A NOTE ON HARMONIZABLE AND STATIONARY SEQUENCES

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In this paper we prove that harmonizable Hilbert sequences are projections of stationary ones, and we use this result to give a sufficient condition for a harmonizable sequence to be deterministic.

$\mathbf{Z}$ will denote the set of integers; T^n the n-dimensional torus for $n = 1, 2, \cdots$ and $C(T^n)$ the space of continuous complex valued functions on T^n. For every $f, g \in C(T), f \otimes g$ will denote the tensor product of f and g, i.e. $(f \otimes g)(s, t) = f(s)g(t)$ for $s, t \in T$. It is clear that $f \otimes g \in C(T^2)$ whenever $f, g \in C(T)$. *c.l.s* will stand for closed linear span and *l.s.* for linear span.

Let H be a Hilbert space with inner product $(\,,)$ and let $x_n \in H$ for $n \in \mathbf{Z}$. Define $H_{+\infty}(x) = c.l.s.\{x_n : n \in \mathbf{Z}\}$; $H_n(x) = c.l.s.\{x_k : k \leq n\}$ for each $n \in \mathbf{Z}$, and $H_{-\infty}(X) = \bigcap_{n \in \mathbf{Z}} H_n(x)$. The Hilbert sequence $\{x_n\}$ is deterministic if $H_{-\infty}(x) = H_\infty(x)$, and it is linearly free if $H_{-\infty}(x) = \{0\}$.

The covariance of a Hilbert sequence $\{x_n\}$ is defined by $B(m, n) = (x_m, x_n)$ for $m, n \in \mathbf{Z}$. The Hilbert sequence $\{x_n\}$ is stationary if its covariance depends only on the difference $m - n$.

A Hilbert sequence $\{x_n\}$ is called harmonizable [1 and 2] if for some complex valued Borel measure μ on T^2

$$(1) \qquad (x_m, x_n) = \int\int_{T^2} e^{2\pi i(mx-ny)}\, d\mu(x, y)$$

for every $m, n \in \mathbf{Z}$. μ is called the covariance measure of $\{X_n\}$.

A Hilbert sequence $\{x_n\}$ is stationary if and only if it is harmonizable and its covariance measure is concentrated on the diagonal of T^2.

Theorem. *If $\{x_n\}$ is a harmonizable Hilbert sequence then there exists a Hilbert space H' containing $H_{+\infty}(x)$ as a subspace, and a stationary Hilbert sequence $\{z_n\} \subset H'$ such that if $P: H' \to H_{+\infty}(x)$ is the orthogonal projection, then $x_n = Pz_n$ for every $n \in \mathbf{Z}$.*

Proof. Let μ be the covariance measure of $\{x_n\}$, and let $|\mu|$ denote the total variation of μ. Then $|\mu|$ is a positive, finite and symmetric Borel measure on T^2. Let μ_0 be the finite positive Borel measure on T defined by

$$(2) \qquad \mu_0(\Delta) = |\mu|(\Delta \times T) = \tfrac{1}{2}(|\mu|(\Delta \times T) + |\mu|(T \times \Delta))$$

for each Borel subset Δ of T. Then for every continuous complex valued function f on T, we have

$$(3) \qquad \int_T f\, d\mu_0 = \tfrac{1}{2}\int\int_{T^2} f \otimes 1\, d|\mu| + \tfrac{1}{2}\int\int_{T^2} 1 \otimes f\, d|\mu|.$$

Let $\varphi \in C(T)$. Then

$$(4) \qquad 0 \leq \int\int_{T^2} \varphi \otimes \bar{\varphi}\, d\mu \leq \int\int_{T^2} |\varphi \otimes \bar{\varphi}|\, d|\mu|$$

But $|\varphi(t)\varphi(s)| \leq \frac{1}{2}(|\varphi(t)|^2 + |\varphi(s)|^2)$ for $t, s \in T$. Therefore

$$(5) \qquad \int\int_{T^2} \varphi \otimes \bar{\varphi}\, d\mu \leq \tfrac{1}{2}\int\int_{T^2} |\varphi|^2 \otimes 1\, d|\mu| + \tfrac{1}{2}\int\int_{T^2} 1 \otimes |\varphi|^2\, d|\mu|,$$

and in using (3) we obtain

$$(6) \qquad \int\int_{T^2} \varphi \otimes \bar{\varphi}\, d\mu \leq \int_{T^1} |\varphi|^2\, d|\mu|.$$

Let μ_1 be the Borel measure on T^2 which is concentrated in the diagonal and satifies

$$(7) \qquad \int\int_{T^2} f(x, y)\, d\mu(x, y) = \int_{T^1} f(x, x)\, d\mu_0(x)$$

for every $f \in C(T^2)$. Now define $\mu_2 = \mu_1 - \mu$.

From (6) it follows that

$$(8) \qquad 0 \leq \int\int_{T^2} \varphi \otimes \bar{\varphi}\, d\mu_2$$

for every $\varphi \in C(T)$. Therefore μ_2 is the covariance measure of some harmonizable Hilbert sequence $\{y_n\}$. (For example let $(f, g)_2 = \int\int_{T^2} f \otimes \bar{g}\, d\mu_2$ for $f, g \in C(T)$. Let $\ker \mu_2 = \{f \in C(T): (f, f)_2 = 0\}$ and define H'' to be the Hilbert space obtained by completing $C(T)/\ker \mu_2$ with respect to the norm $\|f\|_2 = (f, f)_2^{1/2}$. Then let y_n be the image of the function $x \to e^{2\pi i n x}$ under the canonical map $C(T) \to H''$ for each $n \in \mathbf{Z}$.) Now put $H_\infty(y) = c.l.s.\{Y_n : n \in \mathbf{Z}\}$ and let $H' = H_\infty(x) \oplus H_\infty(y)$. Identify $H_\infty(x)$ with the subspace $H_\infty(x) \oplus \{0\}$ of H'.

