DEHN'S CONSTRUCTION ON KNOTS

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1. Introduction

Our main concern is the study of homology 3-spheres obtained by Dehn's method, i.e. by removing a knotted solid torus from S^3 and sewing it back in a different way.

Several people have considered a homology cobordism invariant μ of homology 3-spheres (see e.g. [15] or §2).

In this paper we are interested in the following two questions:

1. What is the μ-invariant of homology spheres obtained by Dehn's method?
2. Can the fundamental group of such a homology sphere be trivial?

The answer to the first question is given by Theorem 4 which says that $\mu = \nu \cdot \chi$ where χ is the Arf invariant of the core of the solid torus we are removing and ν is an integer which indicates how the solid torus is sewn back.

The second question is more difficult. A conjecture might be that the fundamental group of a 3-manifold obtained by removing a knotted solid torus from S^3 and sewing it back differently is never trivial.

This conjecture is true if the core of the solid torus is a torus knot ([39, §13]; see also [13]).

A modification of the proof of [31, Satz 1], using a remark in [2, page 101], shows that the conjecture is true if the core of the solid torous is a composite knot.

We also prove the conjecture if the core is a doubled knot, a cable knot of type r, s with $r \neq 1$ and $rs \neq 2$, or a knot contained in a knotted solid torus with zero winding number and non-zero self-linking in it.

As a consequence we obtain that knots of any of the types mentioned above are characterized by the topological type of their complement or, in view of Waldhausen's results ([43]), by the group system of the knot. Another consequence is that a homotopy 3-sphere is S^3 if it is the disjoint union of a (tame or wild) solid torus and the complement of any of the previously mentioned knots.

As by-products we obtain some contractible 4-manifolds different from D^4, links cobordic to zero which are not slice links in the weak sense (see [16]) and a sharpening of a result of Burde and Zieschang: the only Neuwirth knots of genus 1 are the trefoil and figure eight knots.

Bing and Martin ([3]) have independently obtained some of the results of this paper.

I should like to express my sincerest gratitude to Professor R. H. Fox for his encouragement and supervision of this work which formed part of my thesis at Princeton.

2. The group $\mathcal{H}^3$

We will define the group $\mathcal{H}^3$ of homology 3-spheres.

DEFINITION. Two oriented closed manifolds M_0, M_1 are homology-cobordant if there is an oriented cobordism W between them such that the inclusion of M_i in W induces isomorphisms in all homology groups for $i = 0, 1$.

Clearly homology-cobordism is an equivalence relation.

If M is an oriented manifold we denote by $-M$ the manifold obtained from M by reversing the orientation.

Let $\mathcal{H}^3$ be the set of homology-cobordism classes of oriented homology 3-spheres. (A homology 3-sphere is a closed manifold with the same homology groups as S^3).

It is not difficult to see that the connected sum (see [17]) induces an abelian group structure in $\mathcal{H}^3$. The trivial element of $\mathcal{H}^3$ is represented by any oriented homology 3-sphere which bounds an acyclic manifold; the inverse of the class represented by M is represented by $-M$.

Higher dimensional analogues of $\mathcal{H}^3$ and groups of pairs of homology spheres are studied in [11]. For example, if we are working in the differentiable category, $\mathcal{H}^n$ is isomorphic to the Kervaire-Milnor group θ^n of homotopy n-spheres ([17]) for $n \neq 3$. In the PL category $\mathcal{H}^n$ is trivial for $n \neq 3$ and in the topological category $\mathcal{H}^n$ is trivial for $n \neq 3, 4$.

The structure of $\mathcal{H}^3$ is not known. A homomorphism $\mu : \mathcal{H}^3 \to Z_2$ can be defined using Rohlin's theorem ([37]) as follows:

If M^3 is a homology sphere, let W^4 be a simply connected manifold with even quadratic form which has M^3 as boundary. (See e.g. [14]). The signature (index) of W^4 is a multiple of 8 because its quadratic form is unimodular and even ([41]). Define

$$\mu([M^3]) = \sigma(W^4)/8 \bmod 2$$

where σ denotes signature.

To see that μ is well defined suppose that W_1^4, W_2^4 are 1-connected manifolds with even quadratic form and that $bW_1^4 = bW_2^4 = M^3$. Pasting together W_1 and W_2 by the identity map on M, we obtain a simply connected closed manifold W^4 with even quadratic form. The quadratic form of W is the direct sum of the quadratic forms of W_1 and $-W_2$. Hence W has an even quadratic form and

$$\sigma(W) = \sigma(W_1) - \sigma(W_2).$$

By Rohlin's theorem ([37]), $\sigma(W)$ is a multiple of 16 so that

$$\sigma(W_1)/8 = \sigma(W_2)/8 \bmod 2.$$

If M^3 bounds an acyclic manifold Y^4 we can do surgery on int Y^4 to change Y into a simply connected π-manifold W^4. The quadratic form of W^4 is even (see [27], Lemma 3) and, since surgery does not change the signature, $\sigma(W^4) = 0$.

Hence μ is well defined.

If W^4 is the connected sum along the boundary of W_1^4 and W_2^4 we have $\sigma(W^4) = \sigma(W_1^4) + \sigma(W_2^4)$ so that μ is a homomorphism.

μ is onto. For example, the dodecahedral space bounds a 1-connected manifold with even quadratic form of signature 8. (See [15] and also §5.)

3. Notation

We shall be concerned mainly with the study of homology 3-spheres obtained by Dehn's method i.e. by removing a neighborhood of a knot in S^3 and sewing it back differently.

Let V be the union of a disjoint collection of tame solid tori in S^3. Let $V_1, V_2, \cdots, V_n$ be different (but not necessarily all) components of V and let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be simple closed curves contained in $bV_1, bV_2, \cdots, bV_n$ respectively. Assume that α_i is not contractible in bV_i for $i = 1, \cdots, n$. Now, construct a 3-manifold as follows. Let $\varphi_i: S^1 \times bD^2 \to bV_i\ i = 1, \cdots, n$ be homeomorphisms mapping $1 \times bD^2$ onto α_i. Now, in the disjoint union

$$(S^3 - \text{int } V) + (S^1 \times D^2)_1 + \cdots + (S^1 \times D^2)_n$$

identify $x \in (S^1 \times bD^2)_i$ with $\varphi_i(x)$.

The resulting manifold will be denoted by $M(V, \alpha)$ where $\alpha = \bigcup_i \alpha_i$. It does not depend on the choice of φ_is. Define $M(V, \Phi)$ as $S^3 - \text{int } V$ and $M(\Phi, \Phi)$ as S^3.

We give to S^3 a canonical orientation and orient $M(V, \alpha)$ in such a way that $M(V, \alpha)$ and S^3 induce the same orientation in $S^3 - \text{int } V$.

By [44] or [23], every closed orientable 3-manifold is homeomorphic to $M(V, \alpha)$ for some V and α.

The following properties are easy to establish.

3.1 If V^* is the union of another collection of disjoint solid tori and if there is a homeomorphism h from $S^3 - \text{int } V$ onto $S^3 - \text{int } V^*$ then $M(V, \alpha)$ is homeomorphic to $M(V^*, \alpha^*)$, where $\alpha^* = h(\alpha)$.

3.2 If α_i' is a simple closed curve on bV_i isotopic (or equivalently, by [1], homotopic) to α_i in $bV_i\ i = 1, \cdots, n$, then $M(V, \alpha)$ is homeomorphic to $M(V, \alpha')$ where $\alpha' = \bigcup_i \alpha_i'$.

3.3 If a component α_k of α bounds a disk in a component V_k of V then $M(V, \alpha)$ is homeomorphic to $M(V - V_k, \alpha - \alpha_k)$.

4. The μ invariant of M(V, α)

Let V be a tame solid torus and let k be its core. The Arf invariant $\chi(k)$ of k is defined by

$$\chi(k) = \textstyle\sum_{i=1}^{h} v_{2i-1,2i-1} v_{2i,2i} \bmod 2$$

where (v_{ij}) is a $2h \times 2h$ Seifert Matrix for k ([21], [36], [29]).

It can be shown ([21], [29]) that

$$\chi(k) = \begin{cases} 0 \text{ if the determinant of } k \text{ is } \pm 1 \bmod 8 \\ 1 \text{ if the determinant of } k \text{ is } \pm 3 \bmod 8 \end{cases}$$

An oriented simple closed curve α on the boundary bV of the solid torus V represents an element $m^\rho l^\nu$ of $\pi_1(bV)$, where m and l are represented by a meridian and a longitude respectively. If $\rho \neq 0$ the orientation of α may be so chosen that ρ is positive. Since $H_1(M(V, \alpha))$ is cyclic of order ρ, $M(V, \alpha)$ is a homology sphere if and only if $\rho = \pm 1$, i.e., if and only if $\rho = 1$ when α is properly oriented.

