# THE GENERALIZED VECTOR FIELD PROBLEM AND BILINEAR MAPS

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## 1. Introduction

Let $\xi_n$ be the canonical line bundle over $RP^n$. Then the generalized vector field problem [4] is: Given $k$ and $n$, find the largest integer $s(k, n)$ such that $k\xi_n$ admits $s(k, n)$ independent sections.

Let $R^n$ denote euclidean $n$-space, then the bilinear map problem [9] is: Given $k$ and $n$, find the largest integer $b(k, n)$ such that there exists a non-singular bilinear mapping

$$f: R^n \times R^{b(k,n)} \to R^k.$$

Estimates on the functions $s(k, n)$ and $b(k, n)$ appear in [2, 3, 7, 11, 13, 15, 16, 17].

The two problems are related by the following inequality:

$$s(k, n) \geq b(k, n + 1) \tag{1.1}$$

(see [7]).

It has been shown, in many instances, that equality holds in (1.1). One general result is the tangent vector field problem for $RP^n$, for which one has

$$s(n + 1, n) = b(n + 1, n + 1).$$

(See [1], [6], (10]).

This led the first author to conjecture in [7], that (1.1) is always an equality. More recently the second named author, in [12], has shown that for each fixed $n$, (1.1) is an equality for sufficiently large $k$. He has also verified equality for $k \leq 31$, $n \leq 31$, except possibly when $k = 20, 9 \leq n \leq 14$.

In this note we show $s(32, 27) \neq b(32, 28)$. Our method is to show that $32\xi_{27}$ has 13-sections using the obstruction theory developed by M. Mahowald. On the other hand a non-singular bilinear map $f: R^{28} \times R^{13} \to R^{32}$ cannot exist, for its existence would imply $32\xi_{12}$ has 28-sections, which we prove to be impossible using a non-stable operation in $K$-theory arising from the complex representations of the Spinor groups.

## 2. The positive result

We use the modified Posnikov system of Mahowald (14], [8] to show that $32\,\xi_{27}$ has 13 sections. We consider the fibration $B\mathrm{Spin}(19) \to B\mathrm{Spin}(32)$ and use the method described in [8] to obtain the following:

(2.1) Table of $k$-invariants for $B\mathrm{Spin}(19) \to B\mathrm{Spin}(32)$ in dimension $\leq 27$

First level: $k_1^1 = W_{20}$; $k_2^1 = W_{24}$

Second level

| | dim | defining relation |
|---|---|---|
| $k_1^2$ | 22 | $Sq^{2,1} k_1^1 = 0$ |
| $k_2^2$ | 23 | $(Sq^4 + W_4)k_1^1 = 0$ |
| $k_3^2$ | 24 | $Sq^1 k_2^1 + (Sq^4 + W_4) Sq^1 k_1^1 = 0$ |
| $k_4^2$ | 25 | $Sq^2 k_2^1 + (Sq^4 + W_4) Sq^2 k_1^1 = 0$ |
| $k_5^2$ | 27 | $W_4 K_2^1 + (Sq^8 + W_8)k_1^1 = 0$ |

Third level:

| | dim | defining relation |
|---|---|---|
| $k_1^3$ | 23 | $Sq^2 k_1^2 = 0$ |
| $k_2^3$ | 24 | $Sq^1 k_3^2 + Sq^{2,1} k_1^2 = 0$ |
| $k_3^3$ | 26 | $Sq^2 k_4^2 + Sq^3 k_3^2 + (Sq^4 + W_4)k_2^2 = 0$ |

Fourth level:

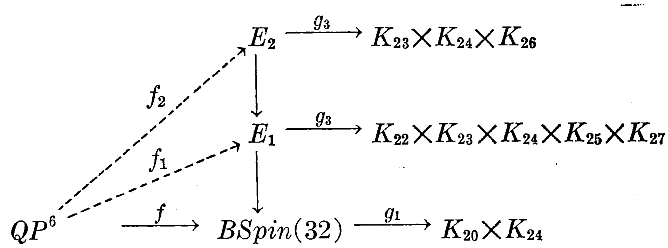
| | dim | defining relation |
|---|---|---|
| $k_1^4$ | 24 | $Sq^1 k_2^3 + Sq^2 k_1^3 = 0$ |

