SOME IMMERSIONS OF REAL PROJECTIVE SPACES

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Introduction

M. Hirsch [6] has shown that the immersion problem for manifolds is just a cross-section problem for the stable normal bundle. Our object here is to give the proofs of the main theorems announced in [5].

We give conditions under which sections of the tangent bundle will imply sections of the normal bundle (and conversely). Given an integer $t = 4a + b$, $0 \leq b \leq 3$, let $j(t) = 8a + 2^b$. If ξ is a stable bundle let $gd(\xi)$ denote the smallest integer n such that there is an n-plane bundle ξ^n which is in the stable class of ξ. The main geometric result of this paper is this theorem.

Theorem A. *Let M^m be an m-manofold, $m \leq 2^t - 1$, whose stable tangent bundle τ_0 is trivial over the $(j(t) - 1)$-skeleton. Then $gd(\tau_0) \leq m - j(t) + 1$ implies $gd(-\tau_0) \leq m - j(t) + 1$.* (Note that $-\tau_0 =$ the normal bundle.)

Applying theorem A to projective spaces, we get:

Theorem B. *Let $m = 2^t$. Then RP^{m-1} immerses in $R^{2m-j(t)+1}$ but not in $R^{2m-j(t)}$.*

The negative part of theorem B is due to James [7]. The positive part follows easily from theorem A. Milgram [10] has obtained immersions of RP^{m-1} which agree with these only if $m = 16$ or 32.

The key step is a technical result stated as the main theorem in §1. We expect this result to have more consequence.

1. The main theorem

Let X be a CW complex, then by $X[k]$ we denote the k^{th} Eilenberg subcomplex of the space X, i.e. $X[k]$ is $(k - 1)$-connected and there is a map $f: X[k] \to X$ such that $f_*: \pi_q(X[k]) \cong \pi_q(X)$ for $q \geq k$. Let BO_n and BO denote, respectively, the classifying spaces of n-plane bundles and stable bundles. The natural map $BO_n \to BO$ induces maps

$$p: BO_n[k] \to BO[k]$$

for all k.

Main Theorem. *For each $m < 2^t$, there exists an H-space E, an H-map $\varphi: E \to BO[j(t)]$ and a fiber map $\psi: BO_n[j(t)] \to E$ such that*

$$\begin{tikzcd} BO_n[j(t)] \arrow[rr, "p"] \arrow[dr, "\psi"'] & & BO[j(t)] \\ & E \arrow[ur, "\varphi"'] & \end{tikzcd}$$

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is commutative, where $n = m - j(t) + 1$. Let F be the fiber of Ψ. If n is odd, then F is m-connected. If n is even, then F is $(n - 1)$-connected and $\pi_k(F)$ for $n \leq k \leq m$ is either zero or a finite group of odd order.

The proof of this theorem will be accomplished in this and the following sections. First we observe how this implies theorem A. By hypothesis we can lift τ_0 to $BO_n[j(t)]$. Thus we can lift to E. Now $-\tau_0 = k\tau_0$, at least up to fiber homotopy type. Thus using the H-space structure we can lift $-\tau_0$ to E. The result of [3] asserts, in this context, that we can lift from E to $BO_n[j(t)]$.

We begin the proof of the main theorem by constructing the space E. As in [4] we can construct a fiber space $\varphi_0: X \to BSO$ as a composite of principal fiber spaces

$$X = X_s \to X_{s-1} \to \cdots \to X_0 = BSO \tag{1.1}$$

where the fiber of $X_k \to X_{k-1}$ is a product of Eilenberg-Maclane spaces of type (Z, q) or (Z_2, q). There is a fiber map $\psi_0: BSO_n \to X$ such that

$$\begin{tikzcd} BSO_n \arrow[rr, "p_0"] \arrow[dr, "\psi_0"'] & & BSO \\ & X \arrow[ur, "\varphi_0"'] & \end{tikzcd} \tag{1.2}$$

is commutative. If F is the fiber of ψ_0, then F satisfies the conditions of the main theorem.

If we take the fiber map $BO[j(t)] \to BSO$ and induce diagram (1.2), we obtain a commutative diagram

$$\begin{tikzcd} BO_n\,[j(t)] \arrow[rr, "p"] \arrow[dr, "\psi"'] & & BO[j(t)] \\ & E \arrow[ur, "\varphi"'] & \end{tikzcd} \tag{1.3}$$

where ψ is again a fiber map with fiber F.

