RANDOM EVOLUTIONS AND PIECING OUT OF MARKOV PROCESSES

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Introduction

In this paper we show the equivalence of two probabilistic methods of solving certain systems of partial differential equations.

Griego and Hersh [1, 3] have shown how to solve systems of differential equations of the form

$$(1) \qquad \frac{\partial u_i}{\partial t} = A_i u_i + \sum_{j=1}^{n} q_{ij} u_j, \; i = 1, \cdots, n,$$

where each A_i is the generator of a (not necessarily stochastic) semigroup $T_i(t)$ on a single Banach space B and $Q = (q_{ij})$ is the infinitesimal matrix of a Markov chain. The system (1) is solved by a method of selecting at random by means of Markov chain from the n semigroups $T_i(t)$ thereby obtaining what is termed a "random evolution". The average of a random evolution determines a semigroup, the so-called expectation semigroup, that gives the solutions of (1). This method is somewhat analytic in that the only probabilistic mechanism employed is that of the Markov chain.

Heath [2] has given a more probabilistic interpretation of essentially the same problem. He used the idea of piecing out of Markov processes, thereby replacing the above random evolution by a "pieced process" whose semigroup is associated with the system (1).

The two respective methods have their advantages. Griego and Hersh apply their form of the solution of (1) to obtain interesting results about hyperbolic partial differential equations including a generalization of a result of Kac about the n-dimensional telegraph equation. They are also able to prove a singular perturbation theorem employing the central limit theorem in a novel way. Heath's method allowed him to study boundary value problems associated with (1) in certain cases. Heath poses and solves a Dirichlet like problem and obtains a representation theorem for the solutions. Also, his method relaxes sign restrictions on the coefficients q_{ij} that the Griego-Hersh method imposed. However, Griego and Hersh can handle more general semigroups $T_i(t)$ and their method is more explicit.

By the fact that the infinitesimal generator determines a semigroup we see that the semigroup obtained from the random evolution and the semigroup obtained from the pieced process are in fact the same semigroup. However, it would be of interest to directly obtain the random evolution from the pieced process. This

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would give insight as to how the piecing mechanism relates to the semigroup structures. In this article we carry out this connection. As a result we will see why the operators in the definition of a random evolution seem to be in reverse order of what one would naturally expect. This point until now seemed somewhat mysterious.

Notation

We use the notation of [4]. For each $i \in I = \{1, \cdots, n\}$ let $X^i = (\Omega^i, \mathcal{B}^i(t), X^i(t), \theta_t^i, P_x^i; x \in E)$ be a right continuous strong Markov process with left limits and state space E, where E is a locally compact Hausdorff space with a countable base. Thus the processes X^i have a common state space. Let $\bar{E} = E \cup \{\Delta\}$ be the one-point compactification of E if E is not compact, otherwise let Δ be an isolated point. For simplicity we assume the sample spaces Ω^i are mutally disjoint and, furthermore, assume $\zeta^i \equiv \infty$ where ζ^i is the lifetime of X^i, that is, the processes X^i are conservative.

The composite process

We construct a new process X^0 from the X^i-processes as follows.

Let $\Omega^0 = \bigcup_{i=1}^n \Omega^i$ and $\mathcal{B}^0(t) = \sigma(\bigcup_{i=1}^n \mathcal{B}^i(t))$, the smallest σ-algebra containing the $\mathcal{B}^i(t)$. Also, let $X^0(t, \omega^0) = (X^i(t, \omega^i), i)$ and $\theta_t^0(\omega^0) = \theta_t^i(\omega^i)$ if $\omega^0 = \omega^i \in \Omega^i$. We define $P_{(x,i)}^0(\Lambda) = P_x^i(\Lambda \cap \Omega^i)$ for $\Lambda \in \mathcal{B}^0(\infty) = \sigma(\bigcup_t \mathcal{B}^0(t))$, $x \in E$ and $i \in I = \{1, \cdots, n\}$. Finally, let $\zeta^0 = \infty$.

