ON THE ISOMORPHISMS OF BOOLEAN RINGS

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It is well known that the Stone isomorphism θ (see below) of a Boolean ring B maps B into a ring of sets. In this paper we prove that if B is an infinite Boolean ring (not necessarily with a unit) then θ is not onto. Moreover, there exists a subset H of B such that

$$\bigcap \theta[H] \qquad \text{or} \qquad \bigcup \theta[H]$$

is not in the range of θ.

Let us recall that a Boolean ring is a ring B such that $x^2 = x$ for every $x \in B$, and that $\leq$ is a partial order in B where $\leq$ is defined by:

$$(1) \qquad x \leq y \quad \textit{if and only if} \quad xy = x$$

for every element x and y of B. Moreover, a nonzero element a of B is called an *atom* [1, p. 27] of B if and only if for every element x of B

$$(2) \qquad x \leq a \quad \textit{implies} \quad x = a \quad \textit{or} \quad x = 0$$

i.e., if and only if for every element x of B

$$(3) \qquad ax = a \quad \textit{or} \quad ax = 0$$

Let us recall that a subset M of B is called an *ultrafilter* of B if and only if $B - M$ is a proper prime ideal of B. It is well known [2, p. 105] that for every subset F of B if $0 \notin F$ and if F is closed under multiplication then there exists an ultrafilter M of B such that $F \subset M$. It can be readily verified that if M is an ultrafilter of B with $a \in M$ and $b \in M$ then $ab \in M$ and vice versa.

There exists always an isomorphism from a Boolean ring B into a ring of sets, where the set-theoretical symmetric difference and intersection are respectively taken as addition and multiplication, and, where

$$(4) \qquad \leq \quad \textit{corresponds to} \quad \subset$$

One of the abovementioned isomorphisms is the Stone isomorphism θ of B defined by:

$$(5) \qquad \theta(x) = \{M \mid M \textit{ is an ultrafilter of } B \textit{ and } x \in M\}$$

Clearly, θ maps B into the ring of sets of all the subsets of the set of all the ultrafilters of B, see [3].

Lemma 1. *Let M be an ultrafilter of a Boolean ring B and let $a = \inf M$. Then either a is an atom of B and $a \in M$ or $a = 0$.*

Proof. Let us observe that if $x \in B$ is a nonzero lower bound of M, then by (1) we have $xm = x \neq 0$ for every element m of M. But then $x \in M$ because otherwise, $M \cup \{x\}$ *with* $0 \notin (M \cup \{x\})$ would be closed under multiplication implying that the ultrafilter M of B is properly contained in another ultrafilter of B, which is a contradiction. Now, let $a \neq 0$ and $a = \inf M$. Thus, $a \in M$. If $x \in B$ is such that $x \neq 0$ and $x \leq a$ then x is a nonzero lower bound of M and hence $x \in M$. But then $a \leq x$. Consequently, $x = a$ which in view of (2) implies that a is an atom of B.

LEMMA 2. *Let B be an infinite Boolean ring with a unit e. Then e is not a finite sum of atoms of B.*

Proof. Assume the contrary, and let $e = a_1 + \cdots + a_n$ where a_i is an atom of B for $i = 1, \cdots, n$. Let x be an element of B. Then $x = ex = (a_1 + \cdots + a_n)x = a_1x + \cdots + a_nx$, which in view of (3) implies that every element x of B is a sum of some of a_i with $i = 1, \cdots, n$. But this contradicts the fact that B is infinite.

LEMMA 3. *Let B be an infinite Boolean ring with a unit e. Then there exists an ultrafilter M of B such that no atom of B is an element of M.*

Proof. Consider the set

$$F = \{e + f \mid f \text{ is a finite sum of atoms of } B\}$$

Let $e + f_1$ and $e + f_2$ be elements of F. In view of (3) it is obvious that f_1f_2 is a finite sum f_3 of atoms of B. Consequently, $(e + f_1)(e + f_2) = e + f_1 + f_2 + f_3$ and therefore, $(e + f_1)(e + f_2)$ is an element of F. Thus, F is closed under multiplication. Moreover, $0 \notin F$ since otherwise $0 = e + f$, i.e., $e = f$, contradicting Lemma 2. Thus, there exists an ultrafilter M of B such that $F \subset M$. We claim that no atom a of B is an element of M. Because, if $a \in M$ then $(e + a)a = a + a = 0$ would be an element of M, contradicting that $0 \notin M$.

LEMMA 4. *Let θ be the Stone isomorphism of a Boolean ring B. Then $\theta(a) = \{M\}$ if and only if a is an atom of B and M is an ultrafilter of B with $a \in M$.*

Proof. Let $\theta(a) = \{M\}$. Since $\{M\} \neq \varnothing$, we see that $a \neq 0$. Now, let x be an element of B such that $x \leq a$. But then by (4), we have $\theta(x) \subset \theta(a) = \{M\}$. Thus, $\theta(x) = \{M\}$ or $\theta(x) = \varnothing$ implying that $x = a$ or $x = 0$. Hence from (2) it follows that a is an atom of B. Clearly, (5) implies that M is an ultrafilter of B and $a \in M$.