What remains to be done is to prove that E is an H-space and that φ is an H-map. The fiber space $\varphi: E \to BO[j(t)]$ is a composite of principal fiber spaces

$$E = E_s \to E_{s-1} \to \cdots \to E_0 = BO[j(t)] \tag{1.4}$$

induced by (1.1). Let F_k be the fiber of $E_k \to E_{k-1}$. Then $E_k \to E_{k-1}$ is classified by a map

$$f_k: E_{k-1} \to BF_k$$

where BF_k is the classifying space of F_k. We will assume that E_{k-1} is an H-space and that $\varphi_{k-1}: E_{k-1} \to E_0$ is an H-map. Then E_k will be an H-space and $\varphi_k: E_k \to E_0$ will be an H-map if the k-invariants, i.e., the images of the fundamental classes of BF_k, are primitive in $H^*(E_{k-1})$. To show this we need some preliminary results.

We have a principal fiber space

$$K(J_t, j(t-1)-1) \to BO[j(t)] \to BO[j(t)-1]$$

where $J_t = Z$ or Z_2, as given by Bott's periodicity [2]. Let

$$(1.5) \qquad \bar{E} = \bar{E}_s \to \cdots \to \bar{E}_0 = K(J_t, j(t-1)-1)$$

be the fibrations induced over $\bar{E}_0$ from (1.4).

Strong, in [12], has determined the mod 2-cohomology of $BO[j]$, and in particular, he has shown that

$$H^*(E_0) \to H^*(\bar{E}_0)$$

is a monomorphism in dimensions $\leq 2^t - 1$.

In section 3, we will prove the following result.

PROPOSITION 1.6. *For $k = 0, 1, \cdots, s$, the natural map $\bar{E}_k \to E_k$ induces a monomorphism*

$$H^*(E_k) \to H^*(\bar{E}_k)$$

in dimensions $\leq 2^t - 1$.

Again, if we assume inductively that E_{k-1} is an H-space, then $\bar{E}_{k-1}$ is an H-space and $\bar{E}_{k-1} \to E_{k-1}$ is an H-map. Therefore from (1.6), we have:

PROPOSITION 1.7. *If the k-invariants of $\bar{E}_k \to \bar{E}_{k-1}$ are primitive, then E_k and $\bar{E}_k$ are H-spaces and $\bar{E}_k \to E_k$ is an H-map.*

In fact, we will show that $\bar{E}_k$ is an r-fold loopspace, for arbitrarily large r.

If $\bar{E}_{k-1}$ is an r-fold loopspace and $\Omega^{-m}\bar{E}_{k-1}$ denotes a space such that $\Omega^m(\Omega^{-m}\bar{E}_{k-1}) = \bar{E}_{k-1}$, it suffices to show that the k-invariants of $\bar{E}_k \to \bar{E}_{k-1}$ are in the image of

$$(1.8) \qquad \Sigma^m \bar{E}_{k-1} \xrightarrow{g} \Omega^{-m}\bar{E}_{k-1} \to \Omega^{-m}BF_k$$

where g is the adjoint of the identity map $\bar{E}_{k-1} \to \bar{E}_{k-1}$.

In order to prove that the k-invariants lie in the image of (1.8), we pass to Thom complexes. Formally, it might be better to phrase the arguments in terms of spectra. We feel, that the geometric flavor of the argument is lost by this device and therefore we will not use spectra.

First observe that

$$\bar{E}_0 \to BO_r[j(t)] \to BO_r[j(t-1)]$$

is a principal fiber space for $r \geq 2^t$. We assume always that $r \geq 2^t$. Let

$$(1.9) \qquad E' = E_s' \to E_{s-1}' \to \cdots \to E_0' = BO_r[j(t)]$$

be the tower induced over E_0' from (1.4). Then the tower (1.5) is induced from (1.9) by the mapping $\bar{E}_0 \to E_0'$. Now let η_r be the canonical bundle over E_0', obtained as the induced bundle of γ_r, the universal bundle over BO_r. Consider the induced bundles over the spaces E_k' and $\bar{E}_k$. Let ME_k' and $M\bar{E}_k$ denote the corresponding Thom complexes. We have a sequence of maps

$$ME' = ME_s' \to ME_{s-1}' \to \cdots \to ME_0'. \tag{1.10}$$

Now observe that through the m-skeleton, the Serre exact sequence holds for the fiber space