THEOREM 4. *Let V be a closed regular neighborhood of k and let α be an oriented simple closed curve on bV representing the element $ml^\nu \in \pi_1(bV)$ where m and l are represented by a meridian and longitude respectively. Then, if $\mu:\mathcal{H}^3 \to Z_2$ is the homomorphism defined in § 2, we have*

$$\mu([M(V, \alpha)]) = \nu \cdot \chi(k)$$

where $\chi(k)$ is the Arf invariant of k

For instance, if M is the dodecahedral space or if M is the example in [2, page 102] then $\mu([M]) \neq 0$.

Proof of Theorem 4. We may assume that k and V are differentiable submanifolds.

Let $E_\nu = B_2^2 \times D_2^2 +_h B_1^2 \times D_1^2$ where B_2^2, D_2^2, B_1^2, D_1^2 are 2-disks and h identifies $bB_2^2 \times D_2^2$ with $bB_1^2 + D_1^2$ as follows. If we think of the 2-disks as copies of the unit disk in C, $h:bB_2^2 \times D_2^2 \to bB_1^2 \times S_1^2$ is defined by

$$h(e^{i\gamma}, \rho e^{i\theta}) = (e^{i\gamma}, \rho e^{i(\theta+\nu\gamma)}).$$

The disks and their boundaries have canonical orientations.

Now define an imbedding $\varphi:B_1^2 \times bD_1^2 \to bD^4 = S^3$ with $\varphi(B_1^2 \times bD_1^2) = V$ $\varphi(\{0\} \times bD_1^2) = k$ and such that the image, under φ, of the oriented curves $bB_1^2 \times 1$ and $1 \times bD_1^2$ are, respectively, an oriented meridian represented by m and an oriented longitude represented by l. Then we have that

(*) $\varphi h\ (bB_2^2 \times \{1\})$ is a curve homotopic to α in bV.

In the disjoint union $E_\nu + D^4$ identify $x \in B_1^2 \times bD_1^2$ with $\varphi(x) \in bD^4$ to obtain a 4-manifold W^4 which can be given a natural differentiable structure. By (*) and 3.2, bW^4 is homeomorphic to $M(V, \alpha)$.

Let $S_1^2 = \{0\} \times D_1^2 \cup Ck$ where Ck denotes the cone over k with vertex $0 \in D^4$. Let $S_2^2 = (B_1^2 \times \{0\}) \cup (B_2^2 \times \{0\})$.

Oriented properly S_1^2 and S_2^2 represent, respectively, classes x and y which generate $H_2(W^4) \approx Z \oplus Z$ with the following intersection numbers

$$x \cdot x = 0 \quad x \cdot y = y \cdot x = 1 \qquad \text{and} \qquad y \cdot y = \pm\nu.$$

If ν is even the quadratic form of W^4 is even and its signature is zero, so that $\mu([M(V, \alpha)]) = 0$.

Now consider the case where ν is odd.

Let Y^4 be a simply connected smooth manifold with even quadratic form and $bY^4 = bW^4$ (see [14]). Let σ be the signature of Y^4. Pasting together W^4 and $-Y^4$ by the identity map on their boundaries we obtain a simply connected closed orientable, smooth manifold N^4. The sphere $S_1^2 \subset W^4 \subset N^4$ is the image of a combinatorial imbedding of the 2-sphere into N^4. It has one singularity at the vertex of Ck. Since $M(V, \alpha)$ is a homology sphere there is a natural isomorphism

$$i_* + j_*: H_2(W^4) + H_2(-Y^4) \xrightarrow{\approx} H_2(N^4)$$

where $i: W^4 \to N^4$ and $j: Y^4 \to N^4$ are the inclusions. If $z \in H_2(N^4)$ is in the image of j_* we have $i_*(x) \cdot z = 0$ and $z \cdot z$ is even. Also $i_*(x) \cdot i_*(y) = 1$ and $i_*(y) \cdot i_*(y) = \pm\nu$. Hence, if $x' \in H_2(N^4; Z_2)$ is the mod 2 reduction of $i_*(x)$, $x' \cdot z = z \cdot z$ for all $z \in H_2(N^4; Z_2)$.

By Wu's formula $i_*(x)$ is therefore dual to the Stiefel Whitney class $w_2(N^4)$ i.e. the natural homomorphisms

$$H_2(N^4; Z_2) \to H_2(N^4; Z_2) \to H^2(N^4; Z_2)$$

(reduction mod 2 followed by Poincaré duality) carry $i_*(x)$ to $w_2(N^4)$.

Hence, there is an imbedding of S^2 into N^4 having S_1^2 as image, which is admissible for the knot k in the sense of [36] and

$$\chi(k) \equiv (i_*(x) \cdot i_*(x) - \sigma(N^4))/8 \equiv \sigma(N^4)/8 \equiv \mu([M(V, \alpha)]) \bmod 2.$$

Consequently, the formula

$$\mu([M(V, \alpha)]) = \nu \cdot \chi(k) \bmod 2$$

holds both for ν even and ν odd. This completes the proof of the theorem.

Let k_0 and k_1 be cobordant knots ([9]). Then there is a locally flat annulus A^2 in $S^3 \times I$, which we may assume is a differentiable submanifold, with $A^2 \cap (S^3 \times \{i\}) = k_i$, $i = 0, 1$. Let W^4 be a tubular neighborhood of A^2 in $S^3 \times I$. Write $V_i = W \cap (S^3 \times \{i\})$. W is homeomorphic to $A^2 \times D^2$ and V_0, V_i are solid tori. Let m_i and l_i be respectively an oriented meridian and longitude in bV_i $i = 0, 1$. We assume that the orientations are chosen in such a way that m_0 is homotopic to m_1 and l_0 is homotopic to l_1 in $bW - \text{int}(V_0 \cup V_1)$. Also m_i, l_i will denote the elements of $\pi_1(bV_i)$ represented by the corresponding curves.

Now, let ν be an integer. Suppose that α_i is a curve on bV_i which represents, with some orientation, the element $m_i l_i^\nu$.

PROPOSITION 4.1. *If (V_0, α_0) and (V_1, α_1) are as described above, then $M(V_0, \alpha_0)$ and $M(V_1, \alpha_1)$ are homology-cobordant.*

Proof. Construct the manifold $Y^4 = \overline{S^3 \times I - W} +_h S^1 \times D^2 \times I$ where $h: bW - \text{int}(V_0 \cup V_1) \to S^1 \times bD^2 \times I$ is a diffeomorphism onto, which maps α_i onto $\{p\} \times bD^2 \times \{i\}$, $p \in S^1$, $i = 0, 1$.

Y^4 is a cobordism between $M(V_0, \alpha_0)$ and $M(V_1, \alpha_1)$. By considering a Mayer-Vietoris sequence it is seen that

$$H_q(Y^4) = \begin{cases} 0 & \text{if } q \neq 0, 3 \\ Z & \text{if } q = 0, 3 \end{cases}$$

It follows, then, from the homology sequence of the pair $(Y^4, M(V_i, \alpha_i))$

$i = 0, 1$, that $M(V_0, \alpha_0)$ and $M(V_1, \alpha_1)$ are homology cobordant. This completes the proof.

Consequently, for every integer ν, a map $D_\nu : \theta^{3,1} \to \mathcal{H}^3$ can be defined by $D_\nu([k]) = [M(V, \alpha)]$ where V is a tubular neighborhood of k and α is an oriented curve on bV representing the element ml^ν. Here we assume that the oriented meridian, represented by m, has linking number 1 with k when k is given the orientation which makes it homotopic in V to an oriented longitude representing l. I do not know whether D_ν is a homomorphism or not.

The diagram

$$\begin{array}{ccc} \theta^{3,1} & \xrightarrow{\;D_\nu\;} & \mathcal{H}^3 \\ & {\scriptstyle \nu\cdot\chi}\searrow \qquad \swarrow{\scriptstyle \mu} & \\ & Z_2 & \end{array}$$

is commutative (Theorem 4).

Proposition 4.1 says, in particular, that if V is a regular neighborhood of a slice knot k, then $M(V, \alpha)$ bounds an acyclic manifold, provided that the linking number of α and k is ± 1.

