PROJECTIVE HOMOTOPY CLASSES OF SPHERES IN THE STABLE RANGE

By Joseph Strutt

Given an element $\alpha \in \pi_n(X)$ we say that α is projective if and only if there exists a map $\bar{f}: RP^n \to X$ such that the diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\ f\ } & X \\ & {\scriptstyle \nu}\searrow \quad \nearrow{\scriptstyle \bar{f}} & \\ & RP^n & \end{array}$$

is homotopy commutative, where f is any map representing α and ν is the standard double covering.

The study of projective homotopy classes is motivated by the question of whether an r-field on S^{n-1} (i.e. a family of r vector fields which are orthonormal at every point) is homotopic to a skew linear r field. Considering an r-field to be a cross section, s, of the fibration

$$\begin{array}{ccc} V_{n-1,r} & \to & V_{n,r+1} \\ & & \downarrow \ \uparrow{\scriptstyle s} \\ & & S^{n-1} \end{array}$$

a skew linear r-field is one which satisfies $s(-x) = -s(x)$. The reader is referred to [7] for a detailed discussion of this question and its connection with projective homotopy classes.

In this paper we investigate projective homotopy classes of spheres in the stable range. Denoting by $\pi_n{}^{\mathrm{Proj}}(X)$ the set of projective homotopy classes of $\pi_n(X)$, we will prove the following:

Theorem 1.

i) $\pi_{2n}{}^{\mathrm{Proj}}(S^{2n}) = 0$

ii) $\pi_{2n+1}{}^{\mathrm{Proj}}(S^{2n+1}) = Z$ *and the sequence*

$$0 \to \pi_{2n+1}{}^{\mathrm{Proj}}(S^{2n+1}) \to \pi_{2n+1}(S^{2n+1}) \to Z_2 \to 0$$

is exact, where i is the inclusion map.

Theorem 2.

i) $\pi_{2n}{}^{\mathrm{Proj}}(S^{2n-1}) = 0$

ii) $\pi_{2n+1}{}^{\mathrm{Proj}}(S^{2n}) = \begin{cases} Z_2 \text{ if } 2 \nmid n \\ 0 \text{ if } 2 \mid n \end{cases}$

Theorem 3.

i) $\pi_{2n+1}{}^{\mathrm{Proj}}(S^{2n-1}) = \begin{cases} Z_2 \text{ if } 2 \nmid n \\ 0 \ \text{ if } 2 \mid n \end{cases}$

ii) $\pi_{2n+2}{}^{\text{Proj}}(S^{2n}) = \begin{cases} Z_2 \text{ if } 2 \nmid n \\ 0 \text{ if } 2 \mid n \end{cases}$

THEOREM 4.

i) $\pi_{2n+3}{}^{\text{Proj}}(S^{2n}) = \begin{cases} Z_{12} \text{ if } 2 \nmid n \\ Z_{24} \text{ if } 2 \mid n \text{ but } 4 \nmid n \\ Z_{12} \text{ if } 4 \mid n \end{cases}$

ii) $\pi_{2n+2}{}^{\text{Proj}}(S^{2n-1}) = \begin{cases} Z_2 \text{ if } 2 \nmid n \\ 0 \text{ if } 2 \mid n \end{cases}$

Since the stable 4- and 5-stems of S^n are zero, $\pi_{n+k}{}^{\text{Proj}}(S^n)$ is known for $k \leq 5$. The results can be conveniently summarized in the following table:

$n \equiv (-) \bmod 8$	0	1	2	3	4	5	6	
$\pi_n{}^{\text{Proj}}\ (S^n)$	0	$2\cdot Z$	0	$2\cdot Z$	0	$2\cdot Z$	0	$2\cdot Z$
$\pi_{n+1}{}^{\text{Proj}}\ (S^n)$	0	0	Z_2	0	0	0	Z_2	0
$\pi_{n+2}{}^{\text{Proj}}\ (S^n)$	0	Z_2	Z_2	0	0	Z_2	Z_2	0
$\pi_{n+3}{}^{\text{Proj}}\ (S^n)$	Z_{12}	Z_2	Z_{12}	0	Z_{24}	Z_2	Z_{12}	0
$\pi_{n+4}{}^{\text{Proj}}\ (S^n)$	0	0	0	0	0	0	0	0
$\pi_{n+5}{}^{\text{Proj}}\ (S^n)$	0	0	0	0	0	0	0	0

The proofs of Theorems 1–4 will make use of a mod 2 Postnikov resolution of S^n and obstruction theory. We recall certain facts about the homology and cohomology of RP^n:

a) $\tilde{H}_i(RP^n) = \begin{cases} 0 \text{ for } i \text{ even} \\ Z_2 \text{ for } i \text{ odd and } i \neq n \\ Z \text{ or } i \text{ odd and } i = n \end{cases}$

b) $\tilde{H}^i(RP^n; Z) = \begin{cases} 0 \text{ for } i \text{ odd and } i \neq n \\ Z \text{ for } i \text{ odd and } i = n \\ Z_2 \text{ for } i \text{ even} \end{cases}$

c) $H^*(RP^n; Z_2) = Z_2[u]/(u^{n+1})\ u \in H^1(RP^n; Z_2)$

d) Let $P{:}RP^n \to RP^n/RP^{n-1} \cong S^n$ be the quotient map and consider $H_n(S^n) \xrightarrow{\nu_*} H_n(RP^n) \xrightarrow{P_*} H_n(S^n)$. For n odd, the groups are Z, ν_* is multiplication by 2, and P_* is the identity. In particular, $P \circ \nu$ is a map of Brower degree 2.

e) The map $\nu^*{:}H^n(RP^n; Z_{2k}) \to H^n(S^n; Z_{2k})$ is zero for all values of k when n is even and the map $\nu^*{:}H^n(RP^n; Z_2) \to H^n(S^n; Z_2)$ is zero for all n.

