ON CERTAIN NON-HOMOTOPY ABELIAN LOOP SPACES

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Introduction

In [3] an example is given of an H-space X such that ΩX has the homotopy type of a product of Eilenberg-MacLane spaces, but when ΩX is considered as an H-space with loop multiplication, it does not split this way. The example in [3] is homotopy abelian In this note we give examples Y with the property that ΩY is not homotopy abelian under loop multiplication, but ΩY has the homotopy type of a product of Eilenberg-MacLane spaces.

The examples

Let Q_i be the unique element of the mod 2 Steenrod algebra A satisfying the following conditions.

a) degree $Q_i = 2^i - 1$

b) $Q_i^2 = 0$

c) Q_i is primitive.

Let $E_{n,i}$ be the principal fibre space over $K(Z_2, n)$ with fibre $K(Z_2, m - 1)$ and k-invariant $b_n \cup Q_i b_n$ where $m = 2n + 2^i - 1$ and b_n is the fundamental class of $K(Z_2, n)$. We assume $n \geq 2^i - 1$.

Our examples are the spaces $\Omega E_{n,i}$. We clearly have

$$\Omega E_{n,i} \simeq K(Z_2, n - 1) \times K(Z_2, m - 2)$$

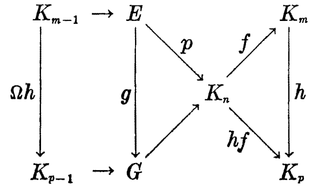
if we ignore the H-structure.

The Theorem

The fact that $\Omega E_{n,i}$ is not homotopy abelian is a consequence of the following theorem giving the structure of $H^* (\Omega E_{n,i}; Z_2)$ as a Hopf algebra over Z_2.

Theorem. *There exists a class $e \in H^{m-2}(\Omega E_{n,i}; Z_2)$ which restricts to the fundamental class of the fibre and with coproduct* $1 \otimes e + b_{n-1} \otimes Q_i b_{n-1} + e \otimes 1$.

Proof. To simplify notation, we drop the subscripts n,i and write $K(Z_2, n)$ as K_n. We also delete the coefficient group which is always Z_2. Define a map $h{:}K_m \to K_p$ by $h^*(b_p) = Qb_m$. Then the composite hf satisfies $f^*h^*(b_p) = (Qb_n)^2 = \mathrm{Sq}^q Qb_n$, with $q = p/2$. Since $n \geq 2^i - 1$, $\mathrm{Sq}^q Q$ is admissible. Furthermore hf is an H-map. Consider the following commutative diagram of fibre spaces,



G is the principal fibre space over K_n induced by hf. The map $g: E \to G$ exists because hfp is homotopically trivial. Now consider the functor Ω applied to (1). $\Omega(hf)$ is homotopically trivial. Hence, ignoring the H-structure, we have

$$\Omega G \simeq K_{n-1} \times K_{p-2} .$$

We also have an H-map $\Omega g: \Omega E \to \Omega G$. The Serre spectral sequence for the path-loop fibration $\Omega G \to PG \to G$ is a spectral sequence of Hopf algebras over Z_2. A straightforward argument (as in [3] or [1]) gives an element $w \in H^{p-2}(\Omega G)$ restricting to the fundamental class of the fibre and having coproduct

$$1 \otimes w + Qb_{n-1} \otimes Qb_{n-1} + w \otimes 1.$$

(This uses the admissibility of $\mathrm{Sq}^q Q$). We shall use this information and the H-map Ωg to make inferences about the H-structure of ΩE. We claim there exists a class $e \in H^{m-2}(\Omega E)$ restricting to the fundamental class of the fibre and with the property that

$$Qe = \Omega g^*(w).$$

To see this consider any class $e' \in H^{m-2}(\Omega E)$ which restricts to the fundamental class of the fibre. Then a glance at the fundamental sequence [2] of ΩE shows

$$\Omega g^*(w) = Qe' + \Omega p^*(r)$$

for some $r \in H^{p-2}(K_{n-1})$. Write the coproduct of e' as

(2) $$1 \otimes e' + \Sigma_i \Theta_i b \otimes \psi_i b + e' \otimes 1 \quad \text{with } \Theta_i, \psi_i \in A,$$

and that of r as

(3) $$1 \otimes r + \Sigma_j r_j' \otimes r_j'' + r \otimes 1, r_j', r_j'' \in H^*(K_{n-1}).$$

It is a known fact that $Qb \otimes Qb$ does not appear as a summand of (3).
Thus we have

(4) $$Qb \otimes Qb = \Sigma_i Q\Theta_i b \otimes \psi_i b + \Sigma_i \Theta_i b \otimes Q\psi_i b + \Sigma_j r_j' \otimes r_j''$$

By considering degrees we obtain the fact that if degree Θ_i, degree ψ_i are both positive, then $Q\Theta_i b \otimes \psi_i b$ appears in $\Sigma_j r_j' \otimes r_j''$. Since $r_j'' \otimes r_j'$ also appears as a summand of $\Sigma_j r_j' \otimes r_j''$ we know that $\psi_i b \otimes Q\Theta_i b$ appears as a summand of (4). Similarly for each appearance of $\Theta_i b \otimes Q\psi_i b$ in (4) we have the appearance of $Q\psi_i b \otimes \Theta_i b$. From these two facts, we obtain the fact that *both* $\Theta_i b \otimes \psi_i b$ and $\psi_i b \otimes \Theta_i b$ appear in (2) if degree Θ_i and degree ψ_i are positive. Hence we can find an element s obtained by summing various $\Theta_i b \cup \psi_i b$ such that

$$r = Qs.$$

Set $e = e' + s$ and we obtain $Qe = \Omega g^*(w)$. By the analysis above we see that the coproduct of e is of the form

$$1 \otimes e + \Sigma_i \Theta_i b \otimes b + \Sigma_j b \otimes \psi b + e \otimes 1.$$

In view of the formula for the coproduct of $Qe = \Omega g^*(w)$, a degree argument limits the possibilities to just two,

$$1 \otimes e + Qb \otimes b + e \otimes 1$$

$$\text{or } 1 \otimes e + b \otimes Qb + e \otimes 1.$$

The result follows by adding $b \cup Qb$ to e if necessary.

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REFERENCES

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