For example, *if k is the stevedore's knot* (see [7], Example 10), *and V is a regular neighborhood of k, then $M(V, \alpha)$ even bounds a contractible manifold.* To see this consider the 2-disk D^2 imbedded in the 4-disk D^4, with $D^2 \cap bD^2 = bD^2 = k$, described in Example 10 in [7]. We remove a neighborhood of D^2 in D^4 and sew it back in such a way that we obtain a manifold W_ν^4 having $M(V, \alpha)$ as boundary. One can see that W_ν^4 is acyclic. The group $\pi_1(D^4 - D^2)$ has a presentation $\{x, a; xa^2 = ax\}$. The fundamental group of W_ν^4 is obtained by adjoining to this group the relation $ml^\nu = 1$ where $m = x$ and $l = x^{-2}a^{-2}xa^2x^{-1}a^{-1}x^{-1}axa^{-2}x^2$. One can see that, in the presence of the relation $xa^2 = ax$, the relation $ml^\nu = 1$ is equivalent to $x = a^{2\nu}$. It is now easy to see that

$$\pi_1(W_\nu^4) = \{x, a; xa^2 = ax, x = a^{2\nu}\}$$

is trivial. It follows that W_ν^4 is contractible.

By Theorem 5, if $\nu \neq 0$ the boundary of W_ν^4 is not simply connected so that W_ν^4 is not the 4-disk. Examples of such contractible 4-manifolds have been given by Poenaru ([35]), Mazur ([26]) and Curtis.

Furthermore, if ν is even $W_\nu^4 \times I \approx D^5$. To verify this it suffices to show, because of the Poincaré Conjecture in dimension 5, that the boundary of $W_\nu^4 \times I$, i.e. the double of W_ν^4, is S^4.

The double of W_ν^4 can be obtained as follows. Take two copies (D_+^4, D_+^2), (D_-^4, D_-^2) of the pair (D^4, D^2) given above. Paste the two copies by the identity map on their boundaries to obtain a knotted 2-sphere $D_+^2 \cup D_-^2$ in $S^4 = D_+^4 \cup D_-^4$. Let N be a tubular neighborhood of $D_+^2 \cup D_-^2$ in S^4. There is a diffeomorphism φ from $(D_+^2 \cup D_-^2) \times D^2$ onto N. Now in the disjoint union $(S^4 - \operatorname{int} N) +$

$(D_+^2 \cup D_-^2) \times D^2$ identify $(z_\epsilon, e^{i\theta}) \in D_\epsilon^2 \times bD^2$ $\epsilon = +, -$ with $\varphi(e^{i\nu\theta}z_\epsilon, e^{i\theta}) \in bN$ where all 2-disks are considered as the unit disk in the complex numbers. The resulting manifold is the double of W_ν^4.

If ν is even, then the autohomeomorphism of $(D_+^2 \cup D_-^2) \times D^2$ which carries $(z_\epsilon, e^{i\theta})$ to $(e^{i\nu\theta}z_\epsilon, e^{i\theta})$ is isotopic to the identity map. It follows that, if ν is even, the double of W_ν^4 is S^4. Hence, by the Poincaré Conjecture in dimension 5, $W_\nu^4 \times I \approx D^5$ if ν is even. The examples given by Poenaru and Mazur also have the property that their product with the unit interval is D^5.

5. Doubled knots

Now we will look at the fundamental group of homology spheres obtained by Dehn's method. We consider in this section doubled knots ([45], [7] page 144).

THEOREM 5. *Let V be a tame solid torus in S^3 whose core is a nontrivial doubled knot and let α be a simple closed curve on bV which does not bound a disk in V. Then $M(V, \alpha)$ is not simply connected.*

We will prove the theorem first for the case where the core is the double of a nontrivial knot. In this case there is a *knotted* solid torus W which contains V in its interior and there is a, not necessarily faithful, homeomorphism from W onto an unknotted solid torus V_1' which maps V onto the unknotted solid torus V_2 as shown in Fig. 1 b).

Let α_2 be the image of α under this homeomorphism; α_2 is not a meridian. Let $V_1 = S^3 - \text{int } V_1'$. The manifold $M(V, \alpha)$ can be expressed as $(S^3 - \text{int } W) \cup (W - \text{int } V) +_h S^1 \times D^2$ where $h: bV \to S^1 \times bD^2$ is a homeomorphism that maps α onto $1 \times bD^2$, $1 \in S^1$.

The homomorphism $\pi_1(b(S^3 - \text{int } W)) \to \pi_1(S^3 - \text{int } W)$ induced by inclusion is a monomorphism since W is knotted.

$(W - \text{int } V) +_h S^1 \times D^2$ is homeomorphic to $(V_1' - \text{int } V_2) +_k S^1 \times D^2$ where $k: bV_2 \to S^1 \times bD^2$ is a homeomorphism which carries α_2 to $1 \times bD^2$; this is precisely $M(V_1 \cup V_2, \alpha_2)$.

Now, there is an autohomeomorphism of S^3 which maps V_1 onto V_2 and V_2 onto V_1. Let α_1 be the image of α_2 under this homeomorphism; α_1 is not a meridian i.e. it does not bound a disk in V_1. By 3.1 $M(V_1 \cup V_2, \alpha_2)$ is homeomorphic to $M(V_1 \cup V_2, \alpha_1)$. If $H_1(M(V_1 \cup V_2, \alpha_1)) \neq Z$, $M(V, \alpha)$ will not even be a homology sphere. The group $H_1(M(V_1 \cup V_2, \alpha_1))$ is Z only if α_1, with some orientation, represents an element of $\pi_1(bV_1)$ of the form ml^{ν} where m is represented by a meridian and l by a longitude of bV_1. We may assume this is the case. Since α_1 is not a meridian $\nu \neq 0$. Then there is a disk D^2 in S^3 such that $D \cap V_1 = bD$ and $D \cap \alpha_1$ consists of one point. We obtain a homeomorphism of $S^3 - \text{int } V_1$ onto itself as follows. Cut $S^3 - \text{int } V_1$ along D. Twist one of the now exposed faces through ν revolutions and sew back together along D. If the twisting is done in the right direction, the image of α_1 under this homeomorphism will be a meridian of V_1.

The homeomorphism changes V_2 to a solid torus V_3 whose core is the double of a trivial knot with twist ν or $-\nu$.

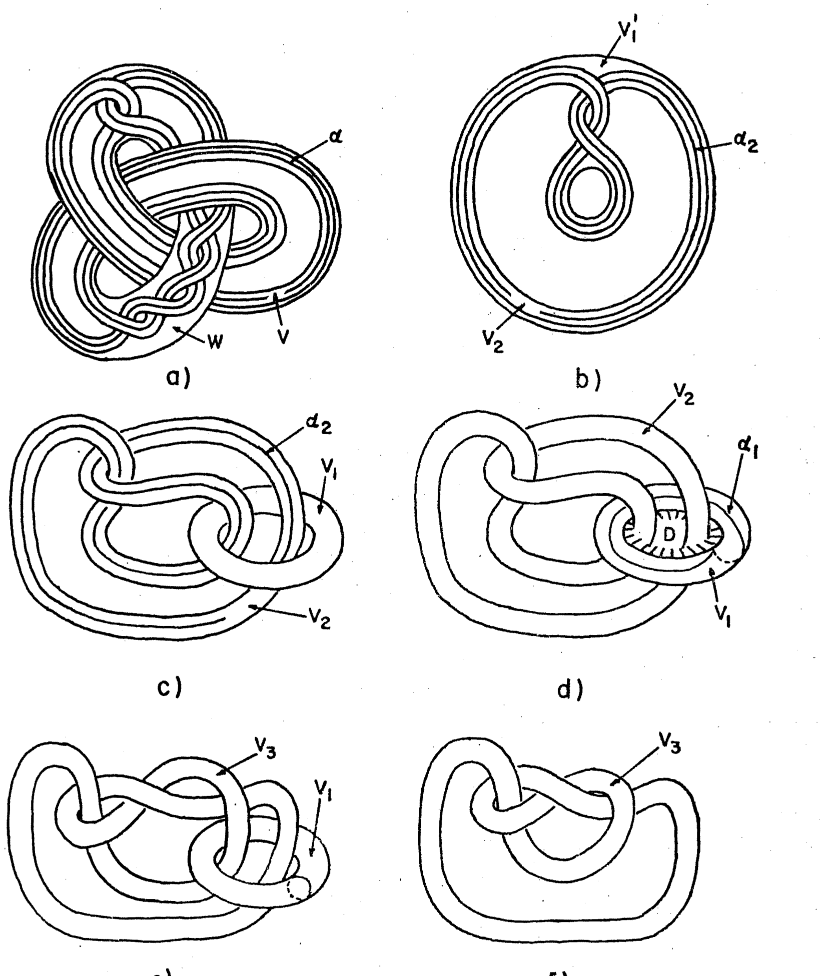


FIG. 1.