PROPOSITION 2.1. *Let X be an $(n - 1)$ connected space where n is even. Then* $\pi_n{}^{\text{Proj}}(X) = 0$.

Proof. Assume $[f] \in \pi_n^{\text{Proj}}(X)$. Then the diagram

$$\begin{array}{ccc} S^n & \xrightarrow{\ f\ } & X \\ & {\scriptstyle \nu}\searrow \quad \nearrow{\scriptstyle \bar{f}} & \\ & RP^n & \end{array}$$

is homotopy commutative, and since $H_n(RP^n) = 0$, the map $f:H_n(S^n) \to H_n(X)$ is zero. In particular, the image of $[f]$ under the Hurewicz isomorphism is zero, implying that $[f] = 0$ in $\pi_n(X)$.

This proves part i of Theorem 1. To prove part ii, we note that the class $[P \circ \nu]$ is projective and so is $[f \circ P \circ \nu]$ where $f:S^n \to S^n$ is a map of arbitrary Brower degree. Therefore $k[P \circ \nu] \in \pi_n^{\text{Proj}}(S^n)$ for all $k \in Z$, and since $P \circ \nu$ has degree 2 when n is odd, we see that $2 \cdot \pi_n(S^n) \subset \pi_n^{\text{Proj}}(S^n)$. Conversely, every projective class must have even degree since $\nu_*:H_n(S^n) \to H_n(RP^n)$ is multiplication by 2.

The following corollary to the proof of Theorem 1 is immediate:

COROLLARY 2.2. *Let X be $(n - 1)$ connected, $n \geq 2$. Then*

$$\pi_n^{\text{Proj}}(X) = \begin{cases} 0 \text{ if } n \text{ is even} \\ 2 \cdot \pi_n(X) \text{ if } n \text{ is odd} \end{cases}$$

The proofs of the remaining theorems will make use of the following mod 2 Postnikov system for S^n:

$$\begin{array}{ccccc} K(Z_8, n+3) & \to & X_{n+3} & & \\ & & \downarrow & & \\ K(Z_2, n+2) & \to & X_{n+2} & \xrightarrow{\ \mathrm{Sq}^4\, i_n\ } & K(Z_8, n+4) \\ & & \downarrow & & \\ L(Z_2, n+1) & \to & X_{n+1} & \xrightarrow{\ \alpha(2)\ } & K(Z_2, n+3) \\ & & \downarrow & & \\ & & K(Z, n) & \xrightarrow{\ \mathrm{Sq}^2\, i_n\ } & K(Z_2, n+2) \end{array}$$

For the construction, see [2], Chapter 12. The symbol "i_n" denotes the fundamental class of $K(Z, n)$ as well as its image in $H^*(X_k; Z_2)$. The symbol "$\alpha(2)$" denotes a cohomology class which pulls back to Sq^2 of the fundamental class of the fibre $K(Z_2, n + 1)$ (usually denoted by $\mathrm{Sq}^2\, i_{n+1}$). We denote by r_j the inclusion $K(Z_{2^m}, j) \to X_j$, by p_j the fibre map $X_j \to X_{j-1}$, and by ρ_j the map $S^n \to X_j$. We recall that the map $\rho_{j\#}:\pi_i(S^n) \to \pi_i(X_j)$ is a $\mathcal{C}_2$-isomorphism for $i \leq j$ and a $\mathcal{C}_2$-epimorphism for $i = j + 1$, where $\mathcal{C}_2$ is the class of abelian torsion groups of finite exponent such that the order of each element is prime to 2.

We note that if X is an H-space or if X is $(m - 1)$ connected and $n < 2m - 1$, then $\pi_n^{\text{Proj}}(X)$ is a subgroup of $\pi_n(X)$. In particular, the X_j's in the Postnikov system are loop spaces since we are in the stable range (see [2], Corollary 2,

p. 153), so $[RP^k, X_j]$ is a group and $\pi_k^{\mathrm{Proj}}(X_j)$ is a subgroup of $\pi_k(X_j)$. Also $\pi_{n+k}^{\mathrm{Proj}}(S^n)$ is a subgroup of $\pi_{n+k}(S^n)$ for $k < n - 1$.