$$F_k \to E_k' \to E_{k-1}'.$$

Thus if we look at the cohomology sequence

$$H^*(ME_{k-1}'/ME_k') \to H^*(ME_{k-1}') \to H^*(ME_k')$$

we see that the Thom isomorphism induces an isomorphism

$$H^*(F_k) \cong H^*(ME_{k-1}'/ME_k') \tag{1.11}$$

with a shift of $r + 1$ dimensions. The isomorphism (1.11) holds in dimensions less than or equal to m in $H^*(F_k)$. Since the Steenrod algebra A_2 acts trivially in the Thom classes, (1.11) is an A_2-isomorphism. Therefore, if we let $\widetilde{ME}_k'$ be the principal fiber space over ME_{k-1}' with fiber $\Omega^{-r}F_k$ and k-invariants $\{U \smile k_i\}$, where k_i are the k-invariants of $E_k' \to E_{k-1}'$ and U is the Thom class of ME_{k-1}' we see that ME_k' and $\widetilde{ME}_k'$ coincide at least through $r + m$ dimensions.

Let Y_q be the universal example space of an integer cohomology class x of dimension q such that

1) all primary cohomology operations vanish on x;
2) all operations with Z_2 coefficients that raise dimension by less than $j(t)$ vanish on x.

For the space Y_q we have the following result.

THEOREM 1.12. *There exists a tower of fiber spaces*

$$\begin{tikzcd} K(Z, r) & A_0 \arrow[l] & \cdots \arrow[l] & A_s \arrow[l] \\ & \Sigma^{r-n} Y_n \arrow[ul, "p"] \arrow[u, "p_0"] \arrow[urr, "p_s"'] & & \end{tikzcd}$$

The fiber G_k of $A_k \to A_{k-1}$ is a product of Eilenberg-Maclane spaces and $G_k = \Omega^{-r}F_k \times S_k$ if $k > 1$, where F_k is the $(k - 1)$-stage of an Adams resolution over Z_2 of V_n through dimension $m - 1$ and S_k is the k-stage of an Adams resolution over Z_2 of S^r. Also $G_1 = \prod_{i=1}^{t-1} K(Z_2, r + 2^i - 1) \times \Omega^{-r}F_1$. The fiber of p_s is $(m - 1)$-connected.

The proof of (1.12) is that of theorem A of [8], restricted to the dimensions stated in (1.12). (Compare Chapter II, section 3 of [8].)

Using (1.12) we can prove

THEOREM 1.13. *There is a mapping* $\lambda_{k-1}: ME'_{k-1} \to A_{k-1}$ *such that under*

$$ME'_{k-1} \xrightarrow{\lambda_{k-1}} A_{k-1} \xrightarrow{f_{k-1}} \Omega^{-r-1} F_k$$

we have

$$\lambda_{k-1}^* f_{k-1}^* (\gamma_i) = U \smile k_i ,$$

where γ_i *ranges over the fundamental classes of* $\Omega^{-r-1} F_k$ *and the* k_i *over the k-invariants of* $E_k' \to E'_{k-1}$.

Theorem (1.13) is a generalization of the principal results of [9].

The spaces E_k' are essentially the spaces E_k in the range of dimensions which are of interest. Clearly we have

PROPOSITION 1.14. *The natural maps* $E_k' \to E_k$ *for* $k = 0, 1, \cdots, s$ *induce isomorphisms* $H^*(E_k) \to H^*(E_k')$ *in dimensions* $\leq m$ *and under these isomorphisms, the k-invariants of* $E_{k+1} \to E_k$ *go over into the k-invariants of* $E_{k+1}' \to E_k'$.

We are now ready to prove the main theorem; it will be by induction. Namely we will show inductively that the spaces E_k in (1.4) are H-spaces for $k = 0, 1, \cdots, s$.

Observe that $M\bar{E}_k = \Sigma^m(\bar{E}_k \cup pt)$, since the induced bundle over $\bar{E}_k$ is trivial. Consider the diagram

$$\begin{array}{ccc} \Sigma^m \bar{E}_0 & \xrightarrow{\rho_0} & K(J_t, r + j(t-1) - 1) \\ \uparrow i_0 \quad \searrow & & \\ \quad M\bar{E}_0 & & \downarrow j_0 \\ \quad \swarrow & & \\ ME_0' & \xrightarrow{\lambda_{1,0}} & A_{1,0} \end{array} \tag{1.15}$$

where $A_{1,0}$ is the fiber space over A_0 with fiber $\prod_{i=1}^{t-1} K(Z_2, r + 2^i - 1)$. The map $\lambda_{1,0}$ is clear. The map i_0 makes the triangle commutative and ρ_0 is the adjoint of the identity map $\bar{E}_0 \to \bar{E}_0$. This leaves the map j_0. We have the diagram