By 3.1 and 3.3, $M(V_1 \cup V_2, \alpha_1)$ is homeomorphic to $M(V_3, \Phi)$. Since V_3 is knotted, the homomorphism $\pi_1(bM(V_3, \Phi)) \to \pi_1(M(V_3, \Phi))$, and therefore the homomorphism $\pi_1(b((W - \text{int } V) +_h S^1 \times D^2)) \to \pi_1((W - \text{int } V) +_h S^1 \times D^2)$, induced by inclusion, is a monomorphism so that $\pi_1(M(V, \alpha))$ is the free product with amalgamation $\pi_1(S^3 - \text{int } W) *_{\pi_1(bW)} \pi_1((W - \text{int } V) +_h S^1 \times D^2)$ which is non trivial. This completes the proof of the theorem for doubles of non trivial knots.

(Alternatively, the theorem for doubles of non trivial knots is a consequence of theorem 9 since $I(W, k) = \{\pm 1, 0, 0, \cdots\}$ where k is a core of V).

We now consider the case of doubles of the trivial knot.

Let V be a tame solid torus whose core is the double of the trivial knot with twist ρ. Let m and l be a meridian and longitude with the orientations indicated in Fig. 2 a).

We also denote by m and l the elements of $\pi_1(bV)$ or of $\pi_1(S^3 - \text{int } V)$ represented by these curves. (Say we have chosen $m \cap l$ as base point). Let α be a curve on bV which, when oriented, represents the element $m^\tau l^{\ \nu}$. The manifold $M(V, \alpha)$ will be a homology sphere if and only if $|\tau| = 1$. We may assume changing the orientation of α if necessary, that α represents the element $ml^{\ \nu}$. Then denote $M(V, \alpha)$ by $M(\rho, \nu)$. In view of 3.2 this is well defined and to complete the proof of the theorem it is sufficient to prove:

PROPOSITION 5.1. *$M(\rho, \nu)$ is not simply connected if both ρ and ν are non-zero.*

Notice that $M(1, \nu)$ is homeomorphic to $M(1, -\nu)$ since the figure eight knot (the double of the trivial knot with twist 1) is amphicheiral.

We give another description of $M(\rho, \nu)$.

Let V_1, m_1, l_1 and D be the solid torus, oriented curves and disk shown in Fig. 2 b). Just as in the proof of the theorem for doubles of non trivial knots, by cutting along D and twisting we define an autohomeomorphism of $S^3 - \text{int } V_1$ which maps V onto an unknotted solid torus V_2, and the oriented meridian and longitude m, l onto the oriented meridian and longitude m_2, l_2. Again, m_1, l_1 m_2, l_2 will also denote the elements of $\pi_1(S^3 - \text{int } (V_1 \cup V_2))$ represented by these curves (after joining them to the base point).

The image of m_1 under this homeomorphism is a curve α_1 which represents the element $m_1 l_1^{\rho}$. The image of α is a curve α_2 which represents $m_2 l_2^{\ \nu}$.

Hence, $M(\rho, \nu)$ can also be defined as $M(V_1 \cup V_2, \alpha_1 \cup \alpha_2)$ where α_1 and α_2 represent, respectively, the elements $m_1 l_1^{\rho}$ and $m_2 l_2^{\ \nu}$ of $\pi_1(S^3 - \text{int } (V_1 \cup V_2))$. Since there is a homeomorphism of S^3 that interchanges V_1 and V_2, m_1 and m_2, l_1 and l_2, it follows that *$M(\rho, \nu)$ is homeomorphic to $M(\nu, \rho)$.*

$\pi_1(S^3 - \text{int } (V_1 \cup V_2))$ has the presentation (see [30])

$$\{m_1, m_2, l_1, l_2 ; l_1 = m_1 m_2^{-1} m_1^{-1} m_2 m_1 m_2 m_1^{-1} m_2^{-1},$$
$$l_2 = m_2 m_1^{-1} m_2^{-1} m_1 m_2 m_1 m_2^{-1} m_1^{-1},$$
$$[m_1, l_1] = 1, \quad [m_2, l_2] = 1\}$$

We obtain $\pi_1 M(\rho, \nu)$ by adjoining the relations $m_1 l_1^{\rho} = 1$, $m_2 l_2^{\ \nu} = 1$.

$$\pi_1 M(\rho, \nu) = \{m_1, m_2, l_1, l_2 ; l_1 = m_1 m_2^{-1} m_1^{-1} m_2 m_1 m_2 m_1^{-1} m_2^{-1},$$
$$l_2 = m_2 m_1^{-1} m_2^{-1} m_1 m_2 m_1 m_2^{-1} m_1^{-1}, \; m_1 l_1^{\rho} = 1, \; m_2 l_2^{\ \nu} = 1\}$$

$$(1) \quad \pi_1 M(\rho, \nu) = \{m_1, m_2 ; m_1 (m_1 m_2^{-1} m_1^{-1} m_2 m_1 m_2 m_1^{-1} m_2^{-1})^{\rho}$$
$$= m_2 (m_2 m_1^{-1} m_2^{-1} m_1 m_2 m_1 m_2^{-1} m_1^{-1})^{\nu} = 1\}$$

LEMMA 5.1. *$\pi_1 M(\rho, \nu)$ has $(2, 3, 6\rho + 1; \nu + 1)$, $(3, 3 \mid 3\rho + 1, 3\nu + 1)$ and $(4\rho + 1, 4 \mid 2\nu + 1, 2)$ as factor groups where*

$$(l, m, n; p) = \{A, B; A^l = B^m = (AB)^n = [A, B]^p = 1\}$$

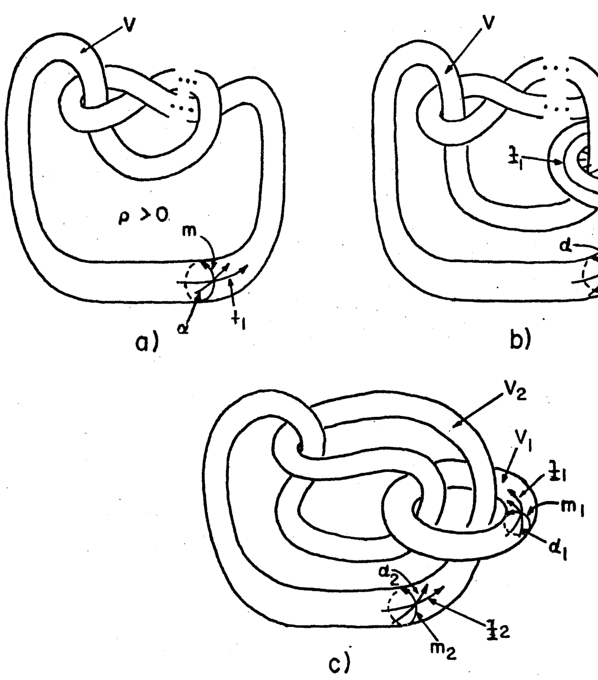


FIG. 2.

and

$$(l, m \mid n, p) = \{A, B; A^l = B^m = (AB)^n = (A^{-1}B)^p = 1\}$$

Proof. Adjoin to the presentation (1) the relation

$$m_2m_1^{-1}m_2^{-1}m_1m_2m_1m_2^{-1}m_1^{-1} = m_2 .$$

We obtain the group

$$(2) \qquad \{m_1, m_2 ; m_1(m_1m_2^{-1}m_1^{-1}m_2m_1m_2m_1^{-1}m_2^{-1})^{\rho} = m_2^{\nu+1} = m_1^{-1}m_2^{-1}m_1m_2m_1m_2^{-1}m_1^{-1} = 1\}$$

The isomorphism φ from the free group $\{m_1, m_2\}$ onto the free group $\{A, B\}$ defined by $\varphi(m_1) = AB$, $\varphi(m_2) = (ABAB^2)^{-1}$ carries $m_1(m_1m_2^{-1}m_1^{-1}m_2m_1m_2 \cdot m_1^{-1}m_2^{-1})^{\rho}$, $m_2^{\nu+1}$ and $m_1^{-1}m_2^{-1}m_1m_2m_1m_2^{-1}m_1^{-1}$ to $AB((AB)^3(A^{-1}B^{-2})^2A^{-1}B)^{\rho}$, $(ABAB^2)^{-\nu-1}$ and AB^2A^2BA respectively. Thus the group (2) can be presented as $\{A, B; AB((AB)^3(A^{-1}B^{-2})^2A^{-1}B)^{\rho} = (ABAB^2)^{\nu+1} = AB^2A^2BA = 1\}$.

Adjoining the relations $A^2 = B^3 = 1$ we obtain the desired group $\{A,B;A^2 = B^3 = (AB)^{6\rho+1} = [A,B]^{\nu+1} = 1\}$.