LEMMA 2.3. *Consider a map* $f:RP^j \to X_j$, $n \le j \le n+3$. *Then*

i) $p_j \circ f$ *has a unique extension* $h:RP^{j+1} \to X_{j-1}$

ii) *If* h *lifts to* X_j, *then* $[f \circ \nu] = 0$ *in* $\pi_j(X_j)$ *provided that* $j < n + 3$ *or* $j = n + 3$ *and* $n + 3$ *even.*

Proof. Part i follows from the Puppe sequence since $\pi_j(X_{j-1})$ and

$$\pi_{j+1}(X_{j-1}) = 0.$$

For part ii we consider the diagram

$$\begin{array}{ccccccc} S^j & \xrightarrow{\nu} & RP^j & \xrightarrow{f} & X_j & \xrightarrow{p_j} & X_{j-1} \\ & & {\scriptstyle k}\downarrow & \nearrow_{\bar{h}} & & \nearrow_{h} & \\ & & RP^{j+1} & & & & \end{array}$$

where $\bar{h}$ is the lifting of h and k is the inclusion. It is not necessarily true that $f \simeq \bar{h} \circ k$, but we claim that $f \circ \nu \simeq \bar{h} \circ k \circ \nu$ (in which case it follows trivially that $f \circ \nu \simeq 0$). We consider the fibre mapping sequence of the fibration $K(Z_{2m}, j) \to X_j \to X_{j-1}$ $(m = 1 \text{ or } 3)$

$$\begin{array}{ccccc} \cdots \to [RP^j, K(Z_{2m}, j)] & \xrightarrow{r_{j\#}} & [RP^j, X_j] & \xrightarrow{p_{j\#}} & [RP^j, X_{j-1}] \to \cdots \\ \downarrow{\scriptstyle \bar{\nu}\#} & & \downarrow{\scriptstyle \nu\#} & & \\ \cdots \to [S^j, K(Z_{2m}, j)] \to & & [S^j, X_j] \to \cdots & & \end{array}$$

Since we are in the stable range, every set in the diagram is an abelian group. Consider $[f] - [\bar{h} \circ k] \in [RP^j, X_j]$. By assumption $p_{j\#}([f] - [\bar{h} \circ k]) = 0$, so $[f] - [\bar{h} \circ k]$ is in the image of $r_{j\#}$. The map $\bar{\nu}\#$ is zero when $j < n + 3$ or when $j = n + 3$ and $n + 3$ is even by remark *e* above, so the commutativity of the square implies that $\nu\#([f] - [\bar{h} \circ k]) = 0$.

LEMMA 2.4. *Consider a map* $f:RP^j \to X_j$ *and denote by* $h:RP^{j+1} \to X_{j-1}$ *the unique extension of* $p_j \circ f$ *to* RP^{j+1} (cf. 2.3). *If the obstruction to lifting* h *is non-zero, then the class* $[f \circ \nu]$ *is non-zero in* $\pi_j(X_j)$. *If* $j = n + 3$ *and the mod 2 reduction of the obstruction to lifting* h *is non-zero, then* $[f \circ \nu]$ *is an odd multiple of the generator of* $\pi_{n+3}(X_{n+3}) = Z_8$.

Proof. By the Puppe sequence $[f \circ \nu]$ is non-zero if and only if f is not extendable to RP^{j+1}. Supposing, to the contrary, that f has an extension $\bar{f}$ to RP^{j+1}, then $p_j \circ \bar{f}$ is an extension of $p_j \circ f$. By the uniqueness of this extension, $p_j \circ f \simeq h$; but this contradicts the fact that h doesn't lift to X_j.

For the case $j = n + 3$, we alter the Postnikov system by killing $\mathrm{Sq}^4 i_n$ in $H^*(X_{n+2})$ as a Z_2 class rather than as a Z_8 class. We get

$$\begin{array}{ccccc} K(Z_2, n+3) & \to & Y_{n+3} & & \\ & & \downarrow & & \\ K(Z_2, n+2) & \to & X_{n+2} & \xrightarrow{\mathrm{Sq}^4 i_n} & K(Z_2, n+4) \\ & & \downarrow & & \\ & & \vdots & & \end{array}$$

By the naturality of pull backs, there is a map $\phi: X_{n+3} \to Y_{n+3}$ which induces mod 2 reduction on $\pi_{n+3}(-)$. ($\pi_{n+3}(X_{n+3}) = Z_8$ and $\pi_{n+3}(Y_{n+3}) = Z_2$.) The same argument then shows that $[\phi \circ f \circ \nu]$ is non-zero in $\pi_{n+3}(Y_{n+3})$; in other words, the mod 2 reduction of $[f \circ \nu]$ is non-zero, or $[f \circ \nu]$ is an odd multiple of the generator.

We can now prove Theorem 2. Since $\pi_{n+1}(S^n) = Z_2$, we ask whether there is an essential composition $S^{n+1} \xrightarrow{\nu} RP^{n+1} \xrightarrow{f} S^n$. By Corollary 2.6.23 of [3], $[RP^{n+1}, S^n] \cong [RP^{n+1}, X_{n+1}]$, so it suffices to find $\pi_{n+1}{}^{\mathrm{Proj}}(X_{n+1})$. Suppose n is equal to $2k-1$, consider any map $f: RP^{2k} \to X_{2k}$. Then the composition $RP^{2k} \xrightarrow{f} X_{2k} \xrightarrow{p_{2k}} K(Z, 2k-1)$ is null-homotopic since $H^{2k-1}(RP^{2k}; Z) = 0$, and by Lemma 2.3, $[f \circ \nu] = 0$. This proves part i.