We easily see that $X^0 = (\Omega^0, \mathcal{B}^0(t), X^0(t), \theta_t^0, P_{(x,i)}^0; (x, i) \in E \times I)$ is a right continuous, conservative, strong Markov process with left limits and state space $S = E \times I$. X^0 is called the *composite process* formed from the X^i's.

The killed process

Let $V = (W, \mathcal{B}(t), v(t), \theta_t, P_i; i \in I)$ be a right continuous (stationary) Markov chain with state space $I = \{1, \cdots, n\}$.

We "kill" the composite process X^0 by means of the chain V as follows: let τ_1 be the first jump time for V.

We define $\tilde{\Omega} = \Omega^0 \times W$, $\tilde{\mathcal{B}}(t) = \mathcal{B}^0(t) \times \mathcal{B}(t)$, $\tilde{\theta}_t(\tilde{\omega}) = (\theta_t^0(\omega^0), \theta_t(w))$. Furthermore,

$$\tilde{X}(t, \tilde{\omega}) = X^0(t, \omega^0) \text{ if } t < \tau_1(w) \text{ where } \tilde{\omega} = (\omega^0, w) \text{ or } \tilde{X}(t, \tilde{\omega}) = \Delta \text{ otherwise.}$$

Let $\tilde{P}_{(x,i)}(\Lambda \times \Gamma) = P_{(x,i)}^0(\Lambda) \cdot P_i(\Gamma)$, for $\Lambda \in \mathcal{B}^0(\infty)$, $\Gamma \in \mathcal{B}(\infty) = \sigma(\bigcup_t \mathcal{B}(t))$ and $(x, i) \in S$. Finally, we let $\tilde{\zeta}(\tilde{\omega}) = \tau_1(w)$.

It is well known that $\tilde{X} = (\tilde{\Omega}, \tilde{\mathcal{B}}(t), \tilde{X}(t), \tilde{\theta}_t, \tilde{P}_{(x,i)}; (x, i) \in S)$ is a right continuous strong Markov process with left limits and state space $S = E \times I$. The lifetime of $\tilde{X}$ is $\tilde{\zeta}$. We call $\tilde{X}$ the *killed process*.

The instantaneous distribution

Below we will want to piece out the killed process $\tilde{X}$ to obtain a new process of interest. In order to accomplish this we must construct an instantaneous distribution. We give the following definition relative to our process $\tilde{X}$, although it is clear that the definition holds for more general Markov processes.

Definition. A real-valued function $\mu(\tilde{\omega}, A)$ on $\tilde{\Omega} \times S$ is an instantaneous distribution for $\tilde{X}$ if

(i) for each $\tilde{\omega} \in \tilde{\Omega}$, $\mu(\tilde{\omega}, \cdot)$ is a probability measure on $\mathcal{B}(S)$, the Borel sets of S, where we put the natural product topology on S;

(ii) for each $A \in \mathcal{B}(S)$, $\mu(\cdot, A)$ is an $\tilde{\mathcal{B}}(\infty)$-measurable function on $\tilde{\Omega}$; and

(iii) for each $\tilde{\mathcal{B}}(t)$-Markov time $\tilde{T}$ and each $(x, i) \in S$, we have,

$$\tilde{P}_{(x,i)}(\mu(\tilde{\omega}, \cdot) = \mu(\theta_{\tilde{T}}\tilde{\omega}, \cdot), \tilde{T} < \tilde{\zeta}) = \tilde{P}_{(x,i)}(\tilde{T} < \tilde{\zeta}).$$

We now construct an instantaneous distribution for $\tilde{X}$. For $\tilde{\omega} \in \tilde{\Omega}$, $A = B \times J \in \mathcal{B}(S)$ where B is a Borel set in E and $J \subseteq I$, we define

$$\mu(\tilde{\omega}, B \times J) = I_{B\times J}(X^{v(0)}(\tilde{\zeta}^{-}), v(\tilde{\zeta})), \tag{2}$$

that is, if $\tilde{\omega} = (\omega^i, w)$ and $v(o, w) = i$, then

$$\mu(\tilde{\omega}, B \times J) = I_{B\times J}(X^i(\tau_1^{-}(w), \omega^i), v(\tau_1(w), w)),$$

where $\tilde{\zeta}^{-}$ and τ_1^{-} denote the left hand limts of $\tilde{\zeta}$ and τ_1, and $I_{B\times J}$ is the indicator function of $B \times J$. By the assumption of the existence of left limits for X^i our definition makes sense.