The homotopy groups for the fiber $V_{32,13}$ in dimensions $\leq 26$ are as follows.

| dim $j$ | 19 | 20 | 21 | 22 | 23 | 24 | 25 | 26 |
|---|---|---|---|---|---|---|---|---|
| $\pi_j(V_{32,13})$ | $Z_2$ | 0 | $Z_2$ | $Z_2 \oplus Z_2$ | $Z_{16}$ | $Z_2$ | $Z_2$ | $Z_2$ |

Let $H_n$ be the 4-dimensional real bundle over $QP^n$ obtained from the symplectic line bundle over $QP^n$. Under the Hopf fibration $q:RP^{27} \to QP^6$, we have that $q^*(8H_6) = 32\xi_{27}$. Let $E_4 \to E_3 \to E_2 \to E_1 \to B$Spin (32) be the modified Postnikov tower for $B$Spin(19) $\to B$Spin(32) in dimensions $\leq 27$. If $f:QP^6 \to B$Spin(32) is the classifying map for $8H_6$, we will see that $f$ admits a lifting to $f_2:QP_6 \to E_2$. Consider then

(2.2)




Here $K_m$ denotes the Eilenberg-Maclane space $K(Z_2, m)$ with generator $\iota_m \in H^m(K_m, Z_2)$. The map $g_1$ is given by $g_1^*\iota_{20} = W_{20}$, $g_2^*\iota_{24} = W_{24}$, hence $g_1 f$ is nul-homotopic and $f$ lifts to $f_1:QP^6 \to E_1$. The second stage $k$-invariants go to zero in $H^*(QP^6)$, except, perhaps, for $k_3^2 \in H^{24}(E_1)$.

We use the generating class theorem of E. Thomas [18] to evaluate $k_3^2$. Consider the relation:

$$\mathrm{Sq}^1\,\mathrm{Sq}^{24} + \mathrm{Sq}^{4,1}\,\mathrm{Sq}^{20} + \mathrm{Sq}^{23}\,\mathrm{Sq}^2 + \mathrm{Sq}^{24}\,\mathrm{Sq}^1 = 0, \tag{2.3}$$

and let $\Phi_{24}$ be a secondary operation associated with this relation. Let $U \in H^{32}(T(\gamma_{32}))$ be the Thom class of the canonical bundle $\gamma_{32}$ over $B\mathrm{Spin}\ (32)$, let $\mathrm{U}_{19} \in H^{19}(T(\gamma_{19}))$ be the Thom class of $\gamma_{19}$ over $B\mathrm{Spin}\ (19)$, and let $\bar{\gamma}$ be the induced bundle over $E_1$, with $\bar{U} \in H^{32}(T(\bar{\gamma}))$ its Thom class.

Then the generating class theorem asserts,

LEMMA 2.4. *There exists a class $b \in H^{24}(B\mathrm{Spin}\ (32))$, such that*

$$\bar{U} \smile (k_3^2 + p_1{}^*b) \in \Phi_{24}(\bar{U}).$$

Now $W(8H_6) = 1$, hence if $T(f_1): T(8\mathrm{H}_6) \to T(\bar{\gamma})$ is the induced mapping on Thom spaces, we obtain for $U' \in H^{32}(\mathrm{T}(8H_6))$ that $\Phi_{24}(U') = U' \smile k_3^2(8\mathrm{H}_6)$ modulo zero. Furthermore since $T(8\mathrm{H}_6) = QP^{14}/QP^7$, all secondary operations $\Phi_{24}$ associated with the above relation coincide when applied to $U'$.