For $n = 2k$ and $2 \mid k$, consider any map $f: RP^{2k+1} \to X_{2k+1}$. The extension of $p_{2k+1} \circ f$ to RP^{2k+2} lifts to X_{2k+1} since the obstruction is either $\mathrm{Sq}^2 u^{2k}$ or $\mathrm{Sq}^2\, 0$ according as $p_{2k+1} \circ f$ is essential or null homotopic. But $\mathrm{Sq}^2 u^{2k} = 0$ since $2 \mid k$, and by Lemma 2.3, $[f \circ \nu] = 0$. For $n = 2k$ and $2 \nmid k$, consider a map $RP^{2k+1} \xrightarrow{h} K(Z, 2k)$ which is not homotopic to zero (note that $[RP^{2k+1}, K(Z, 2k)] \cong H^{2k}(RP^{2k+1}; Z) = Z_2$). h lifts to X_{2k+1} by a map $g: RP^{2k+1} \to X_{2k+1}$, say, however the extension of h to RP^{2k+2} does not lift since the obstruction is $\mathrm{Sq}^2 u^{2k}$, which is non-zero when $2 \nmid k$. By Lemma 2.4, $[g \circ \nu]$ is non-zero, so $\pi_{2k+1}{}^{\mathrm{Proj}}(S^{2k}) \cong Z_2$ when $2 \nmid k$.

The same procedure is used in the proof of Theorem 3. It suffices to find $\pi_{n+2}{}^{\mathrm{Proj}}(X_{n+2})$, and for $n = 2k-1$ where $2 \nmid k$, we consider the composition $RP^{2k+1} \xrightarrow{u^{2k}} K(Z_2, 2k) \xrightarrow{r_{2k}} X_{2k}$. It lifts to X_{2k+1}, but the extension of $r_{2k} \circ u^{2k}$ to RP^{2k+2}, which we also denote by $r_{2k} \circ u^{2k}$, has an obstruction $(r_{2k} \circ u^{2k})^*(\alpha(2)) = (u^{2k})^*(\mathrm{Sq}^2 i_{2k}) = \mathrm{Sq}^2 u^{2k}$ which is non-zero since $2 \nmid k$. This implies, by Lemma 2.4, that $\pi_{2k+1}{}^{\mathrm{Proj}}(S^{2k-1}) \neq 0$, or $\pi_{2k+1}{}^{\mathrm{Proj}}(S^{2k-1}) = Z_2$ when $2 \nmid k$.

When $n = 2k-1$ and $2 \mid k$, we consider any map $f: RP^{2k+1} \to X_{2k+1}$ and look at $p_{2k+1} \circ f$. The map $p_{2k} \circ p_{2k+1} \circ f$ is null homotopic since $H^{2k-1}(RP^{2k+2}; Z) = 0$, so $p_{2k+1} \circ f$ factors through the fibre, $K(Z_2, 2k)$:

$$\begin{array}{ccccc} RP^{2k+1} & \xrightarrow{f} & X_{2k+1} & & \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle p_{2k+1}} & & \\ K(Z_2, 2k) & \xrightarrow{r_{2k}} & X_{2k} & \xrightarrow{\alpha(2)} & K(Z_2, 2k+2). \end{array}$$

$r_{2k} \circ \phi$ has an extension to RP^{2k+2} (by the Puppe sequence) and the obstruction to lifting it is either $\mathrm{Sq}^2 u^{2k}$ or $\mathrm{Sq}^2 0$, in both cases zero since $2 \mid k$. Therefore $[f \circ \nu] = 0$ by Lemma 2.3, and we have that $\pi_{2k+1}{}^{\mathrm{Proj}}(S^{2k-1}) = 0$ when $2 \mid k$.

For $n = 2k$ and $2 \nmid k$, we consider the composition $RP^{2k+2} \xrightarrow{u^{2k+1}} K(Z_2, 2k+1) \xrightarrow{r_{2k+1}} X_{2k+1}$ and notice that it has a lifting to X_{2k+2}. Its extension to RP^{2k+3}, however, does not lift (the obstruction is $\mathrm{Sq}^2 u^{2k+1}$, which is nonzero since $2 \nmid k$). Hence, by Lemma 2.4, $\pi_{2k+2}{}^{\mathrm{Proj}}(S^{2k}) = Z_2$ when $2 \nmid k$.

For $n = 2k$ and $2 \mid k$, we consider any map $f: RP^{2k+2} \to X_{2k+2}$. By the Puppe sequence, the map $p_{2k+2} \circ f$ has an extension to RP^{2k+3}, and we must show that the obstruction to lifting this extension, namely the pull back of $\alpha(2)$, is zero.

LEMMA 2.5. *Let $g: RP^{2k+3} \to X_{2k+1}$ be any map, where $2 \mid k$. Then $g^*(\alpha(2)) = 0$.*

Proof. We need the following facts about the cohomology of X_{2k+1} (see [2], Chapter 12):

1)

j	*generator of* $H^{2k+j}(X_{2k+1}, Z_2)$
0	i_{2k}
1	—
2	—
3	$\alpha(2)$
4	$\mathrm{Sq}^4 i_{2k}$, $\beta(3)$
5	$\gamma(3, 1)$

($\beta(3)$ is a class that pulls back to $\mathrm{Sq}^3 i_{2k+1}$ in the cohomology of the fibre; similarly, $\gamma(3, 1)$ pulls back to $\mathrm{Sq}^3 \mathrm{Sq}^1 i_{2k+1}$)

2) $d_2 \mathrm{Sq}^4 i_{2k} = \gamma(3, 1)$ where d_2 is the secondary Bockstein operator.