THEOREM 1. *μ is an instantaneous distribution for $\tilde{X}$.*

Proof. Properties (i) and (ii) are clearly satisfied. Now, let $\tilde{T}$ be a $\tilde{\mathcal{B}}(t)$-Markov time. If $\tilde{T} < \tilde{\zeta}$ then $\tilde{\zeta} \circ \theta_{\tilde{T}} = \tilde{\zeta} - \tilde{T}$. Thus, if $\tilde{T} < \tilde{\zeta}$

$$X^{v(0,\theta_{\tilde{T}})}(\tilde{\zeta}^{-} \circ \theta_{\tilde{T}}, \theta_{\tilde{T}}) = X^{v(\tilde{T})}(\tilde{\zeta}^{-} - \tilde{T}, \theta_{\tilde{T}}) = X^{v(0)}(\tilde{\zeta}^{-})$$

and

$$v(\tilde{\zeta} \circ \theta_{\tilde{T}}, \theta_{\tilde{T}}) = v(\tilde{\zeta} - \tilde{T}, \theta_{\tilde{T}}) = v(\tilde{\zeta}).$$

Hence, if $\tilde{T} < \tilde{\zeta}$ we have $\mu(\tilde{\omega}, \cdot) = \mu(\theta_{\tilde{T}}\tilde{\omega}, \cdot)$, and this clearly implies (iii), thus showing that μ is an instantaneous distribution.

The pieced process

Based upon the above instantaneous distribution we can now construct our final process, the so-called pieced process.

Let $W' = W_i' = \tilde{\Omega} \times S$ and $\mathcal{B}' = \mathcal{B}_i' = \tilde{\mathcal{B}}(\infty) \times \mathcal{B}(S)$ for $i = 1, 2, \cdots$, where $\tilde{\mathcal{B}}(\infty) = \sigma(\bigcup_t \tilde{\mathcal{B}}(t))$ and $\mathcal{B}(S) =$ Borel sets of S. We define $\Omega = \bigcup_{i=1}^{\infty} W_i'$ and $\mathcal{B} = \bigcup_{i=1}^{\infty} \mathcal{B}_i'$.

Furthermore, we define a stochastic kernel $Q((x, i), dw') = Q_{(x,i)}(dw')$ on $S \times W'$ by

$$Q_{(x,i)}(A) = \int_A \int \tilde{P}_{(x,i)}(d\tilde{\omega})\mu(\tilde{\omega}, d(x, i)) \tag{3}$$

for $(x, i) \in S = E \times I$ and $A \subseteq W' = \tilde{\Omega} \times S$, $A \in \mathcal{B}' = \tilde{\mathcal{B}}(\infty) \times \mathcal{B}(S)$.

As shown in [4] we have the following result.

THEOREM 2. *There exists a unique system* $\{P_{(x,i)}, (x, i) \in S\}$ *of probability measures on* $(\Omega, \mathcal{B})$ *such that for each* $n \geq 1$

$$P_{(x,i)}(dw_1' \times dw_2' \times \cdots \times dw_n') = Q_{(x,i)}(dw_1')Q_{(x_1,i_1)}(dw_2') \cdots Q_{(x_{n-1},i_{n-1})}(dw_n') \tag{4}$$

where

$$w_j' = (\tilde{\omega}_j, (x_j, i_j)) \in W', j = 1, \cdots, n.$$

Now, let $\dot{X}(t, w') = \tilde{X}(t, \tilde{\omega})$ if $t < \tilde{\zeta}(\tilde{\omega})$ and $\dot{X}(t, w') = (x, i)$ if $t \geq \tilde{\zeta}(\tilde{\omega})$, where $w' = (\tilde{\omega}, (x, i))$.