The isomorphism ψ from $\{m_1, m_2\}$ onto $\{A, B\}$ defined by $\psi(m_1) = B^{-1}A^{-1}$, $\psi(m_2) = AB^2$, sends $m_1(m_1m_2^{-1}m_1^{-1}m_2m_1m_2m_1^{-1}m_2^{-1})^{\rho}$ and $m_2(m_2m_1^{-1}m_2^{-1}m_1 \cdot m_2m_1m_2^{-1}m_1^{-1})^{\nu}$ to $B^{-1}A^{-1}(B^{-1}A^{-1}B^{-1}AB^3AB^{-1}A^{-1})^{\rho}$ and $AB^2(AB^2AB^{-1}A^{-2}B^{-1})^{\nu}$ respectively.

Hence $\pi_1 M(\rho, \nu)$ has the presentation

$$\{A,\ B;\ B^{-1}A^{-1}(B^{-1}A^{-1}B^{-1}AB^3AB^{-1}A^{-1})^\rho = AB^2(AB^2AB^{-1}A^{-2}B^{-1})^\nu = 1\}$$

If one adjoins the relations $A^3 = B^3 = 1$ one obtains the group $\{A, B; A^3 = B^3 = (AB)^{3\rho+1} = (A^{-1}B)^{3\nu+1} = 1\}$.

If, instead, one adjoins the relations $A^2 = B^4 = 1$, one obtains $\{A, B; A^2 = B^4 = (AB)^{4\rho+1} = (AB^2)^{2\nu+1} = 1\}$ which is another presentation of the group $(4\rho + 1, 4 \mid 2\nu + 1, 2)$. (See [5], page 77). This completes the proof of the lemma.

LEMMA 5.2. a) *If* $|n| > 10$ *and* $|p| > 5$, $(2, 3, n; p)$ *is non trivial.*
b) *If* $|n| > 5$ *and* $|p| > 5$, $(3, 3 \mid n, p)$ *is non trivial.*
c) *If* $k \geq 1$, $(6k + 1, 4 \mid 3, 2)$ *and* $(6k + 3, 4 \mid 3, 2)$ *are non trivial.*
d) $(2, 3, 11; 5)$ *is non trivial.*

Proof. We will show first, applying theorem IV (i) of [24], that $(2, 3, n; p)$ is neither trivial nor isomorphic to Z_2 for $n > 10$, $p > 5$. We use the terminology of [24]. It suffices to show that B is not in the normal closure N of $\{(AB)^n, [A, B]^p\}$ in the free product $\{A; A^2 = 1\} * \{B; B^3 = 1\}$.

If $n > 10$ and $p > 6$, no element in the symmetrized set

$$R = \{(AB)^n,\ (BA)^n,\ (AB^{-1})^n,\ (B^{-1}A)^n,\ [A, B]^p,\ [B, A]^p,\ [A, B^{-1}]^p,\ [B^{-1}, A]^p\}$$

is a product of less than 6 pieces. Therefore, by Theorem IV (i) of [24], an element w in N must have reduced form bac, where $r = ax_1x_2x_3$ reduced, for some r in R and pieces x_1, x_2, x_3. In our case, $|r| \geq 22$ and pieces have length at most 5. Therefore $|a| \geq 7$ and $|w| \geq 7$. In particular B is not in N.

Since $(3, 3 \mid n, p)$ is a subgroup of index 2 of $(2, 3, 2n; p)$ (see [5] page 90) it follows that $(3, 3 \mid n, p)$ is non trivial if $|n|, |p| > 5$.

For c) we use the presentation

$$\{A, B; A^2 = B^4 = (AB)^l = (AB^2)^3 = 1\} \text{ of } (l, 4 \mid 3, 2).$$

A representation of $(6k + 1, 4 \mid 3, 2)$, with $k \geq 1$, into the symmetric group of degree $6k + 1$ can be defined by sending A to the permutation $\prod_{i=1}^{k} (6i - 5\ 6i - 4)\ (6i - 3\ 6i)$ and B to $\prod_{i=1}^{k} (6i - 4\ 6i - 3\ 6i - 2\ 6i - 1)\ (6i\ 6i + 1)$.

For $k \geq 2$, one can define a representation of $(6k + 3, 4 \mid 3, 2)$ into the symmetric group of degree $6k + 3$ by sending A to

$$\prod_{1 \leq i \leq k-2} (6i - 5\ 6i - 4)(6i - 3\ 6i)(6k - 11\ 6k - 10)(6k - 9\ 6k - 2)\cdot$$
$$(6k - 6\ 6k - 5)(6k - 4\ 6k + 3)(6k - 1\ 6k)(6k + 1\ 6k - 7)$$

and B to

$$\prod_{1 \leq i \leq k-2} (6i - 4\ 6i - 3\ 6i - 2\ 6i - 1)(6i\ 6i + 1)\cdot$$
$$(6k - 10\ 6k - 9\ 6k - 8\ 6k - 7)(6k - 5\ 6k - 4\ 6k - 3\ 6k - 2)\cdot$$
$$(6k\ 6k + 3\ 6k + 2\ 6k + 1)(6k - 6\ 6k - 1).$$

The group $(6k + 3, 4 \mid 3, 2)$ is also non trivial for $k = 1$. In fact it is the group $LF(2, 17)$. (See [5] page 75).

Finally, a representation of $\{A, B; A^2 = B^3 = (AB)^{11} = [A, B]^5 = 1\}$ can be defined by sending A to $(2\ 4)(5\ 7)(6\ 10)(9\ 11)$ and B to $(1\ 2\ 3)(4\ 5\ 6)(7\ 8\ 9)$. The proof of Lemma 5.2 is complete.

Proof of Proposition 5.1.

Case 1) Both ρ and ν are different from -1, 1 and -2.

Then the factor group $(3, 3 \mid 3\rho + 1, 3\nu + 1)$ of $\pi_1 M(\rho, \nu)$ is non trivial by Lemma 5.2 b).

Case 2) One of the numbers ρ, ν (we may assume it is ν) is -1.

Then the factor group $(2, 3, 6\rho + 1; \nu + 1) = \{A, B; A^2 = B^3 = (AB)^{6\rho+1} = 1\}$ is non trivial (see [4] page 67).

Case 3) One of the numbers ρ, ν, say ν, is 1.

Since $M(\rho, 1) \approx M(-\rho, 1)$ we may assume $\rho > 0$.

If $\rho = 3r > 0$, $\pi_1 M(\rho, 1)$ has the factor group $(12r + 1, 4 \mid 3, 2)$; if $\rho = 3r + 1$, $r > 0$, then $\pi_1 M(\rho, 1) \approx \pi_1 M(-\rho, 1)$ has the factor group $\{A, B; A^2 = B^4 = (AB)^{4(-3r-1)+1} = (AB^2)^3 = 1\} \approx (12r + 3, 4 \mid 3, 2)$; if $\rho = 3r + 2 > 0$, then $\pi_1 M(\rho, 1)$ has the factor group $(12r + 9, 4 \mid 3, 2)$. All these groups are non trivial by 5.2 c). If $\rho = 1$, $\pi_1 M(\rho, 1) \approx \pi_1 M(-1, 1)$ which is non trivial by case 2).

Case 4) One of the numbers ρ, ν, say ρ, is -2.

If $\nu > 4$ or $\nu < -6$, the factor group $(2, 3, 6\rho + 1; \nu + 1)$ of $\pi_1 M(\rho, \nu)$ is non trivial by Lemma 5.2 a).

$\pi_1 M(-2, -6)$ has $(2, 3, 11; 5)$ as factor group which is non trivial (Lemma 5.2 d)).

$\pi_1 M(-2, -5)$ has $(19, 4 \mid 3, 2)$ as factor group. This is non trivial by Lemma 5.2 c).

$\pi_1 M(-2, -4)$ has $(15, 4 \mid 3, 2)$ as factor group. This is non trivial by Lemma 5.2 c).

$\pi_1 M(-2, -3)$ has $(3, 3 \mid 5, 8)$ as factor group which in turn has the non trivial (see [5] page 84) group $(3, 3 \mid 5, 4)$ as factor group.

$\pi_1 M(-2, -2)$ has $(4, 7 \mid 3, 2)$ as factor group, which is non trivial ([5] page 83).

$\pi_1 M(-2, -1)$ is non trivial by case 2).

$\pi_1 M(-2, 1)$ is nontrivial by case 3).

$\pi_1 M(-2, 2)$ has $(9, 4 \mid 3, 2)$ as factor group. This is non trivial ([5] page 84).

$\pi_1 M(-2, 3)$ has $(13, 4 \mid 3, 2)$ as factor group. By Lemma 5.2 c) this group is non trivial.