To prove the converse, in view of (5), it is enough to show that if M and N are ultrafilters of B such that $M \neq N$ and $a \in M$ for some atom a of B, then $a \notin N$. Assume on the contrary that $a \in M$ and $a \in N$. Since $M \neq N$, there exists $b \in B$ such that, say, $b \in N$ and $b \notin M$. But then since N is an ultrafilter $ab \in N$, and, since $0 \notin N$, from (3) it follows that $ab = a$. Thus, $ab \in M$ which in view of the fact that M is an ultrafilter implies $b \in M$ which is a contradiction.

LEMMA 5. *Let B be an infinite Boolean ring with a unit. Then there exists an ultrafilter M of B such that $\{M\}$ is not in the range of the Stone isomorphism θ of B.*

Proof. By Lemma 3, there exists an ultrafilter M of B such that no element of

M is an atom of B. But then by Lemma 4, we see that $\{M\}$ cannot be in the range of θ.

THEOREM 1. *Let B be an infinite Boolean ring. Then there exists a set V of ultrafilters of B such that V is not in the range of the Stone isomorphism θ of B.*

Proof. Assume on the contrary that every set of ultrafilters of B is in the range of θ. This would imply (since θ is an isomorphism) that B is a complete Boolean ring. Consequently, B would have a unit. But then by Lemma 5, for some ultrafilter M of B, the set $\{M\}$ would not be in the range of θ, contradicting the above assumption.

In view of Theorem 1, we have the following:

COROLLARY 1. *For no infinite Boolean ring B does the Stone isomorphism of B map B onto the set of all the subsets of the set of all the ultrafilters of B.*

Next, we prove:

THEOREM 2. *Let B be an infinite Boolean ring. Then there exists a subset H of B such that*

$$\bigcap \theta[H] \quad \text{or} \quad \bigcup \theta[H] \tag{6}$$

is not in the range of the Stone isomorphism θ of B.

Proof. Since B does not necessarily have a unit, two cases may occur:

(i) There exists an ultrafilter M of B such that $\{M\}$ is not in the range of θ,

(ii) for every ultrafilter M of B the set $\{M\}$ is in the range of θ.

Let (i) occur. We show that $\bigcap \theta[M]$ is not in the range of θ. Assume on the contrary and let $\theta(a) = \bigcap \theta[M]$ for some element a of B. But then from (4) it follows that $a = \inf M$. Hence, in view of Lemma 1, either a is an atom of B and $a \in M$, or, $a = 0$. But the first alternative is impossible since then (5) and Lemma 4 would imply that $\theta(a) = \{M\}$ which would contradict (i). Likewise, the second alternative is impossible since if $a = 0$ then on the one hand $\theta(a) = \varnothing$ whereas on the other hand, by (5), for every element m of M we would have $\{M\} \subset \theta(m)$ which would imply $\{M\} \subset \bigcap \theta[M] = \theta(a) \neq \varnothing$.

Thus, if (i) occurs then (6) is valid for $H = M$.

Next, let (ii) occur. By Theorem 1 there exists a set $V = \{M, N, K, \cdots\}$ of ultrafilters $M, N, K, \cdots$ of B such that V is not in the range of θ. However, by (ii) there exist elements (in fact atoms) $m, n, k, \cdots$ of B such that $\theta(m) = \{M\}$, $\theta(n) = \{N\}, \theta(k) = \{K\}, \cdots$ let $H = \{m, n, k, \cdots\}$. Clearly,

$$\bigcup \theta[H] = \{M, N, K, \cdots\} = V$$

and since V is not in the range of θ, we see that if (ii) occurs then (6) is valid.

As expected, a Boolean ring is called *complete* if and only if every subset of B has infimum (or supremum) with respect to $\leq$, as given in (1).

Based on Theorem 2 we have the following:

COROLLARY 2. *Let θ be the Stone isomorphism of a complete infinite Boolean* *ring B. Then there exist subsets E and G of B such that*

$$\theta(\inf E) \neq \bigcap \theta[E] \qquad \textit{and} \qquad \theta(\sup G) \neq \bigcup \theta[G]$$

Proof. If $\bigcap \theta[H]$ mentioned in (6) is not in the range of θ, then it is enough to take $E = H$ and $G = \{e + h \mid h \in H\}$. On the other hand, if $\bigcup \theta[H]$ mentioned in (6) is not in the range of θ, then it is enough to take $E = \{e + h \mid h \in H\}$ and $G = H$. Clearly, e is the unit of B which exists since B is complete.

Remark. Corollary 2 shows that the Stone isomorphism of a complete infinite Boolean ring preserves neither infima nor suprema. Thus any isomorphism from a complete infinite Boolean ring B into a ring of sets which preserves infima (or suprema) cannot be the Stone isomorphism of B.

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