# A REMARK ON A THEOREM OF R. C. JAMES

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If $E$ is an infinite dimensional Banach space with dual space $E^*$ then a biorthogonal system $(x_n, f_n)$, i.e. $(x_n) \subset E$, $(f_n) \subset E^*, f_n(x_m) = \delta_{mn}$, is a *Schauder basis* for $E$ if, for each $x \in E$

$$x = \sum_{n=1}^{\infty} f_n(x)x_n \tag{1}$$

convergence in the norm topology of $E$.

A closed subspace $E_0$ of $E$ is *complemented* in $E$ if there is a bounded linear projection $P$ (i.e. $P^2 = P$) from $E$ onto $E_0$. When we use the term "subspace" from now on we mean a "closed infinite dimensional subspace".

Denote the set of all bounded linear operators from $E$ to $F$ by $\mathfrak{L}(E, F)$, $E$ and $F$ Banach spaces. The purpose of this brief note is to give a simple proof of the following known result.

THEOREM 1: *If $E$ has a complemented subspace $E_0$ with a Schauder basis then $\mathfrak{L}(E, E)$ is not reflexive.*

This theorem follows from the Grothendieck-Schatten theory of tensor products [1], [8]. Indeed the theorem has been exploited from this point of view by the author's colleague J. R. Holub [2], [3].

We give here a truly elementary proof, avoiding tensor products, based on the following remarkable result of R. C. James [5] (see also [6], [4]): *A Banach space $E$ is non-reflexive if and only if, for each number $r < 1$, there exists a sequence $\{z_i\}$ of elements in the unit ball of $E$ and a sequence $\{f_i\}$ of continuous linear functionals with unit norms such that*

$$f_n(z_i) > r \quad \text{if} \quad n \leq i, f_n(z_i) = 0 \quad \text{if} \quad n > i. \tag{2}$$

(Geometrically, a Banach space $E$ is reflexive if and only if its unit sphere contains no large flat region.)

We prove the following somewhat stronger statement of the main result.

THEOREM 2: *Suppose $E$ and $F$ are infinite dimensional Banach spaces containing biorthogonal systems $(x_i, f_i)$, $(y_i, g_i)$ respectively such that*

(a) $\| x_i \| \leq 1$ *for each* $i$;

(b) $\| g_i \| = 1$ *for each* $i$; *and,*

(c) $\sup_n \| \sum_{i=1}^{n} f_i(x)y_i \| \leq \| x \|$ *for each* $x \in E$.

*Then $\mathfrak{L}(E, F)$ is not reflexive.*

*Proof.* Define $T_n: E \to F$ by

$$T_n(x) = \sum_{i=1}^{n} f_i(x)y_i \tag{3}$$

By (c) $\| T_n \| \leq 1$ for all $n$. If $A = (a_i) \subset F^*$ is such that $\Sigma \| a_i \| < +\infty$

define $G_A$ by

$$(4) \qquad G_A(T) = \sum_{i=1}^{\infty} a_i(Tx_i),$$

where $T \in \mathfrak{L}(E, F)$ is arbitrary. Then,

$$(5) \qquad | G_A(T) | \leq \sum_{i=1}^{\infty} \| a_i \| \| Tx_i \| \leq (\sum_{i=1}^{\infty} \| a_i \|) \| T \|$$

by (a) and so $G_A \in (\mathfrak{L}(E, F))^*$.

Let $a_{i,n} = \delta_{in} g_n$, $A_n = (a_{i,n})$ and $G_n = G_{A_n}$. Then $\| G_n \| \leq \sum_{i=1}^{\infty} \| a_{i,n} \| = \| g_n \| = 1$ by (5) and (6).

Also, since $\| T_n \| \leq 1$,

$$(6) \qquad \| G_n \| \geq | G_n(T_n) | = | \sum_{i=1}^{\infty} a_{i,n}(T_n x_i) | = | g_n(T_n x_n) | = | g_n(y_n) | = 1,$$

i.e.

$$(7) \qquad \| G_n \| = 1 \quad \text{for all } n.$$

Let $i \leq n$. Then

$$G_i(T_n) = g_i(T_n(x_i)) = \sum_{j=1}^{n} f_j(x_i) g_i(y_j) = 1.$$

If $i > n$ then clearly $G_i(T_n) = 0$. By the theorem of James above, $\mathfrak{L}(E, F)$ is non-reflexive.

Theorem 1 is an immediate corollary; for, if $(z_i, h_i)$ is a Schauder basis for $E_\circ$ and $P$ is a projection from $E$ onto $E_\circ$ and $x_i = y_i = z_i$, $f_i = h_i \circ P$, $g_i = h_i$ then $E$ can be renormed so that a), b) and c) hold. Indeed without loss of generality we may suppose

$$(8) \qquad 0 < \inf_n \| z_n \| \leq \sup_n \| z_n \| < +\infty.$$

For $x \in E_\circ$ define

$$(9) \qquad \| x \|_\circ = \sup_{m \leq n} \| \sum_{i=m}^{n} h_i(x) z_i \|.$$

It is straightforward to check that

$$\| z_n \|_\circ = \| h_n \|_\circ = 1, \quad \text{and} \quad \| \sum_{i=1}^{n} h_i(x) z_i \|_\circ \leq \| x \|_\circ$$

for each $x \in E_\circ$.

Define $| x |$ on $E$ by

$$(10) \qquad | x | = \| Px \|_\circ + \| (I - P(x) \|$$

where $I$ is the identity operator on $E$.

Now

$$(11) \qquad | Px | = \| P(Px) \|_\circ + \| (I - P)(Px) \| = \| Px \|_\circ \leq | x |$$

and so $| P | = 1$. Thus (a), (b) and (c) hold.

We observe that there are infinite dimensional $E$ and $F$ for which $\mathfrak{L}(E, F)$ is reflexive. Indeed by Pitts theorem [7], $\mathfrak{L}(\ell^q, \ell^p)$ is reflexive for $q > p > 1$.

Whether there are infinite dimensional $E$ and $F$ such that both $\mathcal{L}(E, F)$ and $\mathcal{L}(F, E)$ are reflexive appears to be open.

We conclude by remarking that the proof of theorem 2 shows that if $E$ and $F$ satisfy the hypothesis of that theorem, then $\ell^1 \subset (\mathcal{L}(E, F))^*$. Indeed the mapping $A \to G_A$ is an isomorphism of $\ell^1 \hat{\otimes} F^*$ into $(\mathcal{L}(E, F))^*$ for $\ell^1 \hat{\otimes} F^*$ is exactly the absolutely converging sequences in $F^*$ [1],

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