3) $\mathrm{Sq}^2 \alpha(2) = \gamma(3, 1)$

We first extend g to a map $\bar{g}: RP^{2k+5} \to X_{2k+1}$ and note that $g^*(\alpha(2)) = 0$ if and only if $\bar{g}^*(\alpha(2)) = 0$. Assume that $\bar{g}^*(\alpha(2)) \neq 0$. In particular $\bar{g}^*(\alpha(2)) = u^{2k+3}$ and $\mathrm{Sq}^2 \bar{g}^*(\alpha(2)) = u^{2k+5}$ since $2 \mid k$. But $\mathrm{Sq}^2 \bar{g}^*(\alpha(2)) = \bar{g}^*(\gamma(3, 1)) = \bar{g}^*(d_2 \mathrm{Sq}^4 i_{2k}) = d_2 \mathrm{Sq}^4 \bar{g}^* i_{2k}$ and $\mathrm{Sq}^4 \bar{g}^* i_{2k}$ is an even dimensional class of $H^*(RP^{2k+5}; Z_2)$, so it is the mod 2 reduction of an integral class. This means that all Bocksteins vanish, in particular that $d_2 \mathrm{Sq}^4 \bar{g}^* i_{2k} = 0$, contradicting the fact that $\mathrm{Sq}^2 \bar{g}^*(\alpha(2)) = u^{2k+5}$. Therefore $\bar{g}^*(\alpha(2))$ must be 0.

By Lemma 2.5, then, the extension of $p_{2k+2} \circ f$ lifts to X_{2k+2}, and so $[f \circ \nu] = 0$ by Lemma 2.3. In particular, $\pi_{2k+2}{}^{\mathrm{Proj}}(S^{2k}) = 0$ when $2 \mid k$. This concludes the proof of Theorem 3.

The stable 3-stem of S^n is Z_{24}, the 3-primary part of which is lost in the mod 2 Postnikov system. In odd dimensions the 3-primary part, Z_3, must be projective, since, by Remark *d* above, twice any odd dimensional homotopy class is projective. Therefore, remembering that we are in the stable range (so that projective classes are additive), we need only analyze the 2-primary component of $\pi_{2k+3}(S^{2k})$.

Since $2k + 3$ is odd, we have that $2 \cdot \pi_{2k+3}(S^{2k}) \subset \pi_{2k+3}{}^{\mathrm{Proj}}(S^{2k})$, or $Z_4 \subset \pi_{2k+3}(S^{2k})$

(considering only the 2-primary part). For $2 \nmid k$, we must show that $[f \circ \nu]$ is divisible by 2 for any map $f: RP^{2k+3} \to X_{2k+3}$. This is equivalent to showing that $[\phi \circ f \circ \nu]$ is 0 in $\pi_{2k+3}(Y_{2k+3})$ where $\phi: X_{2k+3} \to Y_{2k+3}$ is the map constructed in the proof of Lemma 2.4. We consider $\bar{p}_{2k+3} \circ \phi \circ f$ ($\bar{p}_{2k+3}$ is the projection $Y_{2k+3} \to X_{2k+2}$—see proof of Lemma 2.4) and we must show that its extension to RP^{2k+4}, which we denote by g, lifts to Y_{2k+3}. This amounts to showing that $g^*(\mathrm{Sq}^4 i_{2k}) = 0$. (We are in the altered Postnikov system, so $\mathrm{Sq}^4 i_{2k}$ is considered as a Z_2-class). We claim that $g^*(i_{2k}) = 0$. This is true if and only if the projected map $p_{2k+1} \circ p_{2k+2} \circ g: RP^{2k+4} \to K(Z, 2k)$ pulls the fundamental class i_{2k} back to 0. If it didn't, there would be a non-zero obstruction to lifting $p_{2k+1} \circ p_{2k+2} \circ g$ to X_{2k+1}, namely $\mathrm{Sq}^2 u^{2k}$ (this is a non-zero since $2 \nmid k$), which is clearly a contradiction. Therefore, the extension g lifts, and by Lemma 2.4, $[\phi \circ f \circ \nu] = 0$ or $[f \circ \nu]$ is divisible by 2 in $\pi_{2k+3}(X_{2k+3})$, implying that $\pi_{2k+3}{}^{\mathrm{Proj}}(S^{2k}) = Z_4$ when $2 \nmid k$.

When $2 \mid k$ but $4 \nmid k$, we consider the map $h: RP^{2k+3} \to K(Z, 2k)$ where h is not null homotopic. Since $2 \mid k$, all obstructions to lifting h are zero, so we get a map $f: RP^{2k+3} \to X_{2k+3}$. Denote the extension of $p_{2k+3} \circ f$ to RP^{2k+4} by g; the mod 2 reduction of the obstruction to lifting g is simply $g^*(\mathrm{Sq}^4 i_{2k})$ with $\mathrm{Sq}^4 i_{2k}$ considered as a Z_2 class. The map g pulls i_{2k} back to u^{2k} since h is not null homotopic, so $g^*(\mathrm{Sq}^4 i_{2k}) = \mathrm{Sq}^4 u^{2k} = u^{2k+4}$ since $4 \nmid k$. Therefore, by Lemma 2.4, $[f \circ \nu]$ is an odd multiple of the generator of $\pi_{2k+3}(X_{2k+3})$, which implies that $\pi_{2k+3}{}^{\mathrm{Proj}}(S^{2k}) = Z_8$ when $2 \mid k$ but $4 \nmid k$.

