AN EXAMPLE IN THE THEORY OF ORDERED SEMIGROUPS

By Robert Gilmer

It is easy to find examples of commutative rings R with identity containing an ideal A such that $A \supset A^2 \supset \cdots \supset A^k = A^{k+1} = \cdots$, where k is a positive integer. For example, $A = (2)/(2^k)$ in $Z/(2^k)$, and $A = (18)/(2 \cdot 3^{2k})$ in $Z/(2 \cdot 3^{2k})$ are such ideals. Examples of this type, however, are not so easy to come by if R is an integral domain with identity and $k > 2$. In fact, if D is a Prufer domain and if A is an ideal of D such that $A^k = A^{k+1}$ for some positive integer k, then A is idempotent and $A = \sqrt{A}$; if D is a valuation ring, this implies that A is an idempotent prime ideal [1; p. 135]. Since a finitely generated idempotent ideal of a commutative ring is principal and is generated by an idempotent element [2; p. 58], it is also true that if A is an ideal of a Noetherian domain D such that $A^k = A^{k+1}$ for some positive integer k, then A is an idempotent prime ideal of D. While it is not difficult to find an example of a Prufer domain J containing an idempotent ideal A that is not prime (for example, see [2; p. 275]), the answer to the following question $(^*)$ has not been so easy to determine.

$(^*)$ *Given a positive integer $k > 1$, does there exist an integral domain D_k with identity containing an ideal A such that $A \supset A^2 \supset \cdots \supset A^k = A^{k+1} = \cdots$?*

In seeking to answer question $(^*)$ in the affirmative, we recently discovered that it was sufficient to answer affirmatively the following question $(^{**})$ in the theory of order semigroups.

$(^{**})$ *Does there exist a totally ordered cancellative abelian semigroup T_k with zero, such that S_k, the set of positive elements of T_k, has the following property?*
$S_k \supset 2S_k \supset \cdots \supset kS_k = (k + 1)S_k = \cdots$; here nS_k is
$\{s_1 + s_2 + \cdots + s_n \mid s_i \in S_k\}$.

The relationship between questions $(^*)$ and $(^{**})$ is the following: If S_k is such that $S_k \supset 2S_k \supset \cdots \supset kS_k = (k + 1)S_k = \cdots$, then the semigroup ring D_k of $S = S_k \cup \{0\}$ with respect to any field F is an integral domain with identity; the elements of D_k of positive order form a maximal ideal M_k such that $M_k \supset M_k^2 \supset \cdots \supset M_k^k = M_k^{k+1} = \cdots$ [3]. While the answer to question $(^{**})$ was shown to be affirmative, a simpler example to the original ideal-theoretic question $(^*)$ was found later [1, p. 136]. In communicating with several persons in the area of semigroups, it seems, however, that the example answering question $(^{**})$ may be of independent interest, and hence it is presented here. Our example, in fact, gives a subsemigroup S_k of the additive semigroup S of positive reals with the desired property.

Let T be the subsemigroup of S consisting of all elements $b\theta$, where b is a positive rational and $\theta \in S$ is irrational. T is closed under multiplication by any positive rational. In particular, x/m is in T whenever $x \in T$ and m is any positive integer. Thus T contains arbitrarily small positive elements, and $T = 2T = 3T = \cdots$.

We let A_k be the subsemigroup of S consisting of all rational numbers greater than $1/k$, and we let

$$S_k = T \cup (A_k + T) \cup \{m/(k-1)\}_{m=1}^{k-1} \cup \{1/(k-1) + t \mid t \in T\} \cup (1, \infty).$$

It is easy to check that S_k is a subsemigroup of S. To prove that S_k has the desired properties, it suffices to prove that

$$(k-1)S_k \supset kS_k = (k+1)S_k.$$

We show first that $(1, \infty) \subseteq nS_k$ for any positive integer n; we can assume that $n > 1$. Thus if $x > 1$, then $x - 1 > 0$, and since T contains arbitrarily small positive elements, there is an element t in T such that $(n-1)t < x - 1$ so that $1 < x - (n-1)t$ and $x - (n-1)t \in S_k$. Hence, it follows that $x = (n-1)t + [x - (n-1)t] \in nS_k$.

Now $1 = (k-1)(1/\mathrm{k} - 1)$, where $1/(k-1) \in S_k$, and hence $1 \in (k-1)S_k$. Any element $\alpha = \alpha_1 + \cdots + \alpha_k$ of kV such that $\alpha \leq 1$ must be such that each $\alpha_i < 1$, and some α_j is not greater than $1/k$. Therefore, each α_i is expressible in the form $u_i + t_i$, where u_i is rational and $t_i \in T$, $u_i > (1/k)$ or $u_i = 0$. Since $\alpha_j \leq 1/k$, $t_j \neq 0$ so that $t_1 + t_2 + \cdots + t_k > 0$. We have $t_1 + t_2 + \cdots + t_k \in T$, and hence $t_1 + \cdots + t_k$ is not rational. It follows that $\alpha = \alpha_1 + \cdots + \alpha_k = (t_1 + \cdots + t_k) + (u_1 + \cdots + u_k)$ is not rational. In particular, $1 \neq \alpha$, and $1 \notin kS_k$.

We next prove that $(k+1)S_k \subseteq kS_k$. We have previously shown that $(1, \infty) \subseteq (k+1)S_k$; hence we need only prove that $(0, 1) \cap kS_k \subseteq (k+1)S_k$. As we have just proved, however, each element of $(0, 1) \cap kS_k$ is of the form $r + t$, where r is nonnegative rational and $t \in T$. But it follows from the definition of S_k that $r + t$ in kS_k implies that $r + (t/2) \in kS_k$, and hence $r + t = [r + (t/2)] + (t/2) \in (k+1)S_k$. Therefore, $kS_k = (k+1)S_k$ and our proof is complete.

FLORIDA STATE UNIVERSITY

REFERENCES

[1] J. ARNOLD AND R. GILMER, *Idempotent ideals and union of nets of Prüfer domains*, J. Sci. Hiroshima Univ. Ser. A-I. Math. **31** (1967), 131–45.

[2] R. GILMER, Multiplicative Ideal Theory. Queen's University, Kingston, Ontario, Canada, 1968.

[3] ———, *A note on semigroup rings*, Amer. Math. Monthly **76** (1969), 36–7.