We now define the sample functions for the pieced process. First, for $\omega = (w_1', w_2', \cdots) \in \Omega$ where $w_j' = (\tilde{\omega}_j, (x_j, i_j)) \in W'$ we define $N(\omega) = \inf \{j : \tilde{\zeta}(\omega_j) = 0\}$, or $+\infty$ if the set in brackets is empty. Also, let $\sigma_n(\omega) = \sum_{j=1}^{n} \tilde{\zeta}(\tilde{\omega}_j)$. Finally we define

$$X(t, \omega) = \begin{cases} \dot{X}(t, w_1') & \text{if } 0 \leq t \leq \sigma_1(\omega) \\ \dot{X}(t - \sigma_1(\omega), w_2') & \text{if } \sigma_1(\omega) < t \leq \sigma_2(\omega) \\ \vdots & \\ \dot{X}(t - \sigma_n(\omega), w_{n+1}') & \text{if } \sigma_n(\omega) < t \leq \sigma_{n+1}(\omega) \\ \vdots & \\ \Delta & \text{if } t \geq \sigma_{N(\omega)}(\omega). \end{cases} \tag{5}$$

It is shown in [4] how to define σ-algebras $\mathcal{B}(t)$ on Ω and θ_t so that $X = (\Omega, \mathcal{B}(t), X(t), \theta_t, P_{(x,i)}, (x, i) \in E \times I)$ is a right continuous strong Markov process with left limits and state space $E \times I$. Clearly $\zeta(\omega) = \sigma_{N(\omega)}(\omega)$ is the lifetime of X. We call X the *pieced process* obtained from the processes X^i, $i = 1, \cdots, n$, by means of the Markov chain V. We emphasize that $E \times I$, not E, is the state space of X.

In spite of all the complications, the description of the pieced process is intuitively very simple. If the Markov chain starts out in its i^{th} state then we begin our pieced process X in the i^{th} Markov process X^i and let it run until the time τ_1 of the first jump of the chain (this is the first piece of X), whereupon if the chain jumps to the j^{th} state, X switches to X^j and stays in this process until the next jump of the chain and so on. The instantaneous distribution merely implies that the new piece takes up at the point in E where the old piece left off.

The random evolution

We now introduce the analytic structure of our study. We retain our previous notation.

Let $B(E)$ be the Banach space of bounded real valued $\mathcal{B}$-measurable functions on E, where $\mathcal{B}$ is the σ-algebra of Borel sets of E. The semigroup $\{T_i(t), t \geq 0\}$ of X^i is given by $T_i(t)f(x) = E_x^i[f(X^i(t))]$, for $i \in I$, and $t \geq 0$, $x \in E$, where $f \in \boldsymbol{B}(E)$. We have $T_i(t):\boldsymbol{B}(E) \to \boldsymbol{B}(E)$.

Furthermore we assume each semigroup $\{T_i(t), t \geq 0\}$ is *strongly continuous* on $\boldsymbol{B}(E)$.

Let $\tau_1(w)$, $\tau_2(w)$, $\cdots$ be the successive jump times (for the path w) of the Markov chain V. Also let $N(t, w)$ be the number of jumps for V in the time interval $[o, t]$.

For $t \geq 0$, we define $M(t, w)$, $w \in W$, by

$$M(t) = T_{v(0)}(\tau_1)T_{v(\tau_1)}(\tau_2 - \tau_1) \cdots T_{v(\tau_{N(t)})}(t - \tau_{N(t)}). \tag{6}$$

The family of random operators $\{M(t), t \geq 0\}$ is called the *random evolution* associated with the semigroups T_i and the Markov chain V. The random evolution describes the selection by means of the Markov chain from the n laws of evolutions corresponding to the semigroups. The evolution stays in a semigroup until the chain jumps to a new state whereupon the evolution evolves in the corresponding new semigroup, except that the last semigroup (up to time t) operates first and so on back to the initial semigroup.

As explained in [3] the random evolution $M(t)$ determines a semigroup on $\boldsymbol{B}(E \times I)$, the Banach space of bounded measurable functions on $E \times I$, where the obvious product topology is introduced into $E \times I$. Indeed, if $f \in \boldsymbol{B}(E \times I)$ then

$$\tilde{T}(t)f(x, i) = E_i[M(t)f(x, v(t))] \tag{7}$$

defines the semigroup $\{\tilde{T}(t), t \geq 0\}$ where the expectation refers to the chain starting in state i.

