UNSTABLE DIVISIBILITY OF THE CHERN CHARACTER

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IN MEMORIAM

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1. Introduction

In [1], F. Adams proved certain divisibility properties of the Chern character. We begin by recalling these results. Let X be a finite CW-complex and let $K(X)$ be the Grothendieck ring of complex vector bundles on X, (see [2]). Then $K(X)$ has a filtration by skeletons, namely

$$K_p(X) = \text{Ker } (K(X) \to K(X^{p-1})) \tag{1.1}$$

The Chern character is a ring homomorphism

$$ch: K(X) \to H^{ev}(X; Q) \tag{1.2}$$

Let $i: Z \to Q$ be the inclusion of the integers in the rationals. Then a class $x \in H^q(X; Q)$ is called *integral* if $x = i_* y$ for some $y \in H^q(X; Z)$. Let Q_p be the ring of fractions whose denominators are prime to p and let $i_p: Q_p \to Q$ be the inclusion, then a class $x \in H^q(X; Q)$ is called *integral mod p* if $x = i_{p*} y$ for some $y \in H^q(X; Q_p)$. Let $\rho_p: Q_p \to Z_p$ be the quotient map. We will be interested in mod 2 cohomology, so we set $i = i_2$ and $\rho = \rho_2$. Note that if x is integral mod 2, then $\rho_*(x)$ is well defined. We will let $H^*(X)$ denote cohomology with Z_2 coefficients.

THEOREM 1.3 (Adams). *Let $\mu \in K_{2q}(X)$, then $ch_q(\mu)$ is integral and $2^r ch_{q+r}(\mu)$ is integral* mod 2. *Moreover*

$$\chi(\mathrm{Sq}^{2r}) \rho_*(ch_q \mu) = \rho_*(2^r ch_{q+r} \mu)$$

In (1.3), χ is the canonical antiautomorphism of the Steenrod algebra A.

The object of this paper is to obtain unstable divisibility properties of the Chern character. Given integers m and n, let $[m/n]$ denote the integral part of m/n.

THEOREM 1.4. *Let X be a finite CW-complex such that $H^*(X; Z)$ has no torsion. Suppose that X embeds in the M-sphere S^M. Then for any pair of integers q and r such that $\theta: H^{2q}(X) \to H^{2q+2r}(X)$ vanishes for all $\theta \in A_{2r}$, we have: For any $\mu \in K_{2q}(X)$ $2^{r-t} ch_{q+r}(\mu)$ is integral* mod 2, *where* $t = [4r + 2q + 5 - M/4]$.

Let X be a finite CW-complex, then for any $\mu \in K(X)$, $ch_m(\mu) = s_m(\mu)/m!$, where $s_m(\mu)$ is integral. We say $s_m(\mu)$ is odd if $\rho_*(s_m(\mu)) \neq 0$.

COROLLARY 1.5. *Let X be a finite CW-complex, such that $H^*(X; Z)$ has no torsion. If there exists $\mu \in K_{2q}(X)$ with $s_m(\mu)$ odd, then $q \leq \alpha(m)$.* If $q < \alpha(m)$ *then X does not embed in $S^{4m-4\alpha(m)+2q+1}$.*

Remark 1.6. If $q = \alpha(m)$, we have to add then hypothesis $\theta: H^{2\alpha(m)}(X) \to H^{2m}(X)$ vanishes for all $\theta \in A_{2m-2\alpha(m)}$, and then we obtain X does not embed in $S^{4m-2\alpha(m)+1}$.

Remark 1.7. Corollary (1.5) should be compared with the results of Atiyah-Hirzebruch for manifolds, [2]. Their result does not involve the connectivity of μ, and thus for manifolds, are stronger. Our result (1.5) coincides with the results of Atiyah-Hirzebruch only when $q = \alpha(m) - 1$. However, our results are true not only for embedding of complexes, but also for stable embeddings, where we say that X stably embeds in S^M if there exists a complex Y such that Y is of the homotopy type of $\Sigma^k X$ for some k, and Y embeds in S^{M+k}.

Remark 1.8. Maunder in [4], defines a system of higher order cohomology operations $\chi(\phi_{2r}{}^{(N)})$, and relates them to higher divisibility of the Chern character, for instance, in (1.4), one has:

$$\chi(\phi_{2r}{}^{(t+1)})\rho_*(ch_q\mu) = \rho_*(2^{r-t}ch_{q+r}\mu) \tag{1.9}$$

and this extends the Adams theorem (1.3), regarding the action of cohomology operations.

2. Higher order cohomology operations

In this section we recall the notion of higher order cohomology operations as axiomatized by Maunder in [3].

Let A be the Steenrod algebra over Z_2 and let C:

$$C_0 \xleftarrow{\partial_1} C_1 \xleftarrow{\partial_2} C_2 \leftarrow \cdots \xleftarrow{\partial_N} C_N \tag{2.1}$$

be an A-chain complex, i.e., C_i is a free A-module, ∂_i is of degree 0 and $\partial_{i-1}\partial_i = 0$. Then a pyramid of stable operations $\phi^{r,s}$, where $N \geq r > s \geq 0$, associated with (2.1) is an inductive family of operations satisfying the following axioms:

Axiom I (Induction): The operations $\phi^{r,s}$ for $u \geq r > s \geq v$ are associated with the chain complex

$$C_v \leftarrow \cdots \leftarrow C_u$$

Axiom II (Domain of definition): Let $\epsilon: C_0 \to H^*(X)$ be an A-map. Then $\phi^{N,0}(\epsilon)$ is defined if $\phi^{i,0}(\epsilon)$ is defined and contains the zero map, for $i = 1, \cdots, N-1$.