Finally, $\pi_1 M(-2, 4)$ has $(2, 3, 11; 5)$ as factor group which is non trivial (Lemma 5.2 d)).

Cases 1), 2), 3) and 4) cover all possibilities. The proof of Proposition 5.1, and therefore of Theorem 5, is complete.

Remarks. Since the Arf invariant of a doubled knot with twist ρ is ρ mod 2, it follows from Th. 4 that $\mu([M(\rho, \nu)]) = \nu \cdot \rho \bmod 2$. In particular $M(\rho, \nu)$ does not bound an acyclic manifold if ρ and ν are odd.

$M(2, \nu)$ bounds a contractible manifold for all ν. (See §4).

The manifold $M(1, 1)$ was considered by Bing in [2] page 102. Several descriptions of $M(1, 1)$ can be given as follows.

1) $M(1, 1)$ can be obtained by doing surgery to a trefoil knot (in other words $M(1, 1) \approx M(-1, 1)$).

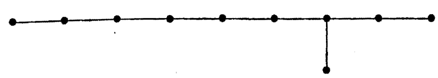
2) $M(1, 1)$ is the Seifert manifold $(O\ o; 0 \mid -1; 2, 1; 3, 1; 7, 1)$ ([39]).

3) $M(1, 1)$ is the p-fold cyclic covering of S^3 branched over the torus knot of type q, r where p, q, r is any permutation of 2, 3, 7.

4) $M(1, 1)$ is the Brieskorn manifold

$$\{(z_0, z_1, z_2) \in C^3 : z_0^2 + z_1^3 + z_2^7 = 0,\ z_0\bar{z}_0 + z_1\bar{z}_1 + z_2\bar{z}_2 = 1\}.$$

5) $M(1, 1)$ is the tree manifold ([42]) which corresponds to the tree



with all vertices weighted by 2.

6) $M(1, 1)$ is Friedgé's generalized dodecahedral space β^7 ([8]). That is to say, $M(1, 1)$ can be obtained from a polyhedron having 2 heptagons as bases and 14 pentagons as side faces, by identifying faces with their opposite ones.

7) $M(1, 1)$ can be obtained by doing surgery to the Borromean rings.

Hence, the fact that $M(1, 1)$ is not simply connected can also be obtained from [39, §13], [39, Satz 12], [10] (see also [11] Prop. II 4.3), [28], [42] or [8].

Bing and Martin ([3]) ask whether the groups $\pi_1 M(1, \nu)$ have finite non trivial homomorphs. Our proof of Proposition 5.1 gives an affirmative answer to this question.

6. Slice links in the weak sense

In [16] the relationships between four possible definitions of slice link are studied, the only open question being whether or not a link cobordic to zero is a slice link in the weak sense. We will see that, for example, the link whose components are the cores of V_1 and V_2 in Fig. 2 c) is a link cobordic to zero which is not a slice link in the weak sense i.e., it does not bound a locally flat surface of genus 0 in D^4.

Suppose that this link is a slice link in the weak sense so that it is the boundary of a locally flat annulus A^2 in D^4. We may assume that A^2 is a differentiable submanifold ([19]). Let W^4 be a tubular neighborhood of A^2, with $W^4 \cap bD^4 = V_1 \cup V_2$ where V_1 and V_2 are as in Fig. 2 c). Let α be a simple closed curve on bV_1 representing, with some orientation, the element $m_1 l_1$ of $\pi_1(bV_1)$ where m_1 represents a meridian and l_1 a longitude of bV_1. Let α_2 be a simple closed curve which with some orientation, is homotopic to α_1 in $bW - \text{int}\,(V_1 \cup V_2)$. Then α_2 represents the element $m_2^{\pm 1} l_2^{\pm 1}$ of $\pi_1(bV_2)$ where m_2 and l_2 are represented by meridian and longitude of bV_2.

As in the proof of Prop. 4.1 remove W and sew it back so as to obtain an acyclic manifold whose boundary is $M(V_1 \cup V_2, \alpha_1 \cup \alpha_2) \approx M(\pm 1, \pm 1)$. However, $M(\rho, \nu)$ does not bound an acyclic manifold if ρ and ν are odd (see remarks after Corollary 5.1). This contradiction proves that the link we are considering is not a slice link in the weak sense; it is a link cobordic to zero by [16, Lemma 8].

In fact one can define an Arf invariant for a certain kind of link called proper in [36], and this definition is such that if λ is a proper link which is a slice link in the weak sense, then $\chi(\lambda) = 0$. (See [36].) Definitions follow.

A differentiably imbedded link λ with oriented components $k_1, \cdots, k_n$ is called a *proper link* if, for every j, the sum of the linking numbers of k_j with the rest of the components is an even integer. Notice this property does not depend on the orientation of the components.

Now, suppose that M^2 is a 2-manifold of genus 0 differentiably imbedded in $S^3 \times I$ with $bM^2 = M^2 \cap b(S^3 \times I)$ and such that $M^2 \cap (S^3 \times \{0\})$ is a proper link λ in $S^3 \times \{0\}$ and $M^2 \cap (S^3 \times \{1\})$ is a knot k in $S^3 \times \{1\}$. Then, define the Arf invariant $\chi(\lambda)$ of λ to be the Arf invariant $\chi(k)$ of k. By [36, Th. 2] this is well defined.

If λ is a proper link which is a slice link in the weak sense, then $\chi(\lambda)$ is the Arf invariant of the trivial knot, and this is zero.

The link λ of two components that we were considering above is a proper link and $\chi(\lambda) \neq 0$. Hence the Arf invariant also shows that λ is not a slice link in the weak sense.

7. Composite knots

Noga ([31, Satz 1)] has proved that if a regular neighborhood of a composite knot is removed from S^3 and sewn back differently, then the manifold obtained is not S^3. A modification of his proof, using a remark in [2, page 101], shows that this manifold is not even simply connected. We give the details.

THEOREM 7. *If the core of a tame solid torus V is a composite (non prime) knot and α is a simple closed curve on bV which does not bound a disk in V, then $M(V, \alpha)$ is not simply connected.*

LEMMA 7.1. *(See [2] page 101) Let T^2 be a two-dimensional torus tamely imbedded in a homotopy 3-sphere M^3. Then the closure of one of the components of $M^3 - T^2$ is homeomorphic to $V^3 \# \Sigma^3$ where V^3 is a solid torus, Σ^3 is a homotopy sphere and $\#$ denotes connected sum. In particular the fundamental group of this component is infinite cyclic.*

Proof of Lemma. Let A and B be the closures of the components of $M - T$. Let $i_*: \pi_1(bA) \to \pi_1(A)$, $j_*: \pi_1(bB) \to \pi_1(B)$ be induced by inclusions. Now i_* and j_* cannot be both monomorphisms because this would imply that $\pi_1(M^3)$ is the free product with amalgamation $\pi_1(A) *_{\pi_1(T)} \pi_1(B)$, which would contradict the fact that M^3 is simply connected. Thus i_*, say, is not a monomorphism. Then by the loop theorem ([33]) and Dehn's lemma ([34]) there is a disk D^2 in A with $bD^2 = D^2 \cap bA$. A regular neighborhood N of $bA \cup D^2$ in A is homeomorphic to a solid torus with a 3-cell removed. One of the components of bN is a 2-sphere S^2. Since M^3 is simply connected, the closures of the components of $M^3 - S^2$ are homotopy cells. One of these components is $A - N$. Thus A is the sum of a homotopy cell $\overline{A - N}$ with N, a solid torus with a 3-cell removed, that is to say, A is the connected sum of a homotopy sphere with a solid torus.

Proof of Theorem 7. Suppose that $k_1 \# k_2$ is the core of V, with k_1 and k_2 non trivial. Let S^2 be a 2-sphere which cuts $k_1 \# k_2$ in two points and such that, for $i = 1, 2$, the component A_i of $S^3 - S^2$ cuts from $k_1 \# k_2$ an arc that together with an arc of S^2, forms a knot equivalent to k_i. We can take S^2 so that $S^2 \cap V$ consists of two meridian cells.

Consider the torus $T^2 = (S^2 - \text{int } V) \cup (A_1 \cap bV)$ imbedded in $M(V, \alpha)$. One of the components of $M(V, \alpha) - T$ is $A_1 - V$, whose fundamental group is not Z since k_1 is not trivial.

The other component can be expressed as $(A_2 - \text{int } V) +_{h'} S^1 \times D^2$, where h' is a homeomorphism from $bV - T$ into $S^1 \times bD^2$ to be described now. $M(V, \alpha)$ is defined by $(S^3 - \text{int } V) +_h S^1 \times D^2$ where the homeomorphism $h: b(S^3 - \text{int } V) \to S^1 \times bD^2$ maps α onto $1 \times bD^2$. We define h' as the restriction of h to $bV - T$. Since α is not a meridian, the annulus $h'(bV - T)$ is not contractible in $S^1 \times D^2$ so that the homomorphism $\pi_1(h'(bV - T)) \to \pi_1(S^1 \times D^2)$ induced by inclusion is a monomorphism.