$$\begin{array}{ccccc} \prod_{i=1}^{t-1} K(Z_2, r + 2^i - 1) & \to & A_{1,0} & \to & K(Z, r) \\ \uparrow p & & \uparrow & & \uparrow \\ Z_0 & \longrightarrow & ME_0' & \to & MO_r[j(t-1)] \end{array}$$

where Z_0 is the fiber of $ME_0' \to MO_r[j(t-1)]$. Let α_i be the fundamental classes of the spaces $K(Z_2, r + 2^i - 1)$ in the top row and α the fundamental class of $K(Z, r)$. Then $\tau\alpha_i = \mathrm{Sq}^{2^i}\alpha$, and p can be chosen so that $p^*\alpha_i = 0$ for $i < t - 1$. Now $p^*\alpha_{t-1} = \beta$ and $\tau\beta = U \smile W_{2^{t-1}}$. Stong in [12] has shown that β can be chosen to be a stable primary operation on the class μ, $\beta = \phi\mu$, where $\tau\mu = U \smile \gamma_{j(t-1)}$, and $\gamma_{j(t-1)} \in H^{j(t-1)}(BO_r[j(t-1)])$ is the fundamental class. Now since $\bar{E}_0$ maps trivially to $BO_r[j(t-1)]$, we have a mapping $l: \Sigma^r \bar{E}_0 \to Z_0$, which induces an isomorphism in the first nontrivial cohomology group. This gives

$$\begin{tikzcd}
 & & K(J_t, r + j(t-1) - 1) \arrow[d, "g"] \\
\Sigma^r \bar{E}_0 \arrow[r, "l"] \arrow[dr, "i_0"'] & Z_0 \arrow[ur, "h"] \arrow[d] & \Pi K(Z_2, r + 2^i - 1) \arrow[d] \\
 & ME_0' \arrow[r] & A_{1,0}
\end{tikzcd}$$

where $h^*(a) = u$, a is the fundamental class of $K(J_t, r + j(t-1) - 1)$ and $g^*\alpha_{t-1} = \phi a$. Then $p \sim gh$. This defines j_0 and hl is homotopic to the adjoint of the identity mapping $\bar{E}_0 \to \bar{E}_0$. Thus (1.15) is homotopy commutative. Assume now that we have constructed a homotopy commutative diagram

$$\begin{tikzcd}
\Sigma^r \bar{E}_{k-1} \arrow[r, "\rho_{k-1}"] \arrow[d, "i_{k-1}"'] & \Omega^{-r} \bar{E}_{k-1} \arrow[d, "j_{k-1}"] \\
ME'_{k-1} \arrow[r, "\lambda_{k-1}"] & A_{k-1}
\end{tikzcd} \tag{1.16}$$

where ρ_{k-1} is homotopic to the adjoint of the identity. Then by (1.8) and (1.13), we can take $\bar{E}_k = \Omega^r D_k$, where D_k is the induced fiber space over $\Omega^{-r}\bar{E}_{k-1}$, of the fiber space $A_k \to A_{k,0}$, relative to a lifting $J_{k-1} : \Omega^{-r}\bar{E}_{k-1} \to A_{k,0}$ of j_{k-1}. It remains, therefore, to show that given (1.16) for $k - 1$, we can produce a diagram such as (1.16) for k. This will be done in section 2.

2. The main commutativity diagram

Suppose then that we are given the following diagram:

$$\begin{tikzcd}
M\bar{E}_{k-1} \arrow[r, "\rho_{k-1}"] \arrow[d, "i_{k-1}"] & \Omega^{-r} \bar{E}_{k-1} \arrow[d, "j_{k-1}"] & \\
ME_{k-1} \arrow[r, "\lambda_{k-1}"] & A_{k-1} \arrow[r, "j_{k-1}"] & BF_k
\end{tikzcd} \tag{I_{k-1}}$$

such that I_{k-1} is homotopy commutative, where ρ_{k-1} is homotopic to the adjoint of the identity and $f_{k-1}\lambda_{k-1}i_{k-1}$ is a homotopy adjoint of the classifying map for $\bar{E}_k$. The main object of this section is then to establish;

THEOREM 2.1. *The existence of diagram I_{k-1} implies the existence of diagram I_k.*

This result will follow from the following two lemmas.

LEMMA 2.2. *Given diagram I_{k-1}, then there exists a mapping $\lambda_k : ME_k \to A_k$ such that*

$$\begin{tikzcd}
M\bar{E}_k \arrow[r, "\rho_k"] \arrow[d, "i_k"'] & \Omega^{-r}\bar{E}_k \arrow[d] \\
ME_k \arrow[r, "\lambda_k"] \arrow[d] & A_k \arrow[d] \\
ME_{k-1} \arrow[r, "\lambda_{k-1}"] & A_{k-1}
\end{tikzcd}$$

is a homotopy commutative diagram.