Axiom III (Values of the operation): Let $\epsilon: C_0 \to H^*(X)$ be an A-map of degree q, then if $\phi^{N,0}(\epsilon)$ is defined, $\phi^{N,0}(\epsilon)$ consists of certain equivalence classes of A-maps $\eta: C_N \to H^*(X)$ of degree $q - N + 1$, where $\eta \sim \eta'$ if there exists $\mu \in \phi^{N,1}(\zeta)$ such that $\eta = \mu + \eta'$ and where $\zeta: C_1 \to H^*(X)$ runs through all A-maps of degree $q - 1$ such that $\phi^{N,1}(\zeta)$ is defined.

Axiom IV (Naturality): Let $\epsilon: C_0 \to H^*(X)$ be an A-map such that $\phi^{N,0}(\epsilon)$ is defined. Then for every map $f: Y \to X$, $\phi^{N,0}(f^*\epsilon)$ is defined and $f^*\phi^{N,0}(\epsilon) \subset \phi^{N,0}(f^*\epsilon)$.

Axiom V (Stability): Let $\Sigma^*:H^*(X) \to H^*(\Sigma X)$ be the suspension isomorphism. Then if $\epsilon:C_0 \to H^*(X)$ is an A-map such that $\phi^{N,0}(\epsilon)$ is defined, then $\phi^{N,0}(\Sigma^*\epsilon)$ is defined and $\Sigma^*\phi^{N,0}(\epsilon) = \phi^{N,0}(\Sigma^*\epsilon)$.

Axiom VI (Second formula of Peterson-Stein): Let (X, Y) be a pair. Let $\epsilon:C_0 \to H^*(X)$ be an A-map, such that $\phi^{N-1,0}(\epsilon)$ is defined and let $\mu:C_{n-1} \to H^*(X, Y)$ be such that $j^*\mu \in \phi^{N-1,0}(\epsilon)$, then $\phi^{N,0}(i^*\epsilon)$ is defined and for every such μ, there exists $\eta:C_N \to H^*(Y)$, $\eta \in \phi^{N,0}(i^*\epsilon)$ such that $\delta\eta = \phi^{N,N-1}\mu$.

Remark 2.2. Given a chain complex, it may have no pyramid of higher order cohomology operations associated with it.

Pyramids of higher order cohomology operations are constructed by means of universal examples.

A realization $\mathcal{E}$ of the chain complex (2.1) is a tower of principal fiber spaces

$$
\begin{array}{ccccccccccc}
BF_N & & & & BF_{k+1} & & & & BF_2 & & BF_1 \\
\uparrow g_N & & & & \uparrow g_{k+1} & & & & \uparrow g_2 & & \uparrow g_1 \\
E_{N-1} & \xrightarrow{p_{N-1}} & \cdots & \xrightarrow{p_{k+1}} & E_k & \xrightarrow{p_k} & \cdots & \xrightarrow{p_2} & E_1 & \xrightarrow{p_1} & E_0 \\
\uparrow i_{N-1} & & & & \uparrow i_k & & & & \uparrow i_1 & & \\
F_{N-1} & & & & F_k & & & & F_1 & &
\end{array}
\tag{2.3}
$$

where BF_k is a product of Eilenberg-Maclane spaces of type $K(Z_2, q)$ and g_k is the classifying map for the fiber space $F_k \to E_k \to E_{k-1}$. Moreover we have A-isomorphisms $\alpha_k:C_k \to H^*(F_k)$ of degree $-k$ and $\beta_k:C_k \to H^*(BF_k)$ of degree $-k+1$ in a certain range of dimensions and such that in

$$
\begin{array}{ccccc}
 & H^*(BF_k) & \xrightarrow{g_k^*} H^*(E_{k-1}) \xrightarrow{i_{k-1}^*} & H^*(F_{k-1}) \\
 & \uparrow \beta_k & & \uparrow \alpha_{k-1} \\
C_{k+1} \xrightarrow{\partial_{k+1}} & C_k & \xrightarrow{\qquad\partial_k\qquad} & C_{k-1}
\end{array}
\tag{2.4}
$$

the rectangle is commutative and $g_k^*\beta_k.\partial_{k+1} = 0$. Let $G_N = C_N \otimes_A Z_2$. Then $g_N^*\beta_N(G_N)$ is a universal example for the operation $\phi^{N,0}$ associated with the chain complex (2.1) and the realization (2.2).

Remark 2.5. Given a chain complex as (2.1) it may admit realizations $\mathcal{E}$ and $\mathcal{E}'$. If E_N and E_N' are spaces that are not of the same homotopy type then the corresponding operations $\phi^{N,0}$ and $\phi'^{N,0}$ bear no relation to each other whatsoever.

3. The Maunder higher order operations

Consider the chain complex $C(N, r)$:

$$
C_0 \xleftarrow{\partial_0} C_1 \leftarrow \cdots \xleftarrow{\partial_N} C_N \tag{3.1}
$$

where

$$\begin{cases} C_0 \text{ has } A\text{-basis } \{c_0\} \\ C_N \text{ has } A\text{-basis } \{c_N\} \\ C_k \text{ has } A\text{-basis } \{c_k, c_{k,0}, \cdots, c_{k,k}\} \\ \qquad \text{for } 1 \leq k \leq N-1 \end{cases} \tag{3.2}$$

and where the boundary operators ∂_i are given by

$$\begin{cases} \partial_1 c_1 = \chi(\mathrm{Sq}^{2r})c_0 \\ \partial_1 c_{10} = \mathrm{Sq}^1 c_0 \\ \partial_1 c_{11} = \mathrm{Sq}^{01} c_0 \end{cases} \tag{3.3}$$

and for $2 \leq k \leq N$

$$\begin{cases} \partial_k c_k = \mathrm{Sq}^1\, c_{k-1} + \sum_{t=0}^{k-1} \chi(\mathrm{Sq}^{2r-2t})c_{k-1,t} & 2 \leq k \leq N \\ \partial_k c_{k,0} = \mathrm{Sq}^1\, c_{k-1,0} & 1 \leq k < N \\ \partial_k c_{k,i} = \mathrm{Sq}^1 c_{k-1,i} + \mathrm{Sq}^{01}\, c_{k-1,i-1} & 0 < i < k < N \\ \partial_k c_{k,k} = \mathrm{Sq}^{01}\, c_{k-1,k-1} & 1 \leq k < N \end{cases} \tag{3.4}$$

where $\mathrm{Sq}^{01} = \mathrm{Sq}^3 + \mathrm{Sq}^2\,\mathrm{Sq}^1$.