When $4 \mid k$, consider any map $f: RP^{2k+3} \to X_{2k+3}$. We will show that $[\phi \circ f \circ \nu] = 0$ in $\pi_{2k+3}(Y_{2k+3})$. The extension of $\bar{p}_{2k+3} \circ \phi \circ f$ to RP^{2k+3} pulls $\mathrm{Sq}^4 i_{2k}$ (as a Z_2-class) back to $\mathrm{Sq}^4 u^{2k}$ or $\mathrm{Sq}^4 0$, which is zero in both cases since $4 \mid k$. Therefore the extension lifts and by the proof of Lemma 2.3 $[\phi \circ f \circ \nu] = 0$, or $[f \circ \nu]$ is divisible by 2. This implies that $\pi_{2k+3}{}^{\mathrm{Proj}}(S^{2k}) = Z_4$ when $4 \mid k$. This finishes the proof of part *i* of Theorem 4.

In the proof of part ii, it will be helpful to have the mod 2 Postnikov system written out explicitly:

$$\begin{array}{ccccc}
K(Z_8, 2k+2) & \to & X_{2k+2} & & \\
 & & \downarrow & & \\
K(Z_2, 2k+1) & \to & X_{2k+1} & \xrightarrow{\mathrm{Sq}^4 i_{2k-1}} & K(Z_8, 2k+3) \\
 & & \downarrow & & \\
K(Z_2, 2k) & \to & X_{2k} & \xrightarrow{\alpha(2)} & K(Z_2, 2k+2) \\
 & & \downarrow & & \\
 & & K(Z, 2k-1) & \xrightarrow{\mathrm{Sq}^2 i_{2k-1}} & K(Z_2, 2k+1)
\end{array}$$

Since $\mathrm{Sq}^4 i_{2k-1}$ is considered as a Z_8-class, it will be a delicate matter to decide under what conditions it will give rise to a non-zero obstruction. We will need certain information about the Z_8 cohomology of $K(Z_2, n)$.

LEMMA. 2.6

$$H_{n+i}(K(Z_2, n)) = \begin{cases} Z_2 & i = 0 \\ 0 & i = 1 \\ Z_2 & i = 2 \\ Z_2 & i = 3 \end{cases}$$

where only the 2*-primary component is considered.*

Proof. The result for $i = 0$ follows from the Hurewicz Theorem. Since $H^*(K(Z_2, n); Z_2)$ is known, we can apply the universal coefficient theorems. Setting $K = K(Z_2, n)$, we have

$$H^{n+1}(K; Z_2) = Z_2 = \text{Hom }(H_{n+1}(K), Z_2) + \text{Ext }(Z_2, Z_2)$$

which implies that $H_{n+1}(K) = 0$. Secondly

$$H^{n+2}(K; Z_2) = Z_2 = \text{Hom }(H_{n+2}(K), Z_2) + \text{Ext }(0, Z_2)$$

which implies that $H_{n+2}(K)$ is equal to Z or Z_{2^r}, $r \geq 1$. But $H^{n+2}(K; Z_2)$ is generated by $\text{Sq}^2\, i_n$ and $d_1 \text{Sq}^2\, i_n = \text{Sq}^3\, i_n$ which is non-zero, so, in particular, $\text{Sq}^2\, i_n$ is not the mod 2 reduction of a Z_4 class. This implies that $H_{n+2}(K)$ must equal Z_2. Finally, we have

$$H^{n+3}(K; Z_2) = Z_2 + Z_2 = \text{Hom }(H_{n+3}(K); Z_2) + \text{Ext }(Z_2, Z_2)$$

implying that $H_{n+3}(K)$ is equal to Z or Z_{2^r}, $r \geq 1$. $H^{n+3}(K; Z_2)$ has generators $\text{Sq}^3\, i_n$ and $\text{Sq}^2\, \text{Sq}^1\, i_n$; the first Bockstein d_1 vanishes on the first and is non-zero on the second, implying that the first generator is the mod 2 reduction of a Z_4 class and the second isn't. Since $\text{Ext }(Z_2, Z_4) \to \text{Ext }(Z_2, Z_2)$ is an isomorphism the Z_2 summand generated by $\text{Sq}^3\, i_n$ must be $\text{Ext }(Z_2, Z_2)$. This leaves $\text{Sq}^2\, \text{Sq}^1\, i_n$ as the generator of $\text{Hom }(H_{n+3}(K); Z_2)$, and since $\text{Sq}^2\, \text{Sq}^1\, i_n$ is not the reduction of a Z_4-class, $H_{n+3}(K)$ must equal Z_2.

LEMMA 2.7. *The composition*

$$RP^{2k+3} \xrightarrow{u^{2k+1}} K(Z_2, 2k + 1) \xrightarrow{r_{2k+1}} X_{2k+1} \xrightarrow{\text{Sq}^4\, i_{2k-1}} K(Z_8, 2k + 3)$$

represents zero in $H^{2k+3}(RP^{2k+3}; Z_8) = Z_8$ *when* $2 \mid k$ *and* 4 *times the generator when* $2 \nmid k$.