Consider now the fibration,

$$g: CP^{29} \to QP^{14}$$

then $g^*: H^*(QP^{14}) \to H^*(CP^{29})$ is a monomorphism, and if $y \in H^4(QP^{14})$, $\omega \in H^2(CP^{29})$ are the generators, $g^*y = \omega^2$. Now $g^*\Phi_{24}(y^8) = \Phi_{24}(\omega^{16}) = 0$ by [3], hence by naturality $\Phi_{24}(U') = 0$, and $k_3^2(8\mathrm{H}_6) = 0$.

Therefore we can lift $f_1$ to $f_2: QP^6 \to E_2$. In $H^*(E_2)$ we find three $k$-invariants and their images in $H^*(QP^6)$ are $(0, \epsilon\, y^6, 0) \in H^{23}(QP^6) \oplus H^{24}(QP^6) \oplus H^{26}(QR^6)$. Let $h_2: RP^{27} \to E_2$ be the composition $f_2 \cdot g$. Then $(k_1^3(32\xi_{27}), k_2^3(32\xi_{27}), k_3^3(32\xi_{27}))$ has as indeterminacy subgroup the subgroup of $H^{23}(RP^{27}) \oplus H^{24}(RP^{27}) \oplus H^{26}(RP^{27})$ generated by $(0, x^{24}, 0)$ and $(0, 0, x^{26})$ as can be easily seen from the table.

Hence there exists a choice $h_2': RP^{27} \to E_2$, lifting $f_0$, the classifying map for $32\xi_{27}$, such that $h_2'(k_1^3, k_2^3, k_3^3) = (0, 0, 0)$, and hence, $h_2': RP^{27} \to E_2$ admits a lifting to

$$h_3: RP^{27} \to E_3$$

There is a last $k$-invariant in $H^{24}(E_3)$. Its image $k_1^4(32\xi_{27})$ in $H^{24}(RP^{27})$ has full indeterminacy, hence there exists a choice of $h_3$ that admits a lifting to $h_4: RP^{27} \to E_4$ and the proof of existence of the sections of $32\xi_{27}$ is complete.

## 3. The negative result

THEOREM. *The bundle $34\xi_{12}$ cannot have 28 independent sections over $RP^{12}$.* (Hence, $32\xi_{12}$ cannot have 28 independent sections).

*Proof.* First recall that the reduced Grothendieck ring of $RP^{12}$ is $\tilde{K}(RP^{12}) = Z_{64}$ with generator $y =$ the complexification of $\xi_{12} - 1$ and ring structure given by $y^2 = -2y$. (See [1]). Suppose the theorem were false so that $34\xi_{12} = 28 \oplus \eta$ for some 6-dimensional bundle $\eta$. Tensoring both sides with $\xi_{12}$ and using $\xi_{12} \otimes \xi_{12} = 1$, we get

$$\eta \otimes \xi_{12} \oplus 28\xi_{12} = 34.$$

Let $\eta \otimes \xi_{12} = \zeta$. Since $W_1(\zeta) = 0 = W_2(\zeta)$, $\zeta$ is a Spin(6)-bundle. Let $\Delta_6^+(\zeta)$ denote the $U(4)$-bundle associated to $\zeta$ through the Spin representation $\Delta_6^+ : \mathrm{Spin}(6) \to U(4)$. As an element in $K(RP^{12})$, $\Delta_6^+(\zeta) = my + 4$ for some integer $m$.

On the other hand, according to a formula in [5, p. 61], the second exterior power $\lambda^2(\Delta_6^+(\zeta))$ is equal to the complexification of $\zeta$, i.e.

$$\lambda^2(my + 4) = -28y + 6 \text{ in } K(RP^{12}). \tag{3.1}$$

Now it is not hard to show that $\lambda^2(my) = -m^2 y$, and consequently (3.1) gives

$$(m^2 - 4m - 28)y = 0.$$

But, by an elementary argument in number theory, $(m^2 - 4m - 28)$ is never divisible by 64. This contradicts the fact that $y$ has order 64, and finishes the proof.

*Remark* 1. One can use the same method to prove other non-sectioning statements (for example, that $16\xi_{10}$ does not admit 10 independent sections). But the most general result which can be obtained by this technique is not yet available to us.

CENTRO DE INVESTIGACIÓN DEL I P N

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