Now, $\pi_1(bV - T) \to \pi_1(A_2 - \text{int } V)$ is also a monomorphism.

Thus, $\pi_1((A_2 - \text{int } V) +_{h'} S^1 \times D^2)$ is a free product with amalgamation $\pi_1(A_2 - \text{int } V) *_{\pi_1(bV - T)} \pi_1(S^1 \times D^2)$ which is not infinite cyclic since its subgroup $\pi_1(A_2 - \text{int } V)$ is not Z. In view of Lemma 7.1 this completes the proof of the theorem.

8. Cable knots

In this section we consider cable knots. Let k_1 be a tame knot in S^3 and let k be a simple closed curve in the boundary of a regular neighborhood T of k_1 such that k represents the element $m^r l^s$ of $\pi_1(bT)$ where r and s are relatively prime positive integers, m is represented by a meridian and l by a longitude of bT. Then k is called a *cable knot of type* r, s about k_1.

THEOREM 8. *Let k be a cable knot of type r, s about a non trivial knot k_1 with $r \neq 1$ and $rs \neq 2$. Let V be a closed regular neighborhood of k and α a simple closed curve on bV which does not bound a disk in V. Then $M(V, \alpha)$ is not simply connected.*

Remark. If k_1 is trivial i.e. if k is a torus knot then the theorem holds provided $r, s \neq 1$. (See [39, §13] or [13]).

Proof. We may assume that α represents an element of the form $ml^\nu \in \pi_1(bV)$ with $\nu \neq 0$, where m is represented by a meridian and l by a longitude of bV.

Let W be a closed regular neighborhood of k_1 which contains V in its interior. Then $M(V, \alpha)$ can be obtained by pasting together along their boundaries $S^3 - \text{int } W$ and $N = (W - \text{int } V) +_\varphi S^1 \times D^2$ where $\varphi: bV \to S^1 \times bD^2$ is a homeomorphism that maps α onto $1 \times bD^2$.

Since W is knotted, $\pi_1(S^3 - W) \neq Z$. Hence, to prove the theorem, it suffices, by Lemma 7.1, to prove that $\pi_1(N)$ is not infinite cyclic. N has a Seifert fibration ([39]) with two exceptional fibers. To see this, we first fiber W in such a way that the ordinary fibers are cable knots of type r, s about k_1 and V is a union of ordinary fibers so that $W - \text{int } V$ is fibered. The fiber in bV represents an element $m^{\pm rs}l^{\pm 1}$ in $\pi_1(bV)$ so that α is not homotopic in bV to a fiber and the image, under φ, of a fiber is not a meridian in $S^1 \times D^2$. Hence the fibration of $S^1 \times bD^2$, induced via φ by the fibration of bV, can be extended to a Seifert fibration of $S^1 \times D^2$. We have then a Seifert fibration of N with two exceptional fibers with multiplicities r and $|\nu rs - 1|$ (compare [39, §13]). Then $\pi_1(N)$ has a presentation of the form (see [32, §1, (1.1)])

$$\{t_1, q_1, q_2, h; [q_1, h] = [q_2, h] = q_1^{\gamma_1}h^{\beta_1} = q_2^{\gamma_2}h^{\beta_2} = t_1q_1q_2h^{-b} = 1\}$$

where γ_1 and γ_2 are the multiplicities of the exceptional fibers i.e. $\gamma_1 = r$ and $\gamma_2 = |\nu rs - 1|$. By the hypothesis on r and s, γ_1 and γ_2 are greater than 1.

If we adjoin the relation $h = 1$ to the presentation of $\pi_1(N)$, we obtain the free product $Z_{\gamma_1} * Z_{\gamma_2}$. Hence $\pi_1(N)$ is not infinite cyclic. This finishes the proof of the theorem.

9. Some knots contained in knotted solid tori

We will prove in this section a theorem for certain knots contained in knotted solid tori with zero winding number, which is analogous to Theorems 5, 7 and 8.

Let W be a tame solid torus in S^3. Let k be a tame simple closed oriented curve contractible in int W. Let V be a closed regular neighborhood of k contained in int W and α simple closed curve on bV representing the element ml^{ν} in $\pi_1(bV)$, where m is represented by a meridian and l by a longitude. Let

$$X = (W - \text{int } V) +_{\varphi} S^1 \times D^2,$$

where $\varphi: bV \to S^1 \times D^2$ is a homeomorphism that carries α onto $\{1\} \times bD^2$. The first homology group of X is infinite cyclic. We want to compute the Alexander polynomial of $\pi_1(X)$.

Consider the universal cover $\tilde{W}$ of W; then $\tilde{W}$ is contractible. Let t be a generator of the group of covering transformations. Take a fixed lift k_0 of k. Define k_m by $k_m = t^m(k_0)$ where m is any integer. Orientations of $\tilde{W}$ and all k_m are determined by the orientations of S^3 and k. Denote by a_i the linking number $L(k_0, k_i)$ for $i \neq 0$; we have $a_i = a_{-i}$. Define the *self-linking* $I(W, k)$ *of* k *in* W to be the sequence $(a_1, a_2, \cdots)$. (Compare [12, §4]). Now, write $b_i = \nu a_i$ for $i \neq 0$ and $b_0 = 1 - \sum_{i \neq 0} b_i$ (notice that only a finite number of b_is are non zero.)

LEMMA 9.1. *The Alexander polynomial of* $\pi_1(X)$ *is* $\sum_{-\infty<i<\infty} b_i t^i$.

Proof. (Compare [22] 140–141) Let $\tilde{X}$ be the universal abelian covering of X. Let $\tilde{V}$ be the inverse image of V under the covering map and let $\tilde{\alpha}_i$ be a lift of α which lies in the same component of V as k_i. Write $\tilde{T} = \tilde{W} - \text{int } \tilde{V}$. We obtain $\tilde{X}$ if we attach solid tori to $\tilde{T}$ along each component of $b\tilde{V}$ in such a way that a meridian from each solid torus goes to a curve $\tilde{\alpha}_i$. Now, we have the exact sequence of JZ-modules

$$0 = H_2(\tilde{W}) \to H_2(\tilde{W}, \tilde{T}) \to H_1(\tilde{T}) \to H_1(\tilde{W}) = 0.$$

By excision $H_2(\tilde{W}, \tilde{T})$, and therefore $H_1(\tilde{T})$, is a free JZ-module on one generator. A generator γ of $H_1(\tilde{T})$ is represented by a curve on $b\tilde{T}$ that has linking number 1 with k_0 and linking number 0 with all other k_i.

In the exact sequence

$$H_2(\tilde{X}, \tilde{T}) \xrightarrow{\Delta} H_1(\tilde{T}) \to H_1(\tilde{X}) \to H_1(\tilde{X}, \tilde{T})$$

we have, by excision, $H_1(\tilde{X}, \tilde{T}) = 0$, and the image of Δ is generated by the class of $\tilde{\alpha}_0$. This class can be expressed as $\lambda \cdot \gamma$ where $\lambda = \sum_i c_i t^i$ and $c_i = L(\tilde{\alpha}_0, k_i) = \nu a_i = b_i$ if $i \neq 0$.

A relation matrix for $H_1(\tilde{X})$ is then the 1×1 matrix $(\lambda(t))$ so that $\lambda(t)$ is the Alexander polynomial of $\pi_1(X)$. Since $H_1(X) = Z$ we have $\lambda(1) = 1$ and therefore $c_0 = 1 - \sum_{i \neq 0} c_i = 1 - \sum_{i \neq 0} b_i = b_0$. This completes the proof of the lemma.

THEOREM 9. *Let W be a closed regular neighborhood of a non trivial knot; let $V \subset$ int W be a tame solid torus contractible in W and let α be a simple closed curve on bV which does not bound a disk in V. If $I(W, k) \neq (0, 0, 0, \cdots)$ where k is an oriented core of V, then $M(V, \alpha)$ is not simply connected.*

Proof. As in the proof of Theorem 8, it suffices to show that the fundamental group of $X = (W - \text{int } V) +_\varphi S^1 \times D^2$ is not infinite cyclic where $\varphi: bV \to S^1 \times bD^2$ is a homeomorphism which maps α onto $1 \times bD^2$.

Since $I(W, k) \neq (0, 0, 0, \cdots)$, the Alexander polynomial of $\pi_1(X)$ is not 1. Hence $\pi_1(X)$ is not infinite cyclic. This completes the proof.