Maunder in [4], constructed a realization $\mathcal{E}$ of (3.1) as follows: Let BU be the classifying space for stable complex vector bundles and let $BU[2q]$ be the space obtained from BU by killing its first $(q-1)$ non-trivial homotopy groups. Adams in [1], showed that there exist natural integral classes $ch_{q,r} \in H^{2q+2r}(BU[2q]; Z)$ such that if $p_q: BU[2q] \to BU$ is the projection, then

$$m(r)p_q^{\,*}(ch_{q+r}) = i_* ch_{q,r} \tag{3.5}$$

where $m(r) = \Pi_p p^{[r/p-1]}$.

Let $BU[2q](r, s)$ be the fiber of the mapping

$$BU[2q] \xrightarrow{r,\,s} K(Z_{2^s}, 2q + 2r)$$

where if $\gamma_{2q+2r,s}$ is the fundamental class of $K(Z_{2^s}, 2q + 2r)$ with Z_{2^s} as coefficients, then $g_{r,s}^{\,*}(\gamma_{2q+2r,s}) = \rho_s^{\,*} ch_{q,r}$, where $\rho_s^{\,*}$ is reduction mod 2^s. Then we have a sequence of fibrations

$$BU[2q](r, N-1) \to \cdots \to BU[2q](r, 1) \to BU[2q] \tag{3.6}$$

where the fiber at each stage is $K(Z_2, 2q + 2r - 1)$.

Maunder then constructed a realization $E_{N-1} \to \cdots \to E_0$ of (3.1) and mappings $f_k: BU[2q](r, k) \to E_k$ such that, with the notation of (2.3),

$$\begin{cases} f_k^{\,*} g_k^{\,*} \beta_k(c_{k,i}) = 0 \quad \text{for} \quad 0 \leq i \leq k \\ f_k^{\,*} g_k^{\,*} \beta_k(c_k) = \rho^* \left(\frac{1}{2^k}\, ch_{q,r}(\xi_{r,k}) \right) \end{cases} \tag{3.7}$$

where $\bar{\xi}_{r,k}$ is the bundle over $BU[2q](r, s)$ induced by the universal bundle ξ over BU.

Let $\{\chi(\phi^{r,s})\}$ $N \geq r > s \geq 0$, be the pyramid of higher order cohomology operations associated with (3.1) and the realization $\mathcal{E}$ satisfying (3.7).

Then we can state Maunder's theorem:

THEOREM 3.7 (Maunder). *Let $\{\chi(\phi_{2r}^{s,t})\}$ $N \geq s > t \geq 0$ be the pyramid of of higher order cohomology operations mentioned above. Let $\mu \in K_{2q}(X)$ be a bundle. Then if $2^{r-N+1}ch_{q+r}(\mu)$ is integral* mod 2, *$\chi(\phi_{2r}^{N})(\rho_* ch_q(\mu))$ is defined and $\rho_*(2^{r-N+1}ch_{q+r}(\mu)) \in \chi(\phi_{2r}^{N})(\rho_* ch_q(\mu))$.*

(Here we have set $\chi(\phi_{2r}^{N,0}) = \chi(\phi_{2r}^{N})$.)

4. Dual cohomology operations

In this section, we describe the duality theorem of Maunder [3], for higher order cohomology operations.

Let M and N be free A-modules with A-basis $\{m_i\}$, $\{n_j\}$ and let $f:M \to N$ be an A-map such that $fm_i = \Sigma\alpha_{ij}n_j$. Then we define $\chi(f):M \to N$ by $\chi(f)m_i = \Sigma\chi(\alpha_{ij})n_j$.

Consider now an A-chain complex C:

$$C_0 \xleftarrow{\partial_1} C_1 \leftarrow \cdots \xleftarrow{\partial_N} C_N \tag{4.1}$$

and let C^* be the chain complex

$$C_N^* \xleftarrow{\chi(\partial_N^*)} C_{N-1}^* \leftarrow \cdots \leftarrow C_1 \xleftarrow{\chi(\partial_1^*)} C_0 \tag{4.2}$$

where $C_k^* = \mathrm{Hom}_{Z_2}(C_k, Z_2)$ and ∂_k^* is the dual of ∂_k. We refer to C^* as the dual complex of C.

Recall from [5], that two finite complexes X and Y are S-dual mod 2 if there exists a mapping $\varphi:S^N \to \Sigma^k Y \wedge \Sigma^l X$ such that φ^* determines a non-singular bilinear pairing

$$H^*(\Sigma^k Y) \otimes H^*(\Sigma X) \xrightarrow{\varphi^*} H^N(S^N) \cong Z_2 \tag{4.3}$$

which we will simply write as $< , >$.