LEMMA 2.3. *There exist mappings g_k, g_k' so that the following diagram homotopy commutes:*

$$\begin{tikzcd}
\Sigma^q MO_n[j] \arrow[rr, "\lambda"] \arrow[dr, "g_k'"] \arrow[dd, "g'_{k-1}"'] & & \Sigma^q Y_n \arrow[d, "g_k"'] \arrow[dd, "g_{k-1}"] & \\
& ME_k' \arrow[r, "\lambda_k"] & A_k \arrow[r, "p_1"'] & LBF_k \arrow[d, "p_1"] \\
ME'_{k-1} \arrow[rr, "\lambda_{k-1}"] & & A_{k-1} \arrow[r, "f"'] & BF_k
\end{tikzcd}$$

Lemma 2.3 shows that we can take the k-invariants of $A_{k+1} \to A_k$ in such a way that they are mapped by $\lambda_k'^*$ to the k-invariants of $E_{k+1} \to E_k$ cupped with the Thom class of ME_k'. Then, using (2.2), we obtain a diagram I_k.

Proof of 2.2. The adjoint of $f_{k-1}\lambda_{k-1}i_{k-1}$ is a map $\bar{E}_{k-1} \to \Omega^r BF_k$. It induces a map $\bar{g} : \bar{E}_k \to \Omega^r LBF_k$, where LBF_k is the space of paths over BF_k, such that we have a commutative diagram

$$\begin{tikzcd}
\bar{E}_k \arrow[r, "\bar{g}"] \arrow[d] & \Omega^r LBF_k \arrow[d] \\
\bar{E}_{k-1} \arrow[r] & \Omega^r BF_k .
\end{tikzcd}$$

Let $g : M\bar{E}_k \to LBF_k$ be the adjoint of $\bar{g}$. Then we have a commutative diagram,

$$\begin{tikzcd}
M\bar{E}_k \arrow[r, "g"] \arrow[d] & LBF_k \arrow[dr] & \\
M\bar{E}_{k-1} \arrow[r] & ME_{k-1} \arrow[r, "\mu"] & BF_k
\end{tikzcd} \tag{2.4}$$

where $\mu = f_{k-1}\lambda_{k-1}$.

We can extend g to $ME_k \to LBF_k$ since LBF_k is contractible. We will also denote by g this extension. We then have

$$\begin{tikzcd} M\bar{E}_k \arrow[r] \arrow[d, "\bar{p}"'] & ME_k \arrow[r, "g"] & LBF_k \arrow[d, "\pi"] \\ M\bar{E}_{k-1} \arrow[r] & ME_{k-1} \arrow[r, "\mu"'] & BF_k \end{tikzcd}$$

Let $p: ME_k \to ME_{k-1}$; it is not necessarily true that $\mu p = \pi g$. We now define a map $\tilde{\mu}$ which is "homotopic" to μ such that $\tilde{\mu}p = \pi g$. We first replace $M\bar{E}_{k-1}$ by the mapping cylinder of $\bar{p}$, $M_{\bar{p}}$, and similarly we replace ME_{k-1} by the mapping cylinder of p, M_p.

We want to define a map

$$F: ME_k \times I \to BF_k$$

such that F is an extension of the map

$$G: M\bar{E}_k \times I \cup ME_k \times \dot{I} \to BF_k$$

where

$$G(x, t) = \begin{cases} \pi g i(x) & \text{if } x \in M\bar{E}_k \\ \pi g(x) & \text{if } x \in ME_k \text{ and } t = 0 \\ \mu p(x) & \text{if } x \in ME_k \text{ and } t = 1. \end{cases}$$

Since BF_k is a product of Eilenberg-Maclane spaces, we can use obstruction theory as given in [1, Chapter III] to prove that G can be extended to F. The obstructions are readily seen to be zero. Using F we define

$$ME_k \times I \cup_p ME_{k-1} \xrightarrow{\tilde{\mu}} BF_k$$

by

$$\begin{cases} \tilde{\mu}(x, t) = F(x, t) \\ \tilde{\mu}(y) = \mu(y). \end{cases}$$