Proof. Recall that $d_2(\text{Sq}^4\, i_{2k-1}) = \gamma(3, 1)$ in $H^*(X_{2k}; Z_2)$, so the $\text{Sq}^4\, i_{2k-1}$ in $H^*(X_{2k}; Z_2)$ is not the reduction of a Z_8 class. Therefore the $\text{Sq}^4\, i_{2k-1}$ in $H^*(X_{2k+1}; Z_8)$ is not in the image of $p_{2k+1}{}^*$, and so $r_{2k+1}{}^*(\text{Sq}^4\, i_{2k-1})$ is non-zero in $H^{2k+3}(K(Z_2, 2k + 1); Z_8)$, which, by Lemma 2.6, is equal to Z_2. It remains only to find the conditions under which the map $(u^{2k+1})^*$ is non trivial on $H^{2k+3}(-; Z_8)$. We have

$$\begin{array}{ccc} H^{2k+3}(RP^{2k+3}; Z_8) & \leftarrow & H^{2k+3}(K(Z_2, 2k + 1); Z_8) \\ \| & & \| \\ Z_8 & & Z_2 \end{array}$$

and by the universal coefficient theorem this is non trivial if and only if $(u_{2k+1})_*$ is non-zero on $H_{2k+3}(-)$, which is true if and only if $(u_{2k+1})^*$ is non trivial on $H^{2k+3}(-; Z_2)$. But this is equivalent to $\mathrm{Sq}^2 u^{2k+1}$ being non-zero, which is equivalent to the condition $2 \nmid k$.

LEMMA 2.8. *Given any map* $f: RP^{2k+3} \to X_{2k}$, *then* $f^*(\mathrm{Sq}^4 i_{2k-1}) = 0$, *where* $\mathrm{Sq}^4 i_{2k-1}$ *is considered as a class in* $H^*(X_{2k}; Z_4)$.

Proof. f factors through $K(Z_2, 2k)$ since $p_{2k} \circ f$ is null homotopic. Therefore it will suffice to check that any map $g: RP^{2k+3} \to K(Z_2, 2k)$ is trivial on $H^{2k+3}(-; Z_4)$. By the universal coefficient theorems, this amounts to showing that the induced maps

$$\mathrm{Hom}\ (H_{2k+3}(RP^{2k+3}), Z_4) \leftarrow \mathrm{Hom}\ (H_{2k+3}(K(Z_2, 2k)), Z_4)$$

and

$$\mathrm{Ext}\ (H_{2k+2}(RP^{2k+3}), Z_4) \leftarrow \mathrm{Ext}\ (H_{2k+2}(K(Z_2, 2k)), Z_4)$$

are zero. The second map is zero since $H_{2k+2}(RP^{2k+3}) = 0$. That the first map is zero will follow if $H_{2k+3}(RP^{2k+3}) \to H_{2k+3}(K(Z_2, 2k))$ is zero, which in turn will follow (by the universal coefficient theorem and Lemma 2.6) if $H^{2k+3}(RP^{2k+3}; Z_2) \leftarrow H^{2k+3}(K(Z_2, 2k); Z_2)$ is zero. But g is homotopic to 0 or u^{2k}, and $\mathrm{Sq}^2 \mathrm{Sq}^1 u^{2k}$ and $\mathrm{Sq}^3 u^{2k}$ are both zero for all values of k.

We first prove part ii of Theorem 4 for the 2-primary component only. When $2 \nmid k$, we consider the composition $RP^{2k+2} \xrightarrow{u^{2k+1}} K(Z, 2k+1) \xrightarrow{r_{2k+1}} X_{2k+1}$, which lifts to a map $f: RP^{2k+2} \to X_{2k+2}$. By Lemma 2.7, the extension of $r_{2k+1} \circ u^{2k+1}$ to RP^{2k+3} does not lift, and so, by Lemma 2.4, $[f \circ \nu] \neq 0$.

The following Proposition completes the proof for the case $2 \nmid k$.

PROPOSITION 2.9. *Given any map* $f: RP^{2k+2} \to X_{2k+2}$, *then the reduction* mod 4 of $[f \circ \nu]$ *is zero.*

Proof. We alter the Postnikov system for S^{2k-1} by killing $\mathrm{Sq}^4 i_{2k-1}$ in $H^*(X_{2k+1}; Z_2)$ as a Z_4 class rather than as a Z_8 class. We get

$$\begin{array}{ccccc} K(Z_4, 2k+2) & \to & Z_{2k+2} & & \\ & & \downarrow & & \\ K(Z_2, 2k+1) & \to & X_{2k+1} & \xrightarrow{\mathrm{Sq}^4 i_{2k-1}} & K(Z_4, 2k+3) \\ & & \downarrow & & \\ & & \vdots & & \end{array}$$

By naturality of pull backs, there is a map $\phi: X_{2k+2} \to Z_{2k+2}$ which is mod 4 reduction on $\pi_{2k+2}(-)$. It will suffice to show that for any map $f: RP^{2k+2} \to Z_{2k+2}$, $[f \circ \nu] = 0$. Consider $p_{2k+2} \circ f$ and denote its extension to RP^{2k+3} by g. The $\mathrm{Sq}^4 i_{2k-1}$ in $H^*(X_{2k+1}; Z_4)$ is the image under $p_{2k+1}{}^*$ of the $\mathrm{Sq}^4 i_{2k-1}$ in $H^*(X_{2k}; Z_4)$, so by Lemma 2.8, $g^*(\mathrm{Sq}^4 i_{2k-1}) = 0$. Therefore g lifts, and it follows from Lemma 2.3 that $[f \circ \nu] = 0$.

