A NOTE ON ANALYTIC STRUCTURAL STABILITY IN COMPACT M^2

By Carlos Perelló

Let M be a compact real analytic manifold of dimension 2, which we may assume properly embedded in an euclidean space of large enough dimension.

Consider now the B-space $\mathfrak{X}^r$ of the C^r-smooth vector fields over M with the C^r norm, where r is some positive integer larger or equal to 1.

We say two vector fields X and Y are equivalent, denoted $X \sim Y$, if there exists a homeomorphism h of M onto M taking the integral curves (orbits) of X onto the orbits of Y.

We say X is C^r-structurally stable if a whole neighborhood of it in $\mathfrak{X}^r$ consists of equivalent vector fields. Let Σ^r denote the set of all C^r-structurally stable vector fields in $\mathfrak{X}^r$. Evidently Σ^r is open.

The following two theorems are due to Peixoto [1]:

I. (Genericity): *Σ^r is open and dense in $\mathfrak{X}^r$ for all $r \geq 1$.*
II. (Characterization): *X belongs to Σ^r if and only if*
 0) $X \in \mathfrak{X}^r$
 1) a) X has a finite number of critical elements
  b) All the critical elements are non-singular (Non-imaginary eigenvalues)
 2) The asymptotic manifolds of X intersect transversally
 3) There is no non-trivial recurrence.

Let now $\mathfrak{X}_r^\omega$ denote the subset of $\mathfrak{X}^r$ consisting of all real analytic vector fields on M. Owing to the Weierstrass approximation theorem, $\mathfrak{X}_r^\omega$ is dense in $\mathfrak{X}^r$. $\mathfrak{X}_r^\omega$ becomes a normed vector space (non complete) under the C^r-norm.

We say X, a member of $\mathfrak{X}_r^\omega$, is C_r^ω-structurally stable if all vector fields in a whole neighborhood of X (in $\mathfrak{X}_r^\omega$) are equivalent to X. Let Σ_r^ω denote the set of all the C_r^ω-structurally stable vector fields on M. Clearly Σ_r^ω is open in $\mathfrak{X}_r^\omega$.

We prove now:

I′. (Genericity): Σ_r^ω is open and dense in $\mathfrak{X}_r^\omega$
II′. (Characterization): X belongs to Σ_r^ω if and only if
 0)′ X belongs to $\mathfrak{X}_r^\omega$, and
 1)′, 2)′, 3)′ the same as 1), 2), 3) in II.

We first prove $\Sigma^\omega = \Sigma^r \cap \mathfrak{X}^\omega$. (We drop the subscripts, because II′ shows that all the Σ_r^ω are the same).

a) $\Sigma^\omega \supset \Sigma^r \cap \mathfrak{X}^\omega$: As we are using for $\mathfrak{X}^\omega$ the relative topology, the result is immediate: if a small perturbation of X in Σ^r gives an equivalent vector field, the same is true for a small analytic perturbation.

b) $\Sigma^\omega \subset \Sigma^r \cap \mathfrak{X}^\omega$: As Σ^r is dense we approximate X in Σ^ω by $\{Y_n\} \subset \Sigma^r$. Each Y_n can in turn (Weierstrass) be approximated by X_n in $\mathfrak{X}^\omega$. Hence, for some n we have $X \sim Y_n$. (Because Σ^ω is open in $\mathfrak{X}^\omega$). Now, using II we see that X must fulfill 1a), 2) and 3). On the other hand it must fulfill 1b), also, or a small analytic perturbation would render a non equivalent vector field. Hence X belongs to Σ^r

To end the proof we have to show the density of Σ^ω in $\mathfrak{X}^\omega$:

Let X be in $\mathfrak{X}^\omega$ and approximate it by $\{Y_n\}$ in Σ^r. Each Y_n, in turn can be approximated by X_n in $\Sigma^r \cap \mathfrak{X}^\omega = \Sigma^\omega$.

The problem remains on what is the situation when we furnish $\mathfrak{X}^\omega$ with a more natural topology turning it into a complete space.

CENTRO DE INVESTIGACION DEL IPN.

REFERENCES

[1] M. M. PEIXOTO, *Structural stability on two-dimensional manifolds*, Topology **1** (1962), 101–20.