interested in solutions of the generalized differential equation

$$\frac{dx}{d\tau} = \mathfrak{D}F(x, t) \tag{2.2}$$

when $F \in \mathfrak{F}(\Omega \times (T_1, T_2), \omega_1, \omega_2, \omega_3, \sigma)$.

Given $t_0 \in (T_1, T_2)$ and a real number $h > 0$ such that $[t_0, t_0 + h] \subset (T_1, T_2)$, let $A(t_0, h)$ denote the set of continuous functions from $[t_0, t_0 + h]$ into $\Gamma(r, a, y_0)$ with a supremum norm denoted by $\| \quad \|_A$. We define the set

$$A_1 = A_1(t_0, h, y_0, \omega_1, \Omega) \subset A(t_0, h)$$

to consist of those functions $y \in A(t_0, h)$ for which the following four conditions are fulfilled:

$$\begin{cases} \text{a) } y(t_0) = y_0 \\ \text{b) } \| y(t) - y(t') \|_\Gamma \leq 2\omega_1(| t - t' |) \quad \text{for all} \quad t, t' \in [t_0, t_0 + h] \quad \text{and} \\ \qquad | t - t' | < \sigma \\ \text{c) } | y(t)(\tau) - y(t)(\tau') | \leq 2\omega_1(| \tau - \tau' |) \quad \text{for all} \quad \tau, \tau' \in [0, a], \\ \qquad t \in [t_0, t_0 + h] \quad \text{and} \quad | \tau - \tau' | < \sigma \\ \text{d) } y(t) \in \Omega \quad \text{for all} \quad t \in [t_0, t_0 + h] \end{cases} \tag{2.3}$$

This set is clearly non-empty since it contains the function $y(t) = y_0$ for all $t \in [t_0, t_0 + h]$. We shall now prove

LEMMA 2.4. *The set $A_1(t_0, h, y_0, \omega_1, \Omega)$ is a compact and convex subset of $A(t_0, h)$.*

Proof. Given $y, z \in A_1$ a straightforward computation will prove that the linear combination $(1 - \alpha)y + \alpha z \in A_1$ for all $\alpha \in [0, 1]$, proving convexity. On the other hand, compactness follows from the extended Ascoli's theorem as given in [1] pag. 382.

We can now prove the following existence theorem for equation (2.2).

THEOREM 2.5. *Suppose there exists an $h_0 > 0$ such that the following conditions are satisfied:*

1) $| \int_{t_0}^{t} \mathfrak{D}F[y(\xi), s](\tau) - \mathfrak{D}F[y(\xi), s](\tau')| \leq 2\omega_1(| \tau - \tau' |)$

2) $y_0 + \int_{t_0}^{t} \mathfrak{D}F[y(\xi), s] \in \Omega$

for all $t \in [t_0, t_0 + h]$, $\tau, \tau' \in [0, a]$, $| \tau - \tau' | < \sigma$, $h \leq h_0$ and $y \in A_1(t_0, h, y_0, \omega_1, \Omega)$; then there exists an $h_1 < h_0$ such that equation (2.2) has a solution on $[t_0, t_0 + h_1]$ with initial condition y_0 at t_0.

Proof. We have to prove the existence of a function $y \in A(t_0, h_1)$ such that

$$y(t) = y_0 + \int_{t_0}^{t} \mathfrak{D}F[y(\xi), s]$$

for all $t \in [t_0, t_0 + h_1]$. Let us now define a transformation $\Phi: A_1 \rightarrow A_1$ given by

$$\Phi(y)(t) = y_0 + \int_{t_0}^{t} \mathfrak{D}F(y(\xi), s]. \tag{2.6}$$

We shall prove that for h sufficiently small Φ maps $A_1(t_0, h, y_0, \omega_1, \Omega)$ continuously into itself, Lemma 2.4 and Schauder's fixed point theorem will then give the desired conclusion of Theorem 2.5.

We have to prove that for a suitably chosen $h < h_0$ $y \in A_1(t_0, h, y_0, \omega_1, \Omega)$ implies $\Phi(y) \in A_1(t_0, h, y_0, \omega_1, \Omega)$. Obviously $\Phi(y)$ satisfies conditions a) and d) of (2.3).