Then

$$\tilde{\mu}p(x) = \tilde{\mu}(x, 0) = \pi g(x).$$

We may extend μ itself to $\mu': M_p \to BF_k$ by setting

$$\begin{cases} \mu'(x, t) = \mu p(x) \\ \mu'(y) = \mu(y). \end{cases}$$

Then $\tilde{\mu}$ is homotopic to μ', since we have

$$ME_{k-1} \xrightarrow{s} M_p \xrightarrow[\mu']{\tilde{\mu}} BF_k$$

and $\bar\mu s = \mu' s$, but s is a homotopy equivalence. We thus have a commutative diagram

$$\begin{tikzcd}
M\bar{E}_k \arrow[r] \arrow[d] & ME_k \arrow[r] \arrow[d] & LBF_k \arrow[d] \\
M_{\bar{p}} \arrow[r] & M_p \arrow[r, "\bar{\mu}"'] & BF_k .
\end{tikzcd}$$

We therefore obtain a commutative diagram

$$\begin{tikzcd}
M\bar{E}_k \arrow[r, "\rho_k"] \arrow[dd] & \Omega_{-}^{r} E_k \arrow[d, "j_k"'] \arrow[ddr] & \\
 & A_k \arrow[d, "\bar{f}"] & \\
ME_k \arrow[r, "{(i,\, g)}"] & \bar{\mu}^!(LBF_k) \arrow[r] & LBF_k
\end{tikzcd}$$

where $\bar{f} j_k \rho_k$ is an adjoint of $\bar{g}$, i.e. $\bar{f} j_k \rho_k = g$.

Now we want to define a mapping $\varphi : M\bar{E}_k \to \mu'^!(LBF_k)$ such that

(2.5)
$$\begin{tikzcd}
 & M\bar{E}_k \arrow[r, "\rho_k"] \arrow[d, "\varphi"'] & \Omega_{-}^{r}\bar{E}_k \arrow[d, "j_k"] & \\
 & \mu'^!(LBF_k) \arrow[r, "\bar{j}"] & A_k \arrow[r, "\bar{f}"'] & LBF_k
\end{tikzcd}$$

is commutative.

We have

$$\begin{tikzcd}
M\bar{E}_k \arrow[r, "i"] \arrow[dr, "\rho_k"'] & ME_k \arrow[r, "g"] & LBF_k \\
 & \Omega_{-}^{r}\bar{E}_k \arrow[r] & A_k \arrow[u] ;
\end{tikzcd}$$

so, if we define $\varphi : M\bar{E}_k \to \mu'^!(LBF_k)$ by

$$\varphi = (si,\, gi),$$

then $\bar{f}\bar{\jmath}\varphi(x) = gi(x)$. Now

$$\begin{tikzcd} M\bar{E}_{k-1} \arrow[r] \arrow[dr] & M_p \arrow[r] & A_{k-1} \\ & ME_{k-1} \arrow[ur] & \end{tikzcd}$$

is commutative, so that (2.5) is commutative.

Because the mappings $\bar{\mu}$ and μ' are homotopic, there is a homeomorphism $\psi: \bar{\mu}^!(LBF_k) \to \mu'^!(LBF_k)$. Consider the diagram

$$\begin{tikzcd} M\bar{E}_k \arrow[r, "\varphi"] \arrow[d, "i"'] & \mu'^!(LBF_k) \\ ME_k \arrow[r, "{(s, g)}"] & \bar{\mu}^!(LBF_k). \arrow[u, "\psi"'] \end{tikzcd}$$

We now show this diagram is homotopy commutative. Using (2.4) we see that the mappings $\mu'\bar{\imath}$ and $\bar{\mu}\bar{\imath}: M\bar{p} \to Mp \to BF_k$ coincide. Then the bundles $(\mu'\bar{\imath})^!(LBF_k)$ and $(\bar{\mu}\bar{\imath})^!(LBF_k)$ are naturally homeomorphic under the homeomorphism ψ_0 which acts as the identity.

Let H be a homotopy between μ' and $\bar{\mu}$. Then H induces a bundle X over $M_{\bar{p}} \times I$ such that above $M_{\bar{p}} \times 0$ and $M_{\bar{p}} \times 1$, we have the same space $(\bar{\mu}\bar{\imath})^!(LBF_k)$ (identified under ψ_0). Now the homeomorphism $\psi_t: X \to (\bar{\mu}\bar{\imath})^!(LBF_k) \times I$ constructed as in [11; 11.4] gives a homotopy of $\psi \mid (\bar{\mu}\bar{\imath})^!(LBF_k)$ to ψ_0. This homotopy extends to a homotopy of ψ giving a map $\psi': \bar{\mu}^!(LBF_k) \to \mu'^!(LBF_k)$ which when restricted to $(\bar{\mu}\bar{\imath})^!(LBF_k)$ is just ψ_0. Then $\psi'(s, g)i = \psi'(si, gi) = (si, gi)$ and thus φ is homotopic to $\psi(s, g)i$.