Given a pyramid of cohomology operations $\phi^{r,s}$ associated with the chain complex C of (4.1), we have that

$$\phi^{r,s}:\mathrm{Ker}\,(\phi^{r-1,s}, \Sigma^l X) \to H^*(\Sigma^l X)/Im(\phi^{r,s+1}, X) \tag{4.4}$$

and we define inductively $\chi(\phi^{r,s})$ and dual pairings

$$\mathrm{Ker}\,(\phi^{r-1,s}, \Sigma^l X) \otimes H^*(\Sigma^k Y)/Im(\chi(\phi^{r-1,s}), \Sigma^k Y) \to Z_2 \tag{4.5}$$

$$\mathrm{Ker}\,(\chi(\phi^{r,s+1}), \Sigma^k Y) \otimes H^*(\Sigma^l X)/Im(\phi^{r,s+1}), \Sigma^l X) \to Z, \tag{4.6}$$

such that

$$\chi(\phi^{r,s}):\mathrm{Ker}\,(\chi(\phi^{r,s+1}), \Sigma^k Y) \to H^*(\Sigma^k Y)/Im(\chi(\phi^{r-1,s}), \Sigma^k Y) \tag{4.7}$$

is defined by:

(4.8) $$\langle \chi(\phi^{r,s})(y), x\rangle = \langle y, \phi^{r,s}(x)\rangle$$

for $y \in \text{Ker}\,(\chi(\phi^{r,s+1}), \Sigma^k Y)$ and all $x \in \text{Ker}\,(\phi^{r-1,s}, \Sigma^l X)$, and now

THEOREM 4.9 (Maunder). *The functions* $\{\chi(\phi^{r,s})\}$ $N \geq r > s > 0$ *define a pyramid of stable cohomology operations associated with the chain complex* (4.2), *and* $\chi(\phi^{r,s})$ *is non-zero in* $\Sigma^k Y$ *if and only if* $\phi^{r,s}$ *is non-zero in* $\Sigma^l X$, *where* X *and* Y *are S-duals.*

5. The dual Maunder cohomology operations

In this section we describe the cohomology operations, for which we will prove in a later section evaluation theorems.

We begin by describing the chain complex $C^*(N, r)$ dual to the chain complex $C(N, r)$ of §3. Let

(5.1) $$C_0^* \xleftarrow{\partial_1} \cdots \xleftarrow{\partial_N} C_N^*$$

be a chain complex, where

(5.2) $$\begin{cases} C_0^* \text{ has } A\text{-basis } \{c_0^*\} \\ C_N^* \text{ has } A\text{-basis } \{c_{N,0}^*\} \\ C_k^* \text{ has } A\text{-basis } \{c_k^*, c_{k,0}^*, \cdots, c_{k,N-k}^*\} \end{cases}$$
$$\text{for } 1 \leq k \leq N-1$$

and where the boundary operators are given by

(5.3) $$\begin{cases} \partial_1 c_1^* = \text{Sq}^1 c_0^* \\ \partial_1 c_{1,k}^* = \text{Sq}^{2r-2k} c_0^* \quad \text{for} \quad 0 \leq k \leq N-1 \end{cases}$$

and for $2 \leq n \leq N-1$

(5.4) $$\begin{cases} \partial_n c_n^* = \text{Sq}^1 c_{n-1}^* \\ \partial_n c_{n,k}^* = \text{Sq}^1 c_{n-1,k}^* + \text{Sq}^{01} c_{n-1,k+1}^* + \text{Sq}^{2r-2k} c_{k-1}^* \end{cases}$$

and for N,

(5.5) $$\partial_N c_{N,0}^* = \text{Sq}^1 c_{N-1,0}^* + \text{Sq}^{01} c_{N-1,1}^* + \text{Sq}^{2r} c_{N-1}^*.$$

The chain complex (5.1) is dual to the chain complex (3.1) which has a realization. Therefore by (4.9), the chain complex (5.1) has a realization, namely the realization associated to the dual operations of the Maunder system of §3.

We will now work with any realization of (5.1). Let

$$
(5.6)\qquad
\begin{array}{ccccccccc}
BF_N & & BF_{N-1} & & & & BF_2 & & BF_1 \\
\uparrow g_N & & \uparrow g_{N-1} & & & & \uparrow g_2 & & \uparrow g_1 \\
E_{N-1} & \xrightarrow{p_{N-1}} & E_{N-2} & \xrightarrow{p_{N-2}} & \cdots & \rightarrow & E_1 & \xrightarrow{p_1} & E_0 \\
\uparrow i_{N-1} & & \uparrow i_{N-2} & & & & \uparrow i_1 & & \\
F_{N-1} & & F_{N-2} & & & & F_1 & &
\end{array}
$$

be a realization of (5.1). We put $K_q = K(Z_2, q)$, thus $E_0 = K_n$ for some n and

$$(5.7)\qquad BF_k = K_{n+1} \times \Pi_{i=0}^{N-k} K_{n+2r-2i}$$

Let $BF_k' = \Pi_{i=0}^{N-k} K_{n+2r-2i}$. Then we can unfold (5.6) to the following tower.

$$
(5.8)\qquad
\begin{array}{ccccccccccccc}
BF_N & & BF_{N-1}' & & K_{n+1} & & & BF_2' & & K_{n+1} & & BF_1' & K_{n+1} \\
\uparrow & & \uparrow & & \uparrow & & & \uparrow & & \uparrow & & \uparrow & \uparrow \\
E_{N-1} & \xrightarrow{p_{N-1}'} & E_{N-2}' & \xrightarrow{q_{N-2}} & E_{N-2} & \xrightarrow{p_{N-2}'} & \cdots \rightarrow & E_1' & \xrightarrow{q_1} & E_1 & \xrightarrow{p_1'} & E_0' \xrightarrow{q_0} & E_0
\end{array}
$$