Let now $k > 0$ be such that $\omega_1(\eta) \geq k\eta$ $0 \leq \eta < \sigma$ and define $k^*(h) = \sup \frac{1}{2}\Psi(\eta)$ for $\eta \in [0, \min (h, \sigma)]$; chose $h_2 \in (0, h_0)$ such that

$$k^*(h_2) < k. \tag{2.7}$$

Making use of Theorem 2.1 in [4] and the fact that

$$F \in \mathfrak{F}(\Omega \times (T_1, T_2), \omega_1, \omega_2, \omega_3, \sigma)$$

we have the following estimate:

$$\begin{aligned} \|\Phi(y)(t) - \Phi(y)(t')\|_\Gamma &= \|\textstyle\int_t^{t'} \mathfrak{D}F[y(\xi), s]\|_\Gamma \leq \|\textstyle\int_t^{t'} \mathfrak{D}F[y(\xi), s] \\ &\quad - F[y(t), t'] + F[y(t), t]\|_\Gamma + \|F[y(t), t'] \\ &\quad - F[y(t), t]\|_\Gamma \leq \frac{| t - t' |}{2} \Psi(| t - t' |) \\ &\quad + \omega_1(| t - t' |) \leq k^*(h_2) \, | t - t' | \\ &\quad + \omega_1(| t - t' |) \leq 2\omega_1(| t - t' |), \end{aligned}$$

for all $t, t' \in [t_0, t_0 + h_2]$, $|t - t'| < \sigma$ and this proves that $\Phi(y)$ satisfies condition b) of (2.3) when $h \leq h_2$.

Moreover condition c) is a direct consequence of hypothesis 1) regarding the integrals of F. Finally we need to prove the continuity of Φ. Let $0 < h < h_2$ and for any given $\epsilon > 0$ chose $\delta = \delta(\epsilon)$ such that $h[\max_{0 \leq \eta \leq \delta} \Psi(\eta)] < \epsilon/2$. Given any $t \in [t_0, t_0 + h]$ let $t_0 = s_0 < s_1 < \cdots < s_m = t \leq t_0 + h$ be any subdivision of $[t_0, t]$ of norm not greater than δ. Let now $y, z \in A_1$ be such that $\| y - z \|_A < \sigma$, then we have

$$\begin{aligned}
\| \Phi(y)(t) - \Phi(z)(t) \|_\Gamma & \\
\leq \textstyle\sum_{i=0}^{m-1} \| \int_{s_i}^{s_{i+1}} \mathfrak{D}\{F[y(\tau), s] - F[z(\tau), s]\} & \\
- F[y(s_i), s_{i+1}] + F[z(s_i), s_{i+1}] + F[y(s_i), s_i] & \\
- F[z(s_i), s_i]\|_\Gamma + \textstyle\sum_{i=0}^{m-1} \| F[y(s_i), s_{i+1}] - F[z(s_i), s_{i+1}] & \\
- F[y(s_i), s_i] + F[z(s_i), s_i]\|_\Gamma \leq \textstyle\sum_{i=0}^{m-1} (s_{i+1} - s_i)\Psi(s_{i+1} - s_i) & \\
+ \textstyle\sum_{i=0}^{m-1} \omega_3[\| y(s_i) - z(s_i) \|_\Gamma]\omega_2(s_{i+1} - s_i) & \\
\leq h[\max_{0 \leq \eta \leq \delta} \Psi(\eta)] + m\omega_3(\| y - z \|_A)\omega_2(\delta). &
\end{aligned}$$

Therefore

$$\| \Phi(y) - \Phi(z) \|_A \leq \frac{\epsilon}{2} + m\omega_3(\| y - z \|_A)\omega_2(\delta).$$

Chose $\eta > 0$ such that $m\omega_3(\eta)\omega_2(\delta) < \epsilon/2$, then it follows from the last estimate that

$$\| \Phi(y) - \Phi(z) \|_A < \epsilon$$

if $\| y - z \|_A < \eta$, proving the continuity of Φ and concluding the proof of the theorem.

3. Functional equations

Let $\Omega = \Omega(\omega_1) \subset \Gamma$ be defined by $\Omega = \{v \in \Gamma \mid v(0) = y_0(0)$ and $|v(t) - v(t')| \leq 2\omega_1(|t - t'|)$ for all $t, t' \in [0, a]$, $|t - t'| < \sigma\}$.