Consider

$$\begin{tikzcd} M\bar{E}_k \arrow[r, "\rho_k"] \arrow[d, "i"'] & \Omega^{-r}\bar{E}_k \arrow[d] \\ ME_k \arrow[r, "\theta\psi'(s, g)"'] \arrow[d, "p"'] & A_k \arrow[d] \\ ME_{k-1} \arrow[r, "\lambda_{k-1}"'] & A_{k-1}. \end{tikzcd}$$

The upper square homotopy commutes from the above. Finally if we take ψ instead of ψ' in the lower square,

$$\theta\psi(s, g)(e_k) = \theta\psi(se_k, ge_k) = \theta(se_k, \psi ge_k) = (\lambda_{k-1}pe_k, (\psi g)e_k)$$

and thus $\theta\psi'(s, g)$ makes the above diagram homotopy commutative. Hence taking $\theta\psi'(s, g) = \lambda_k$, we have established (2.2).

Proof of (2.3). Make λ into an inclusion. Let M_λ be the mapping cylinder of λ. Extend g_{k-1} to M_λ. Now fg_{k-1} is nul-homotopic; let H_0 be a nul-homotopy. Take any lifting $\bar{\varphi}: BSO_n[j] \to E_k$ of $BSO_n[j] \to E_{k-1}$. This lifting induces a lifting $\bar{g}_k': \Sigma^q MO_n[j] \to ME_k$ of $\bar{g}_{k-1}': \Sigma^q MO_n[j] \to ME_{k-1}$. Now $p_1\lambda_k\bar{g}_k'$ represents a null homotopy of $f\lambda_{k-1}g_{k-1}'$. Also let G be a homotopy between $g_{k-1}\lambda$ and $\lambda_{k-1}g_{k-1}'$, then G, together with $p_1\lambda_k\bar{g}_k'$, gives another nul-homotopy H_1 of $fg_{k-1}\lambda$. Since nul-homotopies are liftings

$$\begin{tikzcd} & LBF_k \arrow[d] \\ \Sigma^q MO_n[j] \arrow[ur, dashed] \arrow[r, "fg_{k-1}\lambda"'] & BF_k \end{tikzcd}$$

the two homotopies $H_0\lambda$ and H_1 determine an element $\alpha \in [\Sigma^q MO_n[j], \Omega BF_k]$, the difference class. If we take the Thom isomorphism of the map α, we obtain a map $\Phi: BO_n[j] \to \Omega^{r+1}BF_k$. There exists a lifting $\varphi: BSO_n[j] \to E_k$ whose difference class with $\bar{\varphi}$ is just the cohomology classes determined by Φ. If we now use the lifting given by φ to define g_k', then the difference class between the two liftings $p_1\lambda_k g_k'$ and H_0 is zero. Indeed,

$$\begin{tikzcd} \bar{E} \arrow[r] \arrow[d] & BSO_n[j] \arrow[d] \\ K(Z_t, j-1) \arrow[r] & BSO[j] \end{tikzcd}$$

may be induced. Then, if we look at

$$\begin{tikzcd} \Sigma^q M\bar{E} \arrow[r, "i"] & \Sigma^q MO_n[j] \arrow[r] \arrow[d] & \Sigma^q Y_n \arrow[d] \\ & M\bar{E}_k \arrow[r] & A_k \arrow[r] \arrow[d] & LBF_k \\ & & A_{k-1} \arrow[r] & BF_k, \end{tikzcd}$$

the Thom isomorphism of the map α_i restricted to $\Sigma^q M\bar{E}$ is just the adjoint of α_i. Now, because the difference cohomology classes between liftings are additive and the new φ compares correctly with the lifting going via $\Omega^r\Sigma^q Y_n$, it will do so after taking adjoints, i.e., after the Thom isomorphism. Since i^* is a monomorphism, the claim follows.

Now extend H_0 to a nul-homotopy of fg_{k-1}, which commutes with the nul-homotopy of $f\lambda_{k-1}g_{k-1}'$ given by $p_1\lambda_k\bar{g}_k$. Thus these two nul-homotopies are compatible liftings g_k' and g_k.