$(\,,\,)_1$ will denote the innerproduct in H' and $(\,,\,)_2$ will denote the innerproduct in $H_\infty(y)$.

Let $z_n = x_n + y_n$ for each $n \in \mathbf{Z}$. Then $(z_m, z_n)_1 = (x_m, x_n) + (y_m, y_n)_2$ for $m, n \in \mathbf{Z}$. Hence the covariance measure of $\{z_n\}$ is $\mu_1 = \mu + \mu_2$ which shows that $\{z_n\}$ is stationary.

Finally, if $P: H' \to H_\infty(x)$ denotes the orthogonal projection onto $H_\infty(x)$, it is obvious by construction that $x_n = Pz_n$ for each $n \in Z$]. Q.E.D.

We have shown that harmonizable Hilbert sequences are projections of stationary Hilbert sequences. Furthermore, the measure μ_0 determining the covariance of the stationary sequence obtained is given by $\mu_0(\Delta) = |\mu|(\Delta \times T)$ for each Borel set $\Delta \subset T$, where μ is the covariance measure of the harmonizable Hilbert sequence.

PROPOSITION. *Let H' and H'' be two Hilbert spaces and $A: H' \to H''$ a bounded linear transformation. Let $\{x_n\} \subset H'$ be a deterministic Hilbert sequence. Then $\{Ax_n\} \subset H''$ is also a deterministic Hilbert sequence.*

Proof. $l.s.\{Ax_n ; n \leq k\} = A(l.s.\{x_n : n \leq k\})$ for every k. Since A is continuous it follows that

$$c.l.s.\{Ax_n : n \leq k\} \supset A(c.l.s.\{x_n : n \leq k\}) \quad k \in \mathbf{Z}.$$

But since $\{x_n\}$ is deterministic, $H_k(x) = H_\infty(x)$ and therefore

$$c.l.s.\{Ax_n : n \leq k\} \supset A(H_\infty(x)),$$

and since $A(H_\infty(x))$ is dense in $c.l.s.\{Ax_n : n \in \mathbf{Z}\}$ it follows that for every $k \in \mathbf{Z}$

$$c.l.s.\{Ax_n : n \leq k\} = c.l.s.\{Ax_n : n \in \mathbf{Z}\},$$

therefore $\{Ax_n\}$ is deterministic. Q.E.D.

Combining this proposition with the previous theorem we obtain that if the stationary sequence $\{z_n\}$, whose projection onto $H_\infty(x)$ is $\{x_n\}$, is deterministic, then $\{x_n\}$ is deterministic too.

A well-known result for stationary sequences [3 and 4] says that a necessary and sufficient condition for $\{z_n\}$ to be deterministic is that

$$\int_0^1 \log G'(t)\, dt = -\infty \tag{9}$$

where $G(t) = \mu_0([0, t))$ for every $t \in (0, 1]$. We just proved:

COROLLARY. *If $\{x_n\}$ is a harmonizable Hilbert sequence whose covariance measure is μ, then a sufficient condition for $\{x_n\}$ to be deterministic is that*

$$\int_0^1 \log G'(t)\, dt = -\infty \tag{10}$$

where $G(t) = |\mu|([0, t) \times T)$ for every $t \in (0, 1]$.

This result was proved by Cramér [1 and 2] and later by Dudley [5] using a different approach, this new proof shows clearly that it is essentially an application of the results known for stationary sequences.

The characterization of linearly free and deterministic Hilbert sequences in terms of their covariance distributions has been studied by Dudley [5]. Whether deterministic harmonizable Hilbert sequences can be characterized analytically or not seems to be an open problem.

To conclude this paper we prove that it is not necessary for a Hilbert sequence to be harmonizable in order to be the projection of a stationary Hilbert sequence.

Let $\{e_n\}$, $n \in \mathbf{Z}$ be an orthonormal sequence in a Hilbert space H. Define the Hilbert sequence $\{x_n\}$ as follows: $x_n = 0$ if $n \leq 0$; $x_n = e_n$ if $n > 0$. Let $P : H \to H_\infty(x)$ be the projection operator onto $H_\infty(x) = c.l.s.\{e_n : n > 0\}$. Then $x_n = Pe_n$ for every $n \in \mathbf{Z}$. $\{e_n\}$ is obviously a stationary Hilbert sequence. However, $\{x_n\}$ is not harmonizable. Indeed, suppose μ is the covariance measure of $\{x_n\}$. The Fourier coefficients of μ are

$$\hat{\mu}(m, n) = \int\int_{T^2} e^{2\pi i(mx+ny)}\, d\mu(x, y) = (x_m, x_{-n}) = \begin{cases} 1 & \text{if } m = -n > 0 \\ 0 & \text{otherwise.} \end{cases} \tag{11}$$

Thus $\hat{\mu}(m, n)$ vanishes outside the cone $\{(m, n) \in \mathbf{Z}^2 : m = -n > 0\}$. Therefore by a theorem of Bochner (see [6 or 7]), it follows that μ is absolutely continuous with respect to Lebesgue measure and, by the Riemann-Lebesgue lemma, $\hat{\mu}(m, n) \to 0$ as $m^2 + n^2 \to \infty$. This is false acording to (11).

Hence $\{x_n\}$ is not harmonizable.

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