However, the pieced Markov process X determines its own semigroup $\{T(t), t \geq 0\}$ on $\boldsymbol{B}(E \times I)$. Given $f \in \boldsymbol{B}(E \times I)$ we have

$$T(t)f(x, i) = E_{(x,i)}[f(X(t))] \tag{8}$$

where the expectation refers to the pieced process starting in state (x, i).

The main theorem

The main result of this article states that the semigroups $\tilde{T}(t)$ and $T(t)$ are in fact the same semigroup. Thus we have the following theorem.

Main Theorem. $\{\tilde{T}(t), t \geq 0\}$, *the semigroup determined by the random evolution* $\{M(t), t \geq 0\}$ *and given by* $\tilde{T}(t)f(x, i) = E_i[M(t)f(x, v(t))]$ *is equal to the semigroup* $\{T(t), t \geq 0\}$ *determined by the pieced process* X *that is given by* $T(t)f(x, i) = E_{(x,i)}[f(X(t))]$, *where* $f \in \boldsymbol{B}(E \times I)$.

The proof will depend on the following lemma.

LEMMA. *Let* $\varphi(w, \cdots, w; x, i)$ *be a measurable function on* $W \times \cdots \times W \times$

$E \times I$, where the product involving W is an n-fold product. Then

$$\int_{W_1'} Q_{(x,i)}(dw_1') \int_{W_2'} Q_{(x_1,i_1)}(dw_2') \cdots \int_{W_n'} Q_{(x_{n-1},i_{n-1})}(dw_n') \cdot\varphi(w_1, w_2, \cdots, w_n; x_n, i_n) = E_i[T_{v(0)}(\tau_1)T_{v(\tau_1)}(\tau_2 - \tau_1) \cdots \cdot T_{v(\tau_{n-1})}(\tau_n - \tau_{n-1})\varphi(w, \theta_{\tau_1} w, \cdots, \theta_{\tau_{n-1}} w; x, v(\tau_n))]$$

where τ_j is the j^{th} jump time of the chain V.

Proof. We first consider the case $n = 1$, so that $\varphi(w; x, i)$ is a function on $W \times E \times I$. Assume φ is of the form $\varphi = I_{\Lambda\times B\times J}$, where Λ, B and J are measurable sets in the respective spaces W, E, and I. Then

$$\begin{aligned}
&\int_{W_1'} Q_{(x,i)}(dw_1')\varphi(w_1; x_1, i_1) \\
&= \int_W P_i(dw_1) \int_{\Omega^i} P_x^i(d\omega_1^i) I_\Lambda(w_1)\mu(\omega_1^i, w_1, B \times J) \\
&= \int_W P_i(dw_1) \int_{\Omega^i} P_x^i(d\omega_1^i)\, I_{\Lambda\times B\times J}(w_1, X^{v(0,w_1)}(\tau_1^-(w_1), \omega_1^i), v(\tau_1(w_1), w_1)) \\
&= E_i\{E_x^i[I_{\Lambda\times B\times J}(w_1, X^{v(0,w_1)}(\tau_1^-(w_1), \omega_1^i), v(\tau_1(w_1), w_1))]\} \\
&= E_i[T_{v(0)}(\tau_1^-) I_{\Lambda\times B\times J}(w_1, x, v(\tau_1))] \\
&= E_i[T_{v(0)}(\tau_1)\varphi(w_1, x, v(\tau_1))],
\end{aligned}$$

by the strong continuity of $T_i(t)$. The result for general $\varphi(w; x, i)$ now follows easily. Now suppose $n = 2$. Applying the above to the function

$$\psi(w_1; x_1, i_1) = \int_{W_2'} Q_{(x_1,i_1)}(dw_2')\varphi(w_1, w_2; x_2, i_2)$$

we have

$$\begin{aligned}
I &\equiv \int_{W_1'} Q_{(x,i)}(dw_1') \int_{W_2'} Q_{(x_1,i_1)}(dw_2')\varphi(w_1, w_2; x_2, i_2) \\
&= E_i[T_{v(0,w_1)}(\tau_1(w_1)) \int_{W_2'} Q_{(x,v(\tau_1(w_1),w_1))}(dw_2')\varphi(w_1, w_2, x_2; i_2)] \\
&= E_i\{T_{v(0,w_1)}(\tau_1(w_1))E_{v(\tau_1(w_1),w_1)}[T_{v(0,w_2)}(\tau_1(w_2))\varphi(w_1, w_2; x, v(\tau_1(w_2), w_2))]\}.
\end{aligned}$$