3. Proof of the remaining propositions

Proof of (1.6). Consider

$$\begin{array}{ccccccc} \bar{E} & = & \bar{E}_s \to & \cdots & \to \bar{E}_0 & = & K(J_t, j(t-1)-1) \\ i\downarrow & & \downarrow i_s & & \downarrow i_0 & & \\ E & = & E_{s'} \to & \cdots & \to E_0 & = & BO[j(t)] \end{array}$$

and let $\bar{p}_k:\bar{E}_k \to \bar{E}_{k-1}$ and $\bar{q}_k:\bar{E} \to \bar{E}_k$ be the fiber maps of the $\bar{E}$-tower. Let $p_k:E_k \to E_{k-1}$ and $q_k:E \to E_k$ be the corresponding ones for the E-tower.

We first observe that $H^*(E) \cong H^*(E_0) \otimes H^*(V_n)$ additively in dimensions $\leq 2^t$, so that $H^*(\bar{E}) \cong H^*(\bar{E}_0) \otimes H^*(V_n)$ in this range, and therefore $i^*:H^*(E) \to H^*(\bar{E})$ is a monomorphism in this range. Again $H^*(E_1) \cong H^*(E_0) \otimes H^*(F_1)$ and therefore $H^*(\bar{E}_1) \cong H^*(\bar{E}_0) \otimes H^*(F_1)$, so that

$$i_1^*:H^*(E_1) \to H^*(\bar{E}_1)$$

is a monomorphism in dimensions $\leq 2^t$. Moreover $q_1^*:H^*(E_1) \to H^*(E)$ and $\bar{q}_1^*:H^*(\bar{E}_1) \to H^*(\bar{E})$ are epimorphisms. Then, it follows by [4; (2.8)], that for $k \geq 1$

$$\operatorname{Ker} q_{k-1}^* = \operatorname{Ker} p_k^*$$

and

$$\operatorname{Ker} \bar{q}_{k-1}^* = \operatorname{Ker} \bar{p}_k^*.$$

Suppose that we have proved that i_j^* is a monomorphism for $j < k$ and in dimensions $< 2^t$. We have the Serre exact sequence in the fiber space

$$F_k \to E_k \xrightarrow{p_k} E_{k-1}$$

so that if $x \in H^*(E_k)$ restricts non-trivially to the fiber, $i_k^*(x) \neq 0$. Now suppose $x = p_k^*(y)$ and $i_k^*p_k^*(y) = 0$. Then $\bar{q}_k^*\bar{p}_k^*i_{k-1}^*(y) = i^*q_k^*p_k^*(y) = 0$, so $q_k^*p_k^*(y) = 0$, i.e., $q_{k-1}^*(y) = 0$, but then $p_k^*(y) = 0$ and so $x = 0$. Thus i_k^* is a monomorphism.

Proof of (1.7). Consider

$$\begin{array}{ccccc} \bar{E}_{k-1} \times \bar{E}_{k-1} & \xrightarrow{m} & \bar{E}_{k-1} & \xrightarrow{f_k} & BF_k \\ i_{k-1} \times i_{k-1}\downarrow & & \downarrow i_{k-1} & & \downarrow \text{identity} \\ E_{k-1} \times E_{k-1} & \xrightarrow{\bar{m}} & E_{k-1} & \xrightarrow[\bar{f}_k]{} & BF_k. \end{array}$$

Then if $\alpha \in H^*(BF_k)$ is a fundamental class, we assume that $(i_{k-1} \times i_{k-1})^*m^*f_k^*(\alpha) = \bar{f}_k^*\alpha \otimes 1 + 1 \otimes \bar{f}_k^*\alpha$. But since $(i_{k-1} \times i_{k-1})^*$ is a monomorphism, $m^*f_k^*\alpha = f_k^*\alpha \otimes 1 + 1 \otimes f_k^*\alpha$, and we may extend i_{k-1} to i_k such that

$$\begin{array}{ccc} \bar{E}_k \times \bar{E}_k & \to & \bar{E}_k \\ \downarrow & & \downarrow \\ E_k \times E_k & \to & E_k. \end{array}$$

Proof of (1.13). We have a mapping square

$$\begin{array}{ccc} ME' & \to & \Sigma^{r-n}Y_n \\ \downarrow & & \downarrow \\ ME_0' & \to & K(Z, r) \end{array}$$

and up to the range of dimensions of interest, the left fiber space is induced from the one on the right. Hence when we take Postnikov decompositions of both fiber spaces, they will map into each other and the k-invariants will map across. But as remarked after (1.11), the k-invariants for $ME_k' \to ME_{k-1}'$ are $\{U \smile k_i\}$, where the k_i are the k-invariants of $E_k' \to E_{k-1}'$.

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