A REMARK ON ASYMPTOTIC STABILITY

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1. Introduction

Let A be a bounded, linear operator from the Hilbert space H into itself. Henceforth we will assume that A is self-adjoint and *positive*, i.e., $\langle Au, u\rangle > 0$ for all $u \neq 0$. If there is a constant $c > 0$ such that $\langle Au, u\rangle \geq c \| u \|^2$, then A is said to be *strongly positive*. If H has finite dimension, both definitions are clearly equivalent. Let us consider the differential equation

$$\dot{x} = -Ax, \quad x(0) = x. \tag{1.1}$$

Its solution, as is well known, is

$$x(t) = e^{-At}x. \tag{1.2}$$

A sufficient condition for the convergence of e^{-At} to the zero operator when $t \rightarrow \infty$ is that A be strongly positive. This note has been prompted by a remark made by M. K. Gavurin [1; p. 14] to the effect that "this condition is also close to being necessary". We will prove in this note the following

THEOREM 1.3. *Let A be a self-adjoint, positive operator from the Hilbert space H into itself. Then $e^{-At}u$ tends to zero as $t \rightarrow \infty$ for every $u \in H$.*

The proof depends on some known facts about the spectrum of linear operators. For the sake of completeness we will indicate below the main definitions and theorems which will be used in the proof. The standard reference will be Taylor [2].

2. A review of spectral theory.

If A is an operator from H into H we define:

(a) $c\sigma(A)$ (the continuous spectrum of A) is the set of all complex numbers λ such that the range of $\lambda I - A$ is dense in H and $(\lambda I - A)^{-1}$ exists but is not continuous.

(b) $R\sigma(A)$ (the residual spectrum) is the set of all complex numbers λ such that $(\lambda I - A)^{-1}$ exists but the range of $\lambda I - A$ is not dense in H.

(c) $P\sigma(A)$ (the point spectrum) is the set of all complex λ such that $(\lambda I - A)u = 0$ for some $u \neq 0$.

(d) $\sigma(A)$ (the spectrum of A) is the union of the sets defined in (a), (b) and (c).

(e) $\rho(A)$ (the resolvent set of A) is the set of all λ such that range of $\lambda I - A$ is dense in H and $(\lambda I - A)^{-1}$ exists and is continuous.

LEMMA 2.1. *If $A:H \rightarrow H$ is positive but not strongly positive, then $0 \in C\sigma(A)$.*

Proof. If $Au = 0$ for some $u \neq 0$ then $\langle Au, u\rangle = 0$ which is impossible since A

is positive. Therefore A^{-1} exists. We show next that the range R of A is dense in H. If it is not so, we can find $v \neq 0$ in the orthogonal complement of $\bar{R}$. This would imply that $\langle Av, v\rangle = 0$. This is impossible because A is positive.

THEOREM 2.2 [2; Th 6.2-B, p. 330]. *Let A be bounded and self-adjoint and let*

$$m(A) = \inf_{\|x\|=1} \langle Ax, x\rangle, \quad M(A) = \sup_{\|x\|=1} \langle Ax, x\rangle \tag{2.3}$$

Then $\sigma(A)$ is contained in the closed interval $[m(A), M(A)]$. The end points belong to $\sigma(A)$.

Let $\{E_\lambda\}$ be a resolution of the identity for the operator A, i.e., for each λ E_λ is a projection of H into H and
i) $E_\lambda E_\mu = E_\mu E_\lambda$ if $\lambda \leq \mu$
ii) The function $\lambda \rightarrow E_\lambda$ is right continuous
iii) $E_\lambda = 0$ if $\lambda < m(A)$, $E_\lambda = I$ if $M(A) \leq \lambda$
iv) $E_\lambda A = AE_\lambda$
Every bounded self-adjoint operator admits a resolution of the identity.

If A is self-adjoint and $f:[\alpha, \beta] \rightarrow$ Reals is continuous, $\alpha < m < M \leq \beta$, then we can define the operator

$$f(A) = \int_\alpha^\beta f(\lambda)\, dE_\lambda \tag{2.4}$$

where the integral is interpreted as the limit of operator-valued Riemann-Stieltjes sums. Then one can show [2 Th 6.5-C, p. 351] that $f(A)$ is self-adjoint and

$$\| f(A)u \|^2 = \int_\alpha^\beta | f(\lambda) |^2 \, d \, \| E_\lambda u \|^2 \tag{2.5}$$

LEMMA 2.6 *Assume that $A:H \rightarrow H$ is self-adjoint and positive. Let $\{E_\lambda\}$ be a resolution of the identity for A. Then $E_0 = 0$ and $\lambda \rightarrow E_\lambda$ is continuous at $\lambda = 0$.*

Proof. It can be shown [2; Th 6.5-E, p. 353] that $C\sigma(A)$ consists of those points μ for which $\lambda \rightarrow E_\lambda$ is continuous at μ but which are such that E_λ is not constant in any neighborhood of μ. Since $m(A) = 0$ and $E_\lambda = 0$ if $\lambda < m(A)$, it follows that $E_0 = \lim_{\lambda \to 0} E_\lambda = 0$ since $0 \in C\sigma(A)$ by lemma 2.1.

3. The proof of the theorem

We have to show that $e^{-At}u \rightarrow 0$ when $t \rightarrow \infty$ for every $u \in H$. From (2.4) we obtain

$$\| e^{-At}u \|^2 = \int_\alpha^M | e^{-\lambda t} |^2 \, d \, \| E_\lambda u \|^2$$

By lemma 2.6 $\lambda \rightarrow \| E_\lambda u \|^2$ is continuous at $\lambda = 0$, therefore we can write

$$\| e^{-At}u \|^2 = \int_0^\eta e^{-2\lambda t} \, d \, \| E_\lambda u \|^2 + \int_\eta^M e^{-2\lambda t} \, d \, \| E_\lambda u \|^2$$

Let $\epsilon > 0$ be given. Since $e^{-2\lambda t} \leq 1$ for all t, we can make the first integral less than $\epsilon/2$ by taking η small enough. Thus, we can assume that η is such that the first integral is less than $\epsilon/2$ for every t. Let

$$\kappa = \int_0^M d \, \| E_\lambda u \|^2$$

Then we can find $r > 0$ such that $e^{-2\lambda t} \leq 2^{-1}k^{-1}\epsilon$ for all $\lambda \geq \eta$ and all $t \geq r$. Therefore, the second integral is less than $2^{-1}\epsilon$ for all $t \geq r$. We have thus shown that $\| e^{-At}u \|^2 < \epsilon$ if $t \geq r$, i.e., $e^{-At}u \to 0$ as $t \to \infty$.

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REFERENCES

[1] M. K. GAVOURIN, Non linear functional equations and continuous analogs of iterative methods, University of Maryland Technical Report 68–70 (translation from the Russian) (1968).

[2] A. E. TAYLOR, Introduction to Functional Analysis, John Wiley & Sons, New York (1967).