Let $F(w_2) = T_{v(0,w_2)}(\tau_1(w_2))\varphi(w_1, w_2; x, v(\tau_1(w_2), w_2))$, where w_1 is fixed. We easily check that $\tau_1 \circ \theta_{\tau_1} = \tau_2 - \tau_1$, so that

$$F(\theta_{\tau_1}(w_2)) = T_{v(\tau_1(w_2),w_2)}(\tau_2(w_2) - \tau_1(w_2))\varphi(w_1, \theta_{\tau_1}w_2; x, v(\tau_2(w_2), w_2)).$$

By the strong Markov property of the chain V we have

$$E_{v(\tau_1(w_1),w_1)}[F(w_2)] = E_i[F(\theta_{\tau_1}(w_2)) \mid \mathfrak{F}_{\tau_1}](w_1).$$

Hence,

$$\begin{aligned}
I &= E_i[T_{v(0)}(\tau_1)E_i[T_{v(\tau_1)}(\tau_2 - \tau_1)\varphi(\cdot, \theta_{\tau_1}\cdot; x, v(\tau_2)) \mid \mathfrak{F}_{\tau_1}]] \\
&= E_i[T_{v(0)}(\tau_1)T_{v(\tau_1)}(\tau_2 - \tau_1)\varphi(w, \theta_{\tau_1}w; x, v(\tau_2))]
\end{aligned}$$

finishing the proof for $n = 2$.

The proof in general proceeds in the same manner, so we omit it.

Returning to the proof of the theorem we have

$$T(t)f(x,i) = E_{(x,i)}[f(X(t))]$$
$$= \sum_{n=0}^{\infty} E_{(x,i)}[f(X(t)); \tilde{N}(t) = n]$$

where we define $\tilde{N}(t, \omega)$ by $\sigma_{\tilde{N}(t,\omega)}(\omega) \leq t < \sigma_{\tilde{N}(t,\omega)}(\omega)$, so that $\tilde{N}(t)$ is equal to the number of piecings up to time t.

Recall that for $\omega = (w_1', w_2', \cdots)$ we have, $\sigma_n(\omega) = \sum_{j=1}^{n} \tilde{\zeta}(\tilde{\omega}_j) = \sum_{j=1}^{n} \tau_1(w_j)$ where $\tilde{\omega}_j = (w_j^0, w_j)$. Also

$$X(t, \omega) = \dot{X}(t - \sigma_n(\omega), w'_{n+1}) \text{ if } \sigma_n(\omega) < t \leq \sigma_{n+1}(\omega).$$

Now, $\Lambda_n \equiv \{\omega: \tilde{N}(t, \omega) = n\} = \{\omega: \sum_{j=1}^{n} \tau(w_j) \leq t < \sum_{j=1}^{n+1} \tau_1(w_j)\}$. On this set, $t - \sum_{j=1}^{n} \tau_1(w_j) < \tau_1(w_{j+1})$, so that by the definition of $\dot{X}$ we have on this set $\dot{X}(t - \sigma_n(\omega), w_{n+1}') = X^0(t - \sigma_n(\omega), \omega_{n+1}^0)$, or $\dot{X}(t - \sigma_n(\omega), w_{n+1}') = (X^i(t - \sum_{j=1}^{n} \tau_1(w_j), \omega_{n+1}^i), i)$ for starting point (x,i).