10. Complements of knots

We will prove in this section that the knots considered in §5, §7, §8 and §9 are determined by their complements or by their (external) group system.

Let k be a tame non trivial knot in S^3. We state:

Conjecture k. If V is a closed regular neighborhood of k and α is a simple closed curve in bV which does not bound a disk in V, then $M(V, \alpha)$ is not simply connected.

Special cases of this conjecture have been proved in §5, §7, §8 and §9. It is not true if k is a trivial knot.

The *exterior* of a knot k in S^3 is the closure of the complement of a regular neighborhood of k. We observe that the complement and the exterior of a knot are equivalent invariants. More precisely

PROPOSITION 10.1. *The complements of two tame knots in S^3 are homeomorphic if and only if their exteriors are homeomorphic.*

Proof. The exterior of a knot k is a manifold $E(k)$ with boundary. It is easy to see that the complement of k is homeomorphic to the interior of $E(k)$. Now

Theorem 3 in [6] says that two compact 3-manifolds with boundary are homeomorphic if and only if their interiors are homeomorphic. The proposition follows.

Remark. The proposition also holds for differentiable n-knots with $n \geq 3$. To prove this, one uses the fact that an h-cobordism between $S^1 \times S^n$ and itself is diffeomorphic to $S^1 \times S^n \times I$. (See [18] and [38]).

Two knots k and k' are said to have isomorphic (external) group systems if there are isomorphisms

$$\varphi: \pi_1(S^3 - k) \to \pi_1(S^3 - k') \; \psi: \pi_1(bV) \to \pi_1(bV')$$

such that the diagram

$$\begin{array}{ccc} \pi_1(S^3 - k) & \xrightarrow{\varphi} & \pi_1(S^3 - k') \\ \uparrow i_* & & \uparrow j_* \\ \pi_1(bV) & \xrightarrow{\psi} & \pi_1(bV') \end{array}$$

commutes, where V and V' are closed regular neighborhoods of k and k' respectively, and i_*, j_* are induced by inclusions.

PROPOSITION 10.2. *Let k and k' be tame knots such that Conjecture k is true. Then the following are equivalent*

a) *The group systems of k and k' are isomorphic.*

b) *The complements of k and k' are homeomorphic.*

c) *k and k' are equivalent (i.e. there is an autohomeomorphism of S^3 which maps k onto k').*

Proof

c) $\Rightarrow$ a) This is clear.

a) $\Rightarrow$ b) This has been proved by Waldhausen ([43]), Corollary 6.5).

b) $\Rightarrow$ c) By Prop. 10.1 there is a homeomorphism h from the exterior $S^3 -$ int V' of k' to the exterior $S^3 -$ int V of k. Let $\alpha' \subset bV'$ be a meridian and let $\alpha = h(\alpha')$. By 3.1, $M(V, \alpha)$ is homeomorphic to $M(V', \alpha')$ which is homeomorphic to S^3. By hypothesis this implies that α is a meridian. Since $h \mid bV$ maps a meridian onto a meridian, h can be extended to an autohomeomorphism of S^3. We can choose the extension so that k is mapped onto k', since any two cores in the interior of a solid torus are isotopic under an ambient isotopy leaving the boundary fixed. Thus k and k' are equivalent. This finishes the proof.

Remark. By Dehn's Lemma, the conclusion of Proposition 10.2 is valid if k is the trivial knot.

From theorems 5, 7, 8, and 9 we obtain

COROLLARY 10.1. *Let k be any of the following knots*

1) *a doubled knot,*

2) *a composite knot,*

3) *a cable knot of type* r, s *around a knot, where* r, s *are relatively prime positive integers and* $r \neq 1$, $rs \neq 2$,

4) *an oriented knot contractible in a knotted solid torus* W *with* $I(W, k) \neq 0$.

Let k' *be an arbitrary tame knot.*

Then, conditions a), b), c) of Prop. 10.2 are equivalent.

In other words, any of the knots 1), 2), 3), 4) is characterized by the topological type of its complement or by its group system.

Burde and Zieschang ([46]) have proved that a Neuwirth knot (a knot whose group has a finitely generated commutator subgroup) of genus 1 is either the trefoil knot or its complement is homeomorphic to that of the figure eight knot. By Corollary 10.1 (the figure eight knot is a doubled knot) this result can now be improved to

Theorem. The only Neuwirth knots of genus 1 *are the trefoil and the figure eight knots.*

11. Relations with the Poincare Conjecture

Conversations with Prof. Moise were very helpful to obtain the results of this section.

THEOREM 11.1. *Let* k *be a tame knot in* S^3 *such that Conjecture* k *is true (see* §10). *Then a homotopy 3-sphere is* S^3 *if it is the disjoint union of a (tame or wild) solid torus and the complement of* k.

Proof. Let T be a solid torus topologically imbedded in the homotopy 3-sphere M^3 and let h be a homeomorphism from $M - T$ onto $S^3 - k$.

By [25], (see the remark after Corollary 1.3), there is a cube with handles C in M^3 which contains T in its interior. Take a closed regular neighborhood V of k in S^3 which does not intersect $h(M^3 - C)$. Let W be the closure of the component of $M^3 - h^{-1}(bV)$ which contains T. Then bW is a tame torus in M^3 and $M^3 - \text{int } W$ is homeomorphic to $S^3 - \text{int } V$. Since k is non trivial, it follows, by Lemma 7.1, that W is the connected sum of a solid torus and a homotopy sphere. But W is contained in a cube with handles so that every homotopy 3-disk contained in W is a 3-disk. Hence W is a solid torus and M^3 is homeomorphic to $M(V, \alpha)$ for some α.

Since we are assuming that Conjecture k holds, α must be a meridian so that $M(V, \alpha)$ is a 3-sphere by 3.3.

An alternative proof of Theorem 11.1, which shows also that its conclusion is true when k is a trivial knot, may be given as follows. Construct C, V and W as above. Since $W - T$ is homeomorphic to $S^1 \times S^1 \times [0, 1)$, any arc in $W - T$ with end points on bW is homotopic, with fixed end points, to an arc on bW. By Lemma 2 of [25], given any arc A with $bA \subset bW$, there is a homeomorphism from W onto itself which maps A onto an arc disjoint from T. It follows that $\pi_1(W, bW) = 1$. Now the proof of Theorem 19.1 of [33], using that any homotopy 3-disk in W is a

3-disk, shows that W is a solid torus. If k is not trivial continue as in the proof of the theorem. If k is trivial then we have that the 1-connected manifold M^3 is obtained by pasting two solid tori along their boundaries. It is well known (see for example [2]) that this implies M^3 must be the 3-sphere.

COROLLARY 11.1. *Let k be a knot belonging to any of the classes 1), 2), 3), 4) of Corollary 10.1. Then a homotopy 3-sphere is S^3 if it is the disjoint union of a (tame or wild) solid torus and the complement of k.*

Finally, we mention a class of links which do not lead to counterexamples to the Poincaré Conjecture when surgery is done on them.

Let T be a tree (a connected graph without circuits) with vertices $v_1, \cdots, v_n$. Let $D_1, \cdots, D_n$ be disks in S^3 such that

a) $bD_i \cup bD_j$ is a pair of simply linked circles if v_i and v_j are joined by an edge in T,

b) $D_i \cap D_j = \Phi$ if v_i and v_j are not joined by an edge in T and $i \neq j$.

Then we call the link $bD_1 \cup \cdots \cup bD_n$ a *tree link* associated to the tree T.

Now, suppose that $\varphi_1, \cdots, \varphi_n: S^1 \times D^2 \to S^3$ are differentiable imbeddings with disjoint images such that $\varphi_1(S^1 \times \{0\}) \cup \cdots \cup \varphi_n(S^1 \times \{0\})$ is a tree link. Consider the manifold $\chi(\varphi_1, \cdots, \varphi_n)$ obtained from the disjoint union

$$(S^3 - \cup_i \varphi_i(S^1 \times \operatorname{int} D^2)) + (D^2 \times S^1)_1 + (D^2 \times S^1)_2 + \cdots + (D^2 \times S^1)_n$$

by identifying $\varphi_i(u, v)$, for $u \in S^1, v \in S^1$ with $(u, v) \in (D^2 \times S^1)_i$, $i = 1, \cdots, n$.

Then $\chi(\varphi_1, \cdots, \varphi_n)$ is a tree mainfold ([42]) and by [42, VI, 1.5] $\chi(\varphi_1, \cdots, \varphi_n)$ *is S^3 if it is simply connected.*

Thus counterexamples to the Poincaré Conjecture cannot be obtained by doing surgery to tree links.

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