Obviously Ω is a convex subset of Γ. Define as usual $v_t: [-r, 0] \to R^n$ by $v_t(\tau) = v(t + \tau)$ for all $\tau \in [-r, 0]$, $t \in [0, a]$ and $v \in \Gamma$. Let Δ be the class of functions defined as follows

$$\Delta = \{v_t \mid v \in \Omega, t \in [0, a]\}.$$

Denote $\mathcal{G} = \mathcal{G}(\Delta, \omega_1, \omega_2, \omega_3, \sigma) = \{g: \Delta \times [0, a] \to R^n \mid$ a) g is lebesgue integrable in $[0, a]$ for all $v \in \Omega$; b) $| \int_{t'}^{t} g(v_s, s)\, ds | \leq \omega_1(|t - t'|)$ and

$$| \int_{t'}^{t} [g(v_s^1, s) - g(v_s^2, s)]\, ds | \leq \omega_2(|t - t'|)\omega_3(\| v^1 - v^2 \|_\Gamma)$$

for all $v, v^1, v^2 \in \Omega$, $t, t' \in [0, a]$, $|t - t'| < \sigma$ and $\| v^1 - v^2 \|_\Gamma < \sigma\}$.

A function $v \in \Omega$ is said to be a solution of

$$\frac{dv}{dt} = g(v_t, t) \tag{3.1}$$

in $[0, a]$ with initial condition y_0 if and only if

$$v(t) = y_0(0) + \int_0^t g(v_s, s)\, ds \tag{3.2}$$

for all $t \in [0, a]$ and $v(t) = y_0(t)$ for all $t \in [-r, 0)$.

Exactly in the same way as was done in [7] one can prove that if

$$F(x, t)(\tau) = \begin{cases} y_0(\tau); & \tau \in [-r, 0], \quad t \in [0, a] \\ \int_0^\tau g(x_s, s)\, ds; & \tau, t \in [0, a], \quad \tau \leq t \\ \int_0^t g(x_s, s)\, ds; & \tau, t \in [0, a], \quad t \leq \tau \end{cases} \tag{3.3}$$

and $x(t)$ is a solution of (2.2), with F defined by (3.3), with initial condition $x(0) = y_0$ then the function $v(t)$ defined by

$$v(t) = \begin{cases} y_0(t); & t \in [-r, 0] \\ x(t)(t); & t \in [0, a] \end{cases} \tag{3.4}$$

is a solution of (3.1) on $[0, a]$ and initial condition y_0.

Observe that if $g \in \mathcal{G}(\Delta, \omega_1, \omega_2\ \omega_3, \sigma)$ then the function F defined by (3.3) is an element of $\mathfrak{F}(\Omega \times [0, a], \omega_1, \omega_2, \omega_3, \sigma)$. Therefore it follows from Remark 2.1 that

$$\int_0^t \mathfrak{D}F[x(\xi), s] \quad \text{exists if} \quad t \in [0, a]$$

and $x \in A_1(0, a, y_0, \omega_1, \Omega)$.

Let us now prove:

LEMMA 3.5. *Let* $x \in A_1(0, a, y_0, \omega_1, \Omega)$, *then*
$[\int_0^t \mathfrak{D}F[x(\xi), s]](\tau) = [\int_0^t \mathfrak{D}F[x(\xi), s]](t)$ *for all* $\tau \geq t$ *and*
$[\int_0^t \mathfrak{D}F[x(\xi), s]](\tau) = [\int_0^\tau \mathfrak{D}F[x(\xi), s]](\tau)$ *for all* $\tau \leq t$ *and* $t, \tau \in [0, a]$.

Proof. Let $\tau \geq t$ and take a subdivision $0 = s_1 < s_2 < \cdots < s_n = t$, then

$$\begin{aligned}[\int_0^t \mathfrak{D}F[x(\xi), s]](\tau) &= [\textstyle\sum_{i=1}^{n-1} F[x(s_i), s_{i+1}] - F[x(s_i), s_i]](\tau) + 0(1) \\ &= [\textstyle\sum_{i=1}^{n-1} F[x(s_i), s_{i+1}](t) - F[x(s_i), s_i](t)] + 0(1) \\ &= [\int_0^t \mathfrak{D}F[x(\xi), s]](t) + 0(1),\end{aligned}$$

where $0(1)$ tends to zero with the norm of the partition tending to zero, proving the first relation. A similar argument yields the second relation. We shall now prove the following existence result.

THEOREM 3.6. *Let* $g \in \mathcal{G}(\Delta, \omega_1, \omega_2, \omega_3, \sigma)$, *then there exists at least one solution* $v \in \Omega$ *of equation* (3.1) *in* $[0, a]$, *with a sufficiently small, which fullfils the initial condition* $v(t) = y_0(t)$ *for* $t \in [-r, 0]$.