Hence,

$$E_{(x,i)}[f(X(t)); \tilde{N}(t) = n]$$
$$= \int_{W_1'} Q_{(x,i)}(dw_1') \cdots \int_{W_n'} Q_{(x_{n-1},i_{n-1})}(dw_n')$$
$$\cdot \int_{W_{n+1}'} Q_{(x_n,i_n)}(dw_{n+1}') f(X^i(t - \sum_{j=1}^{n} \tau_1(w_j)), i) \cdot I_{\Lambda_n}$$

Now let

$$\varphi(w_1, \cdots, w_n; x_n, i_n)$$
$$= \int_{W_{n+1}'} Q_{(x_n,i_n)}(dw_{n+1}') f(X^i(t - \sum_{j=1}^{n} \tau_1(w_j)), i) \cdot I_{\Lambda_n}.$$

By the lemma we see that

$$E_{(x,i)}[f(X(t)); \tilde{N}(t) = n]$$
$$= E_i[T_{v(0)}(\tau_1) \cdots T_{v(\tau_{n-1})}(\tau_n - \tau_{n-1})\varphi(w, \theta_{\tau_1}w, \cdots, \theta_{\tau_{n-1}}w; x, v(\tau_n))]$$

However, in general $\tau_1 \circ \theta_{\tau_{j-1}} = \tau_j - \tau_{j-1}$, so that $\sum_{j=1}^{n} (\tau_1 \circ \theta_{\tau_{j-1}}) = \tau_n$, where we define $\tau_0 = 0$.

Thus, for fixed w,

$$\varphi(w, \theta_{\tau_1}w, \cdots, \theta_{\tau_{n-1}}w; x, v(\tau_n(w), w))$$
$$= \int_{W_{n+1}'} Q_{(x,v(\tau_n(w),w))}(dw_{n+1}') f(X^i(t - \tau_n(w), \omega_{n+1}^i), i) I_{\Lambda_n'}$$

where

$$I_{\Lambda_n'} = \{w_{n+1}: \tau_n(w) \leq t < \tau_n(w) + \tau_1(w_{n+1})\}.$$

Since $Q_{(x,i)}(dw') = P_x^i(d\omega^i) P_i(dw) \mu(\tilde{\omega}, d(x, i))$, we have letting $v_n \equiv v(\tau_n(w), w)$ that

$$\varphi(w, \theta_{\tau_1}w, \cdots, \theta_{\tau_{n-1}}w; x, v_n) = E_x^{v_n}[f(X^{v_n}(t - \tau_n(w), \omega_{n+1}^{v_n}), v_n)] P_{v_n}(\Lambda_n')$$
$$= T_{v_n}(t - \tau_n) f(x, v_n) P_{v_n}(\Lambda_n').$$

We note by the strong Markov property of the chain V that

$$P_{v_n}(\Lambda_n') = P_i(\{\tau_n \leq t < \tau_{n+1}\} \mid \mathfrak{F}_{\tau_n}).$$

Therefore

$$\begin{aligned} &E_{(x,i)}[f(X(t)); \tilde{N}(t) = n] \\ &= E_i[T_{v(0)}(\tau_1) \cdots T_{v(\tau_{n-1})}(\tau_n - \tau_{n-1}) T_{v(\tau_n)}(t - \tau_n) f(x, v(\tau_n)) E_i[I_{\Gamma_n} \mid \mathfrak{F}_{\tau_n}]] \\ &= E_i[T_{v(0)}(\tau_1) \cdots T_{v(\tau_n)}(t - \tau_n)\, f(x, v(\tau_n)); \tau_n \leq t < \tau_{n+1}] \\ &= E_i[T_{v(0)}(\tau_1) \cdots T_{v(\tau_n)}(t - \tau_n) f(x, v(t)); N(t) = n] \end{aligned}$$

where $N(t, w)$ is the number of jumps in $[0, t]$ for the chain V.

Summing over n, we obtain finally that

$$\begin{aligned} T(t)f(x, i) &\equiv E_{(x,i)}[f(X(t))] \\ &= E_i[T_{v(0)}(\tau_1) T_{v(\tau_1)}(\tau_2 - \tau_1) \cdots T_{v(\tau_{N(t)})}(t - \tau_{N(t)}) f(x, v(t))] \\ &= E_i[M(t) f(x, v(t))] \\ &\equiv \tilde{T}(t) f(x, i), \end{aligned}$$

terminating the proof of the theorem.

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