The next Proposition completes the proof of part ii for the 2-primary component.

PROPOSITION 2.10. *If $2 \mid k$, then for any map $f : RP^{2k+2} \to X_{2k+2}$, we have $[f \circ \nu] = 0$.*

Proof. If we can show that $p_{2k+2} \circ f$ factors through the fibre $K(Z_2, 2k+1)$, we are done by Lemmas 2.7 and 2.3. The composition $p_{2k} \circ p_{2k+1} \circ p_{2k+2} \circ f : RP^{2k+2} \to K(Z, 2k-1)$ is null homotopic, so $p_{2k+1} \circ p_{2k+2} \circ f$ factors through the fibre $K(Z_2, 2k)$ by a map g, say:

$$\begin{array}{ccccccc}
 & & & & \downarrow & & \\
 & & K(Z_2, 2k+1) & \rightarrow & X_{2k+1} & & \\
 & \nearrow^{p_{2k+2} \circ f} & & & \downarrow p_{2k+1} & & \\
RP^{2k+2} & \xrightarrow{g} & K(Z_2, 2k) & \xrightarrow{r_{2k}} & X_{2k} & & \\
 & & & & \downarrow & & \\
 & & & & K(Z, 2k-1) & \xrightarrow{Sq^2 i_{2k-1}} & K(Z_2, 2k+1)
\end{array}$$

It remains only to show that $r_{2k} \circ g$ is null homotopic. We consider the fibre mapping sequence of the fibration $K(Z_2, 2k) \to X_{2k} \to K(Z, 2k-1)$:

$$\cdots \to [RP^{2k+2}, K(Z, 2k-2)] \xrightarrow{Sq^2} [RP^{2k+2}, K(Z_2, 2k)] \xrightarrow{(r_{2k})_\#} [RP^{2k+2}, X_{2k}] \to \cdots$$

Since $2 \mid k$, the map "Sq^2" is an isomorphism, so $(r_{2k})_\# = 0$. It follows that $r_{2k} \circ g$ is null homotopic.

In order to study the 3-primary part of $\pi_{2k+2}{}^{\mathrm{Proj}}(S^{2k-1})$, we must construct a mod 3 Postnikov system. It is known that $H^*(K(Z, 2k-1); Z_3)$ has a fundamental class i_{2k-1}, and the next group is a Z_3 in dimension $2k+3$ generated by $P^1 i_{2k-1}$ where P^1 is the first reduced power operation of Steenrod. We construct a space Y_{2k+2} by killing this class:

$$\begin{array}{ccc}
K(Z_3, 2k+2) \to Y_{2k+2} & & \\
\downarrow & & \\
K(Z, 2k-1) & \xrightarrow{P^1 i_{2k-1}} & K(Z_3, 2k+3).
\end{array}$$

It can be verified that the Z_3-cohomology of Y_{2k+2} is Z_3 in dimension $2k-1$ and 0 in dimensions $2k+1$ through $2k+3$. The map $S^{2k-1} \to K(Z, 2k-1)$ representing the fundamental class of S^{2k-1} lifts to Y_{2k+2} and the lifting induces an isomorphism on $H^i(-; Z_3)$ for $i \leq 2k+2$ and a monomorphism for $i = 2k+3$. By the $\mathcal{C}_p$ approximation theorem (see [2] Chapter 10), it induces an isomorphism on homotopy groups through dimension $2k+2$. Therefore the above is a mod 3 Postnikov resolution of S^{2k-1}.

By the fibre mapping sequence, $[RP^{2k+2}, Y_{2k+2}] = 0$, so, in particular,

$\pi_{2k+2}{}^{\mathrm{Proj}}(Y_{2k+2}) = 0$, implying that the 3-primary part of $\pi_{2k+2}{}^{\mathrm{Proj}}(S^{2k-1})$ is zero. This concludes the proof of Theorem 4.

TULANE UNIVERSITY OF LOUISIANA

REFERENCES

[1] H. CARTAN, Seminaire H. Cartan 1954–1955, Paris.
[2] R. E. MOSHER AND M. C. TANGORA, Cohomology Operations and Applications in Homotopy Theory, Harper and Row, New York, 1966.
[3] E. H. SPANIER, Algebraic Topology, McGraw-Hill, New York, 1966.
[4] N. E. STEENROD AND D. B. A. EPSTEIN, *Cohomology Operations*, Annals of Mathematical Studies, 50, Princeton University Press, Princeton, 1962.
[5] J. STRUTT, Projective Homotopy Classes, Thesis, University of Illinois, 1970.
[6] ———, "Projective homotopy classes of Stiefel manifolds" (submitted)
[7] P. ZVENGROWSKI, "Skew linear vector fields on spheres" (to appear, J. London Math. Soc.)