Proof. With F defined by (3.4) we know that if there exists a solution of (2.2) then also (3.1) has a solution. We shall prove that Theorem 2.5 can be used with $t_0 = 0$ and $a = h_0$. Let $y \in A_1(0, a, y_0, \omega_1, \Omega)$ and let us prove that condition

1) in Theorem 2.5 is satisfied. For this, let $\tau, \tau', t \in [0, a]$ and $|\tau - \tau'| < \sigma$, we shall consider the following three possibilities: i) $\tau' \leq \tau \leq t$, ii) $\tau \geq \tau' \geq t$ and *iii)* $\tau \geq t \geq \tau'$. In case i) we have, using Lemma 3.5, that

$$\begin{aligned}
&|[\textstyle\int_0^t \mathfrak{D}F[y(\xi), s]](\tau) - [\int_0^t \mathfrak{D}F[y(\xi), s]](\tau')| \\
&= |[\textstyle\int_0^\tau \mathfrak{D}F[y(\xi), s]](\tau) - [\int_0^{\tau'} \mathfrak{D}F[y(\xi), s]](\tau)| \\
&= |[\textstyle\int_{\tau'}^\tau \mathfrak{D}F[y(\xi), s]](\tau)| \leq \|\int_{\tau'}^\tau \mathfrak{D}F[y(\xi), s]\|_\Gamma \\
&\leq \| F[y(\tau), \tau'] - F[y(\tau), \tau]\|_\Gamma + \frac{\tau - \tau'}{2} \Psi(\tau - \tau') \leq 2\omega_1(\tau - \tau').
\end{aligned}$$

In case ii) we have, again making use of Lemma 3.5, that

$$\begin{aligned}
&|[\textstyle\int_0^t \mathfrak{D}F[y(\xi), s]](\tau) - [\int_0^t \mathfrak{D}F[y(\xi), s]](\tau')| \\
&\qquad = |[\textstyle\int_0^t \mathfrak{D}F[y(\xi), s]) (t) - [\int_0^t \mathfrak{D}F[y(\xi), s]](t)| = 0.
\end{aligned}$$

Finally, in case iii) we have, according to Lemma 3.5, that

$$\begin{aligned}
&|[\textstyle\int_0^t \mathfrak{D}F[y(\xi), s]](\tau) - [\int_0^t \mathfrak{D}F[y(\xi), s]](\tau')| \\
&\qquad = |[\textstyle\int_0^t \mathfrak{D}F[y(\xi), s]](t) - [\int_0^{\tau'} \mathfrak{D}F[y(\xi), s]](t)| \\
&\qquad = |[\textstyle\int_{\tau'}^t \mathfrak{D}F[y(\xi), s]](t) = |[\int_{\tau'}^\tau \mathfrak{D}F[y(\xi), s]](t)| \leq 2\omega_1(\tau - \tau'),
\end{aligned}$$

proving condition 1) of Theorem 2.5. Condition 2) of Theorem 2.5 follows from what we have just proved and the definition of y_0, ending the proof of Theorem 3.6.

Remark 3.7. It is not difficult to see that the solution that exists on the interval $[0, a]$ may be continued as far as $g \in \mathcal{G}(\Delta, \omega_1, \omega_2, \omega_3, \sigma)$.

Remark 3.8. Observe that the conditions on the functions $g \in \mathcal{G}$ are made for a fixed element $v \in \Omega$, as we shall see these conditions are satisfied if similar conditions hold for a fixed element $\omega \in \Delta$. In fact we have

THEOREM 3.9. *Let $g(z, t)$ be defined for all $z \in \Delta$ and $t \in [0, a]$ and let $\varphi_1(\eta)$, $\varphi_2(\eta)$ be increasing continuous functions defined on $[0, \sigma]$ and $[0, 2\omega_1(\sigma)]$ respectively, with $\varphi_1(0) = \varphi_2(0) = 0$ and $\varphi_1(\eta) \geq \varphi_2[2\omega_1(\sigma)]\eta$ on $[0, \sigma]$. Suppose the following conditions hold:*

a) *g is lebesgue integrable in $[0, a]$ for all fixed $z \in \Delta$*

b) *$|\int_{t'}^t g(z, \tau)\, d\tau| \leq \varphi_1(|t - t'|)$; for all $z \in \Delta$, $t, t' \in [0, a]$ and $|t - t'| < \sigma$*

c) *$|g(z_1, t) - g(z_2, t)| \leq \varphi_2(\|z_2 - z_1\|_\Delta)$, for all $z_1, z_2 \in \Delta$, $t \in [0, a]$ and $\|z_1 - z_2\|_\Delta \leq 2\omega_1(\sigma)$,*

then $g \in \mathcal{G}(\Delta, \omega_1, \omega_2, \omega_3, \sigma)$ with $\omega_1 = 2\varphi_1$, $\omega_2(\eta)\eta$ and $\omega_3 = \varphi_2$.

