RESOLUTIONS OF FIBRATIONS

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If X is a space, a resolution of X is usually understood to mean a sequence of fibrations $\cdots \to X_2 \to X_1 \to X_0 = X$, where the fibers $Y_n \to X_n \to X_{n-1}$ have some desirable property involving X. The most famous resolution of this type perhaps is the Postnikov resolution, where Y_n has the property that $\pi_i(Y_n) = 0$ if $i \neq n-1$, $\pi_{n-1}(Y_n) = \pi_n(X_{n-1}) = \pi_n(X)$, the latter isomorphisms being induced by the obvious maps.

If $F \to E \to B$ is a fibration, a resolution of the fibration is again a sequence of fibrations $\cdots \to E_2 \to E_1 \to E_0 = E$, with the fibers having some desirable property involving F.

Adams has shown how to construct a resolution of a space stably which arises from consideration of the Hurewicz map for a generalized cohomology theory [1]. There is an obvious unstable generalization of this, and Mahowald has shown that for certain ("orientable") fibrations, one can also produce resolutions of this type. The purpose of this note is to show that one can always make resolutions of this, or any other functorial, sort of a fibration. We give up one thing to do this—our fibrations are not principal in the sense that E_{n+1} is not induced from a loop fibering from a map of E_n into something.

If $F \to E \to B$ is a Serre fibration, we can apply the singular complex functor S and obtain a Kan fibration $S(F) \to S(E) \to S(B)$. Every Kan fibration contains a minimal fibration, which is a deformation retract, and every minimal fibration is a fiber bundle in the simplicial sense with structure group $G(B)$, where G is Kan's loop group functor (see [2] for details). Thus, if we are willing to work simplicially, up to homotopy tupe, we may assume that we have a fiber bundle, and that $E = F \times_{G(B)} E(B)$, where $E(B) = G(B) \times_\tau B$ is the principal $G(B)$ fibration over B with contractible total space.

A simplicial set functor ϕ is a functor from simplicial sets to simplicial sets, and from simplicial maps to simplicial maps (one can also consider continuous functors and avoid the simplicial theory, but this tends to produce surmountable but unpleasant and distracting technical difficulties later). A natural transformation $\eta : \phi_1 \to \phi_2$ of simplicial set functors is a natural transformation of functors (such that $\eta(X)$ is a simplicial map for all simplicial sets X). Notice that if $\eta(X)$ is a fiber bundle for all X, with fiber $\phi_0(X)$, ϕ_0 is a simplicial set functor, and $\phi_0(F)$ is the fiber of the fiber bundle. $\phi_1(F) \times_{G(B)} E(B) \to \phi_2(F) \times_{G(B)} E(B)$, where $G(B)$ acts on the $\phi_i(F)$ by means of ϕ_i:Hom $(F, F) \to$ Hom $(\phi_i(F), \phi_i(F))$. Thus the following is obvious.

Theorem 1. *If $\mathcal{E} = F \times_{G(B)} E(B)$ is a simplicial fiber bundle, the function $\phi \to \phi(\mathcal{E}) = \phi(F) \times_{G(B)} E(B)$ from the category of simplicial set functors and natural transformations to the category of $G(B)$-bundles over B is a functor, and if a natural transformation $\eta : \phi_1 \to \phi_2$ is functorially a fiber bundle with fiber ϕ_0 (in the sense above), the fibration $\phi_1(\mathcal{E}) \to \phi_2(\mathcal{E})$ is a fiber bundle with fiber $\phi_0(F)$.*

Notice that if we wish to apply Theorem 1, the ϕ's must be functors, not just functors up to homotopy.

COROLLARY 1. *If ϕ_0 is the identity functor, and if ϕ_n is a simplicial set functor for $n \geq 0$, with natural transformations $\eta_n : \phi_n \to \phi_{n-1}$, and if the maps $\phi_n(X) \to \phi_{n-1}(X)$ are fiber bundles in a functorial fashion with fibers $\psi_n(X)$, then for any fiber bundle $\mathcal{E} = F \times_G E$, where E is a principal G-bundle, there is a natural resolution $\cdots \to \phi_2(\mathcal{E}) \to \phi_1(\mathcal{E}) \to \phi_0(\mathcal{E})$, where $\phi_0(\mathcal{E})$ is the total space of $\mathcal{E}$, and the fiber of $\phi_n(\mathcal{E}) \to \phi_{n-1}(\mathcal{E})$ is naturally isomorphic to $\psi_n(F)$.*

We now show how to produce two types of functorial fibrations of a "Hurewicz type". Suppose that $\mathfrak{M} = \{M_n\}$ is a spectrum [4]. Then $\mathfrak{M}(X) = \lim G^n(X \wedge M_n)$ is a functor on spaces X with basepoint (simplicially this is bovious; topologically lim must be defined with care). Notice that $\pi_*(\mathfrak{M}(X))$ is the group called the generalized homology of X with coefficients in $\mathfrak{M}$[5].

A unit for $\mathfrak{M}$ is a basepoint preserving map $S^0 \to M_0$—that is, a point in M_0. Given a unit for $\mathfrak{M}$, we have an induced map $X = X \wedge S^0 \to X \wedge M_0 \to \mathfrak{M}(X)$. This map is functorial. Let $\phi_0(X) = X$, and let $\phi_1(X)$ be the mapping fiber space over X induced from the universal $G(\mathfrak{M}(X))$ bundle over $\mathfrak{M}(X)$.

At this point, we have two choices. We can iterate this construction by iterating the functor ϕ_1 so that $\phi_n = \phi_1(\phi_{n-1})$. We call this the iterated Hurewicz resolution. This is the straightforward extension of the technique of Adams into the unstable range.

The second choice is the technique of Kan and Bousfield. Let $\phi_2(X)$ be the fibration over $\phi_1(X)$ induced from the universal $G(\phi_1(\mathfrak{M}(X)))$ bundle by the map $\phi_1(X) \to \phi_1(\mathfrak{M}(X))$, obtained by applying ϕ_1 to the map $X \to \mathfrak{M}(X)$. Continue on in this fashion—$\phi_{n+1}(X)$ is induced from the map $\phi_n(X) \to \phi_n(\mathfrak{M}(X))$. This resolution we simply call the Hurewicz resolution.

THEOREM 2. *If $\mathcal{E}$ is a fiber bundle with fiber F, then there is a resolution $\{\phi_i(\mathcal{E})\}$ of the total space, together with a map $\phi_i(F) \to \phi_i(E)$ of resolutions such that the successive fibers $\psi_i(F) \to \psi_i(\mathcal{E})$ are the same, if $\{\phi_i\}$ is either the iterated Hurewicz resolution or the Hurewicz resolution.*

Our techniques have further applications. In [3], Kan and Bousfield show how to "complete" a space with respect to a ring R. This is done by associating to the space, a diagram of spaces and maps. These spaces and maps are defined by functors and natural transformations. Thus, one can complete a fiber bundle along the fiber, by taking the diagram of associated fiber bundles.

THEOREM 3. *If $\mathcal{E}$ is a fiber bundle with fiber F, and if R is a ring, there is an associated fiber bundle $\mathcal{E}_R$ with fiber the R-completion F_R of F, and with the same base space.*

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