We construct a tower

$$
(5.9)\qquad
\begin{array}{ccccccccccc}
BF_N & & BF_{N-1}' & & BF_{N-2}' & & & & BF_2' & & BF_1' \\
\uparrow & & \uparrow & & \uparrow & & & & \uparrow & & \uparrow \\
D_{N-1} & \rightarrow & D_{N-2} & \rightarrow & D_{N-3} & \rightarrow & \cdots & \rightarrow & D_1 & \rightarrow & D_0
\end{array}
$$

over $D_0 = K(Z, n)$ in such a way that we have mappings $f_k': D_k \rightarrow E_k'$ and $D_{k+1} \rightarrow D_k$ is induced from $E_{k+1} \rightarrow E_k'$ by f_k'. We proceed inductively as follows. For $k = 0$,

$$
\begin{array}{ccccc}
 & & E_0' & & \\
 & & \downarrow & & \\
D_0 & \xrightarrow{f_0} & E_0 & \rightarrow & K_{n+1}
\end{array}
$$

where $f_0^*\gamma_n = \alpha_n$, where $\alpha_n \in H^n(D_0)$ is the fundamental class. Recall that $H^{n+1}(D_0) = 0$, hence f_0 admits a lifting $f_0': D_0 \rightarrow E_0'$ and we take now

$$
\begin{array}{ccc}
D_1 & \xrightarrow{f_1} & E_1 \\
\downarrow & & \downarrow \\
D_0 & \xrightarrow{f_0'} & E_0'
\end{array}
$$

where $D_1 \rightarrow D_0$ is induced from $E_1 \rightarrow E_0'$ by f_0'. Note that $H^n(D_1) = Z_2$ and $H^{n+1}(D_1) = 0$. Suppose inductively we have constructed the tower (5.9) through

k-stages, such that $H^n(D_k) = Z_2$, $H^{n+1}(D_k) = 0$. Then we have

$$\begin{array}{ccc} D_k & \xrightarrow{f_k} & E_k \\ \downarrow & & \downarrow \\ D_{k-1} & \xrightarrow{f_{k-1}'} & E_{k-1}' \end{array}$$

Now consider

$$\begin{array}{ccccc} & & E_k' & & \\ & & \downarrow & & \\ D_k & \xrightarrow{f_k} & E_k & \rightarrow & K_{n+1} \end{array}$$

because $H^{n+1}(D_k) = 0$, f_k admits a lifting $f_k': D_k \rightarrow E_k'$ and we take the induced fiber space

$$\begin{array}{ccc} D_{k+1} & \xrightarrow{f_{k+1}} & E_{k+1} \\ \downarrow & & \downarrow \\ D_k & \xrightarrow{f_n'} & E_k' \end{array}$$

and because of (5.7), we see that $H^n(D_{k+1}) = Z_2$, $H^{n+1}(D_{k+1}) = 0$ and the tower now exists through $(k+1)$-stages. We continue this construction so as to obtain (5.9).

Remark 5.10. The spaces D_k so constructed are unique up to homotopy type. The tower (5.9) "defines" a stable cohomology operation defined on integral classes.

Let $x \in H^n(X; Z)$ be a class, where X is a finite CW-complex. Let $w: X \rightarrow D_0$ be a representation of x. Then $f_0 w: X \rightarrow E_0$ is a representation of x mod 2 and we clearly have:

PROPOSITION 5.11. *Suppose that $w: X \rightarrow D_0$ admits a lifting to $w_{N-1}: X \rightarrow D_{N-1}$, then $f_{N-1} w_{N-1}: X \rightarrow E_{N-1}$ is a lifting of $f_0 w$.*

Remark 5.12. For cohomology operations (5.11) states that if on an integral class, the stable operation defined for integral classes, is defined, then so is the mod 2 stable operation and the values of the first are a subset of the values of the second. Let us denote by $\{\psi^{r,s}\}$ the pyramid of operations having (5.9) as its universal example. In particular we denote by ψ_{2k}^N, the operation $\psi^{N,0}$.

6. Evaluation of the dual Maunder cohomology operations

In this section we will show that for integral classes of sufficiently low dimension, the operations $\chi(\phi_{2r}^{N,0})$ are universally defined and thus are primary operations.

We begin by establishing a result on the Steenrod algebra A over Z_2. Let $B(q)$ be the left ideal in A generated by all those admissible monomials of excess greater than q (see [6]).

PROPOSITION 6.1. *Let $a \in A_n$, $b \in A_{n-2}$ and $n - 2 > q$ be such that*

$$\mathrm{Sq}^1 a + \mathrm{Sq}^{01} b \equiv 0 \bmod A\,\mathrm{Sq}^1 + B(q)$$

then

$$\left.\begin{aligned} a &\equiv \mathrm{Sq}^1 a_1 + \mathrm{Sq}^{01} a_2 \\ b &\equiv \mathrm{Sq}^1 b_1 + \mathrm{Sq}^{01} b_2 \\ a_2 &\equiv b_1 \end{aligned}\right\} \bmod A\,\mathrm{Sq}^1 + B(q - \epsilon_{n,q})$$

where $\epsilon_{n,q}$ is the smallest nonnegative mod 4 *reduction of $n + q - 1$.*

For simplicity, we write $I = (a_1, \cdots, a_k)$ for Sq^I. Then

$$\mathrm{Sq}^1 (a_1, \cdots, a_k) = \begin{cases} (a_1 + 1, a_2, \cdots, a_k) & \text{if } a_1 \text{ is even} \\ 0 & \text{if } a_1 \text{ is odd} \end{cases} \tag{6.2}$$

and

$$\mathrm{Sq}^{01} (a_1, \cdots, a_k) = \begin{cases} (a_1 + 3, a_2, \cdots, a_k) + (a_1 + 2, a_2 + 1, \cdots, a_k) \\ \qquad \text{if } a_1 \text{ and } a_2 \text{ are even} \\ (a_1 + 3, a_2, \cdots, a_k) \\ \qquad \text{if } a_1 \text{ is even and } a_2 \text{ is odd} \\ (a_1 + 2, a_2 + 1, \cdots, a_k) \\ \qquad \text{if } a_1 \text{ is odd and } a_2 \text{ is even} \\ 0 \qquad \text{if } a_1 \text{ and } a_2 \text{ are odd} \end{cases} \tag{6.3}$$

Now we have the following:

LEMMA 6.4. *Let $a \in A_n$, $b \in A_{n-2}$ and suppose that*

$$\mathrm{Sq}^1 a + \mathrm{Sq}^{01} b \equiv 0 \bmod A\,\mathrm{Sq}^1$$

then

$$\left.\begin{aligned} a &\equiv \mathrm{Sq}^1 a_1 + \mathrm{Sq}^{01} a_2 + \epsilon_1 \mathrm{Sq}^n \\ b &\equiv \mathrm{Sq}^1 b_1 + \mathrm{Sq}^{01} b_2 + \epsilon_2 \mathrm{Sq}^{n-2} \\ a_2 &\equiv b_1 \end{aligned}\right\} \bmod A\,\mathrm{Sq}^1$$

Proof. Write $a = \Sigma I_k + \Sigma \bar{I}_k$, $b = \Sigma J_j + \Sigma \bar{J}_j$ as sums of admissible monomials, where the second summands lie in $A\,\mathrm{Sq}^1$. By (6.2) and (6.3), we obtain

$$\Sigma\, \mathrm{Sq}^1 I_k + \Sigma\, \mathrm{Sq}^{01} J_j \equiv 0 \bmod A\,\mathrm{Sq}^1. \tag{6.5}$$

By (6.3), for every j, $\mathrm{Sq}^{01} J_j = \mathrm{Sq}^1 I_k$, for some I_k, unless $J_j = \mathrm{Sq}^{n-2}$. Moreover $\mathrm{Sq}^1 \mathrm{Sq}^n + \mathrm{Sq}^{01} \mathrm{Sq}^{n-2} + \mathrm{Sq}^n \mathrm{Sq}^1 = 0$. Hence (6.5) splits up into two relations if Sq^{n-2} is an element of ΣJ_j,

$$\Sigma\, \mathrm{Sq}^1 I_k' + \Sigma\, \mathrm{Sq}^{01} J_j' = 0$$

and

$$\mathrm{Sq}^1\,\mathrm{Sq}^n + \mathrm{Sq}^{01}\,\mathrm{Sq}^{n-2} \equiv 0 \bmod A\,\mathrm{Sq}^1$$

Hence

$$\Sigma I_k' = \mathrm{Sq}^1\,a_1 + \mathrm{Sq}^1\,a_2$$

$$\Sigma J_j' = \mathrm{Sq}^1\,b_1 + \mathrm{Sq}^{01}\,b_2$$

with

$$b_1 = a_2$$

since A is a free module over the sub Hopf algebra generated by Sq^1 and Sq^{01}. Therefore $a \equiv \mathrm{Sq}^1\,a_1 + \mathrm{Sq}^{01}\,a_2 + \epsilon_1\,\mathrm{Sq}^n$, $b \equiv \mathrm{Sq}^1\,b_1 + \mathrm{Sq}^{01}\,b_2 + \epsilon_2\,\mathrm{Sq}^{n-2}$ and (6.4) follows.

Proof of (6.1). Recall that for a monomial $I = (a_1, \cdots, a_k)$, we define $e(I) = a_1 - \sum_{i=2}^k a_i$ and $n(I) = \sum_{i=1}^k a_i$, hence $n(I) + e(I) = 2a_1$ and thus $e(I)$ and $n(I)$ have the same parity.

Given a and b, we write them as sums of admissible monomials

$$a = \Sigma I_k + \Sigma \bar{I}_k \text{ where } e(I_k) \leq q,\ e(\bar{I}_k) > q$$

$$b = \Sigma J_j + \Sigma \bar{J}_j \text{ where } e(J_j) \leq q,\ e(\bar{J}_k) > q,$$

then from $\mathrm{Sq}^1\,a + \mathrm{Sq}^{01}\,b \equiv 0 \bmod A\,\mathrm{Sq}^1 + B(q)$ we obtain

$$\mathrm{Sq}^1\,(\Sigma I_k) + \mathrm{Sq}^{01}(\Sigma J_j) \equiv 0 \bmod A\,\mathrm{Sq}^1 + B(q) \tag{6.6}$$

where $e(I_k) \leq q$, $e(J_j) \leq q$.

We now consider four cases

Case I, $n + q \equiv 0 \bmod 4$. For every monomial I_k in (6.6) we have $n(I_k) = n$, $e(I_k) \leq q$, and for every monomial J_j, $n(J_j) = n - 2$ and $e(J_j) \leq q$. From the above $e(I_k) \neq q - 1, q - 3$, and $e(J_j) \neq q - 1, q - 3$. If $e(J_j) = q - 2$, then $2a_1(J_j) = n - 2 + q - 2 \equiv 0 \bmod 4$, so that $a_1(J_j)$ is even and from (6.3), $\mathrm{Sq}^{01}\,J_j$ contains an admissible monomial of excess $q + 1$ and possibly a monomial of excess $q - 1$. If $e(J_j) = q - 4$, then $2a_1(J_j) = n - 2 + q - 4 \equiv 2 \bmod 4$, so that $a_1(J_j)$ is zero or an admissible monomial of excess $q - 3$. This implies that

$$\mathrm{Sq}^1\,(\Sigma' I_k) + \mathrm{Sq}^{01}\,(\Sigma'' J_j) \equiv 0 \bmod A\,\mathrm{Sq}^1 \tag{6.7}$$

where we take all the summands of (6.6) such that $e(I_k) < q - 2$ and $e(J_j) < q - 2$.

We apply (6.4) to (6.7) and since

$$a \equiv \Sigma' I_k \bmod A\,\mathrm{Sq}^1 + B(q - 3) \qquad b \equiv \Sigma'' J_j \bmod A\,\mathrm{Sq}^1 + B(q - 3)$$

the result follows.

Case II, $n + q \equiv 1 \bmod 4$. In this case we exclude in (6.6) those monomials with excess q or $q - 2$. If $e(I_k) = q - 1$, we have $e(\mathrm{Sq}^1 I_k) = q$ and if $e(J_j) = q - 1$, we have $n - 2 + q - 1 \equiv 2 \bmod 4$ so $a_1(J_j)$ is odd, then by (6.3), $\mathrm{Sq}^{01} J_j$ is zero or has excess q. It follows then that (6.6) holds mod $A\,\mathrm{Sq}^1$ and (6.4) establishes this case.