Proof. Since $v \in \Omega$ it has the modulus of continuity $2\omega_1$ in $[0, a]$, $v_t \in \Delta$ is continuous in $t \in [0, a]$ and it may be approximated uniformly on $[0, a]$ by a sequence $\{e_i(t)\}$, $i = 1, 2, 3, \cdots$, of functions piecewise constant in Δ. Therefore

$$|g(v_t, t) - g(e_i(t), t)| \leq \varphi_2(\|v_t - e_i(t)\|_\Delta)$$

for $t \in [0, a]$, $i = 1, 2, 3, \cdots$, and it follows that $|g(e_i(t), t)| \to |g(v_t, t)|$ with $i \to \infty$ uniformly on $[0, a]$, thus $g(v_t, t)$ is lebesgue integrable in $[0, a]$.

Now let $t, t' \in [0, a]$, $|t - t'| < \sigma$ and $v \in \Omega$, then

$$|\int_{t'}^{t} g(v_s, s)\, ds| \leq |\int_{t'}^{t} [g(v_s, s) - g(v_t, s)]\, ds| + |\int_{t'}^{t} g(v_t, s)\, ds|$$
$$\leq \varphi_2[2\omega_1(|t - t'|)]|t - t'| + \varphi_1(|t - t'|) \leq 2\varphi_1(|t - t'|).$$

Moreover, if $t, t' \in [0, a]$, $|t - t'| < \sigma$ and $v^1, v^2 \in \Omega$ then

$$|\int_{t'}^{t} [g(v_s^1, s) - g(v_s^2, s)]\, ds| \leq \int_{t'}^{t} \varphi_2(\| v_s^1 - v_s^2 \|_\Delta)\, ds \leq \varphi_2(\| v^1 - v^2 \|_\Gamma)|t - t'|.$$

proving that the primitive of g has the required moduli of continuity.

4. Example

Let us consider the scalar equation

$$\frac{dx(t)}{dt} = \frac{x(t-1)}{(t-p)^\alpha} \sin t \tag{4.1}$$

for $0 \leq \alpha < \frac{1}{2}$, $0 < p < 1$, $t \geq 0$, $t \neq p$, with initial condition $x = \varphi_0(\tau)$, $\tau \in [-1, 0]$, where

$$\varphi_0(\tau) = \begin{cases} \frac{1}{2}; & \tau \in [-1, -\frac{1}{2}] \\ 1; & \tau \in (-\frac{1}{2}, 0). \end{cases}$$

Let us define

$$\Gamma = \{v:[-1, 2] \to R^1 \mid v(t) = \varphi_0(t) \quad \text{for} \quad t \in [-1, 0), v \quad \text{continuous in} \quad [0, 2]\},$$

and

$$\Omega = \{v \in \Gamma \mid v(0) = \varphi_0(0); |v(t) - v(t')| \leq 2\omega_1(|t - t'|), |t - t'| < 1\},$$

where $\omega_1(\eta) = \dfrac{2K}{1-\alpha}\eta^{1-\alpha}$ and $K = 1 + \dfrac{p^{1-\alpha} + (1-p)^{1-\alpha}}{1-\alpha}$. Put

$$\Delta = \{v_t \mid v \in \Omega, t \in [0, 2]\}$$

and finally

$$\mathcal{G} = \Big\{ g:\Delta \times [0, 2] \to R^1 \mid a)\ g \in L_2[0, 2] \text{ for all } v \in \Omega;$$

$$\text{b) } |\int_{t'}^{t} [g(v_s, s)\, ds| \leq \frac{2K}{1-\alpha} |t - t'|^{1-\alpha};$$

$$\text{c) } |\int_{t'}^{t} [g(v_s^1, s) - g(v_s^2, s)]\, ds| \leq \frac{2}{1-\alpha} \| v^1 - v^2 \|_\Gamma |t - t'|^{1-\alpha} \Big\}.$$

It is not difficult to prove that in these conditions if $g \in \mathcal{G}$ the F defined by (3.3) is an element of $\mathfrak{F}(\Omega \times [0, 2], \omega_1, \omega_2, \omega_3, 1)$ with $\omega_1(\eta) = 2K/(1-\alpha)\eta^{1-\alpha}$, $\omega_2(\eta) = 2/(1-\alpha)\eta^{1-\alpha}$, $\omega_3(\eta) = \eta$; and that in view of Remark 3.7 the solution of

(4.1) exists on [0, 2]. Such an existence result does not follow from other known results.

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