Case III, $n + q \equiv 2 \bmod 4$. In this case, we exclude from (6.6) those monomials with excess $q - 1$ or $q - 3$. Now if $e(J_j) = q - 2$, $n - 2 + q - 2 \equiv 2 \bmod 4$, so $a_1(J_j)$ is odd and by (6.3), $\mathrm{Sq}^{01} J_j$ is zero or has excess $q - 1$. This implies that if we restrict the sum in (6.6) to monomials of excess $< q$, we obtain a relation mod $A\,\mathrm{Sq}^1$, and (6.4) then implies this case.

Case IV, $n + q \equiv 3 \bmod 4$. In this case, we exclude from (6.6) those monomials with excess q or $q - 2$. If $e(J_j) = q - 1$, then $n - 2 + q - 1 \equiv 0 \bmod 4$, so that $a_1(J_j)$ is even and (6.3) gives that $\mathrm{Sq}^{01} J_j$ has an admissible monomial of excess $q + 2$ and possibly one of excess q. If $e(J_j) = q - 3$, then $\mathrm{Sq}^{01} J_j$ has excess $q - 2$ or is 0. Hence if we restrict the sum of (6.6) to those monomials with excess less than $q - 1$, we obtain a relation mod $A\,\mathrm{Sq}^1$ and (6.4) establishes the last case.

COROLLARY 6.9. *Let $a_{2m-2i} \in A_{2m-2i}$, $i = 0, \cdots, k$ satisfy the following system of relations*

(a) $\mathrm{Sq}^1 a_{2m-2i} + \mathrm{Sq}^{01} a_{2m-2i-2} \equiv 0 \bmod A\,\mathrm{Sq}^1 + B(2q)$

for $i = 0, 1, \cdots, k$ then if $2m - 2k + 2q \equiv 0 \bmod 4$, we have

(b) $$\left.\begin{aligned} a_{2m-2i} &\equiv \mathrm{Sq}^1 a_{2m-2i}^{\;1} + \mathrm{Sq}^{01} a_{2m-2i}^{\;2} \\ a_{2m-2i}^{\;2} &\equiv a_{2m-2i-2}^{\;1} \end{aligned}\right\} \bmod A\,\mathrm{Sq}^1 + B(2q - 1)$$

for $i = 0, 1, \cdots, k$.

Proof. By induction on k. If $k = 1$, we have $2m + 2q \equiv 2 \bmod 4$ and the result is true by (6.1). Assume (6.9) holds for sequences $\{a_i\}$ containing k elements. Then in particular if (a) holds for $i = 1, \cdots, k$, then (b) holds for $i = 1, \cdots, k$. Consider the relation $\mathrm{Sq}^1 a_{2m} + \mathrm{Sq}^{01} a_{2m-2} \equiv 0 \bmod A\,\mathrm{Sq}^1 + B(2q)$. By induction, $a_{2m-2} \equiv \mathrm{Sq}^1 a_{2m-2}^{\;1} + \mathrm{Sq}^{01} a_{2m-2}^{\;2} \bmod A\,\mathrm{Sq}^1 + B(2q - 1)$. Therefore, we obtain,

$$\mathrm{Sq}^1 (a_{2m} + \mathrm{Sq}^{01} a_{2m-2}^{\;1}) \equiv 0 \bmod A\,\mathrm{Sq}^1 + B(2q - 1)$$

Let $x = a_{2m} + \mathrm{Sq}^{01} a_{2m-2}^{\;1}$, then $x \equiv \Sigma I_k$, where the I_k are admissible monomials of degree $2m$, excess $\leq 2q - 1$ and $I_k \notin A\,\mathrm{Sq}^1$. Because $n(I_k)$ and $e(I_k)$ have the same parity, there is no monomial I_k with $e(I_k) = 2q - 1$, and therefore $\mathrm{Sq}^1 (\Sigma I_k) \equiv 0 \bmod A\,\mathrm{Sq}^1 + B(2q - 1)$ implies $\mathrm{Sq}^1 (\Sigma I_k) = 0$ in A, i.e. $\Sigma I_k = \mathrm{Sq}^1 a_{2m}^{\;1}$ and hence $a_{2m} + \mathrm{Sq}^{01} a_{2m-2}^{\;1} \equiv \mathrm{Sq}^1 a_{2m}^{\;1} \bmod A\,\mathrm{Sq}^1 + B(2q - 1)$ and (6.9) follows.

Consider now the tower (5.9). Let $v(n, t, k) \in H^{n+2r-2k}(D_{t-1})$, for $k = 0, 1, \cdots, N - t$ and $1 \leq t \leq N - 1$, be the images of the fundamental classes of $H^*(BF_t^{\,1})$. Then they satisfy,

$$\mathrm{Sq}^1 v(n, t, k) + \mathrm{Sq}^{01} v(n, t, k + 1) = 0. \tag{6.10}$$

We now consider the operations ψ_{2r}^{N} (see (5.12)) defined on integral classes and associated with the chain complex (5.1). We may now state the main theorem of this section.

THEOREM 6.11. *For any integral class $x \in H^q(X)$ of dimension $q \leq 2r - 4N + 5$ and any choice of the operation ψ_{2r}^{N}, we have that $\psi_{2r}^{N}(x)$ is defined and is a primary operation. If $q < 2r - 4N + 5$, then the primary operation is stable.*

Proof. We prove (6.11) in the universal example. By abuse of notation, we denote by $X(t)$ any suspension of $K(Z, t)$. Consider the tower (5.9),

$$D_0(n, r) \leftarrow \cdots \leftarrow D_{N-1}(n, r)$$

where $D_0(n, r) = K(Z, n)$. If $t < t'$, we denote by $X(t) \to X(t')$ the mappings induced by $\Sigma^{t'-t}K(Z, t) \to K(Z, t')$. Consider

$$X(2r - 2(N - 1) - 1) \xrightarrow{f_0} D_0(n, r) \to BF_1^{\,1}(n, r),$$

it is nul-homotopic, so f_0 admits a lifting to $f_1: X(2r - 2(N - 1) - 1) \to D_1(n, r)$. Assume inductively that the mapping

$$X(2r - 2(N - 1) - 2k + 1) \xrightarrow{f_0} D_0(n, r)$$

lifts to $f_k: X(2r - 2(N - 1) - 2k + 1) \to D_k(n, r)$. Then $\{f_k^{\,*}(v(n, k + 1, s)\}$ for $s = 0, 1, \cdots, N - k - 1$ are cohomology operations on $\gamma_{2r-2(N-1)-2k+1}$. Hence if we consider

$$X(2r - 2(N - 1) - 2k) \xrightarrow{g_0} X(2r - 2(N - 1) - 2k + 1)$$

then $g^*f_k^{\,*}v(n, k + 1, s) = a_{2r-2s}(\gamma_{2r-2(N-1)-2k})$ are then k-invariants, where $a_{2r-2s} \in A_{2r-2s}$. The family $\{a_{2r-2s}\}$ $s = 0, 1, \cdots, N - k - 1$ satisfy the relations (a) of (6.9), furthermore, $2r - 2(N - k - 1) + 2r - 2(N - 1) - 2k \equiv 0 \bmod 4$, hence (b) of (6.9) holds. If we further consider

$$X(2r - 2(N - 1) - 2k - 1) \xrightarrow{g_1} X(2r - 2(N - 1) - 2k),$$

we obtain $\{g_1^{\,*}g_0^{\,*}f_k^{\,*}v(n, k + 1, s)\}$ lies as a set in the diagonal indeterminacy, hence the mapping

$$X(2r - 2(N - 1) - 2k - 1) \to D_0(n, r)$$

may be lifted to

$$X(2r - 2(N - 1) - 2k - 1) \to D_{k+1}(n, r).$$

Continuing with this argument, we obtain

$$X(2r - 2(N - 1) - 2(N - 2) - 1) \to D_0(n, r)$$

admits a lifting to

$$X(2r - 2(N - 1) - 2(N - 2) - 1) \to D_{N-1}(n, r)$$

7. Proof of (1.4) and (1.5)

Let X be a finite CW-complex, whose integral cohomology has no torsion. Suppose X embedds in S^M. Let Y be an M-dual of X. Then for any class $y \in H^{M-1-2q-2r}(Y)$, $\psi_{2r}{}^k(y)$ is universally defined for $k \leq N$ and is a primary stable cohomology operation if $M - 1 - 2q - 2r \leq 2r - 4N + 4$. In particular, the above holds for $\phi_{2r}{}^k$, the dual of the Maunder operations of §5. If we assume that

$$\theta: H^{M-1-2q-2r}(Y) \to H^{M-1-2q}(Y)$$

is trivial for all $\theta \in A_{2r}$, then

$$\phi_{2r}{}^k: H^{M-1-2q-2r}(Y) \to H^{M-1-2q}(Y)/Ind$$

is also trivial for $k \leq N$. Therefore

$$\chi(\phi_{2r}{}^k): N^{2q}(X, \phi_{2r}{}^k) \to H^{2q+2r}(X) \tag{7.1}$$

is the trivial homomorphism for $k \leq N$.

Let $\mu \in K_{2q}(X)$, then $ch_q(\mu) \in H^{2q}(X)$ satisfies

$$c(\mathrm{Sq}^{2r})\rho_*(ch_q(\mu)) = \rho_*(2^r ch_{q+r}(\mu) = 0$$

hence $2^{r-1} ch_{q+r}(\mu)$ is integral mod 2. Hence by Maunder's theorem, (4.9), $\chi(\phi_{2r}{}^2)(\rho_* ch_q(\mu))$ is defined and with zero indeterminacy,

$$\chi(\phi_{2r}{}^2)(\rho_* ch_q \mu) = \rho_*(2^{r-1} ch_{q+r}(\mu) = 0$$

by (7.1) Since any class $x \in H^{2q}(X)$ is $ch_q(\mu)$ for some μ, it follows that $N^{2q}(X, \phi_{2r}{}^2) = H^{2q}(X)$, and we proceed in this way, until we obtain,

$$\chi(\phi_{2r}{}^N(\rho_* ch_q(\mu)) = \rho_*(2^{r-N+1} ch_{q+r}(\mu)) = 0$$

and hence $2^{r-N} ch_{q+r}(\mu)$ is still integral mod 2. Now from $M - 1 - 2q - 2r \leq 2r - 4N + 4$, we obtain, $4N \leq 4r + 2q - M + 5$, and (1.4) follows.

Proof of (1.5). Suppose $ch_m(\mu) = s_m(\mu)/m\,!$, with $\rho_*(s_m(\mu)) \neq 0$. Now the highest power of 2 present in $m\,!$ is $2^{m-\alpha(m)}$. Therefore, $\rho_*(2^{m-\alpha(m)} ch_m(\mu)) \neq 0$, and by Maunder's theorem (4.9),

$$\rho_*(2^{m-\alpha(m)} ch_m(\mu)) \in \chi(\phi_{2m-2q}{}^N)(ch_q(\mu)),$$

where $m - \alpha(m) = m - q - N + 1$, i.e. $N = \alpha(m) - q + 1$. However if $X \subset S^M$, and $M - 1 - 2m \leq 2(m - q) - 4N + 4$, $\chi(\phi_{2(m-q)}{}^N)(ch_q(\mu)) = 0$ with zero indeterminacy. This is a contradiction. Hence X does not embedd in S^M, where $M = 4m - 2q - 4(\alpha(m) - q + 1) + 5 = 4m + 2q - 4\alpha(m) + 1$.

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