ORDINARY DIFFERENTIAL EQUATIONS IN A BANACH SPACE AND RETARDED FUNCTIONAL DIFFERENTIAL EQUATIONS

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Introduction

Ordinary differential equations in a Banach space (Krein [3], Massera-Schäffer [4] and many others) and functional differential equations of retarded type (Krasovski [2], Hale [1] and many others) have been studied as two independent subjects. It has been proved, under quite general conditions on the function f of problem (1), that such problem is equivalent to an initial value problem for generalized ordinary differential equations in a Banach space of continuous functions with the supremum norm (Oliva-Vorel [5]). If one wants to write problem (1) as a classical ordinary equation (2) and still preserve the simple relations (4) and (5) between solutions of (1) and (2), it is inconvenient to work with the space of continuous functions, because the function $G[y(s), s]$ in (2) is discontinuous at every s, although f in (1) is continuous. To avoid this difficulty one is naturally led to a space with integral norm, e.g. L_1. The principal result is contained in Theorems 1 and 2 from which properties of solutions of ordinary equations (2) imply corresponding properties of solutions of (1).

1. Description of the equations

Let a and h denote some fixed positive real numbers and let $\mathcal{B}$ stand for a Banach space whose elements are functions with domain in $[-h, a]$ and range in $\mathbf{R}^n$. When $x \in \mathcal{B}$, then for any $t \in [0, a]$ the symbol x_t denotes another function with domain $[-h, a]$ defined by

$$x_t(\tau) \begin{cases} x(\tau);\ \tau \in [-h, t) \\ x(t);\ \rho \in [t, a]; \end{cases}$$

that is, x_t is the same function as x except from t on where it is made constant equal to $x(t)$. Let $A \subset \mathcal{B}$ be any subset of $\mathcal{B}$ with the property that $x \in A$ implies $x_t \in A$ for all $t \in [0, a]$. Notice that $(x_t)_t = x_t$.

Finally, let f and G be two functions with domains and ranges as specified below:

$$f{:}A \times [0, a] \to \mathbf{R}^n$$

$$G{:}A \times [0, a] \to \mathcal{B}.$$

Given fixed elements $x^0, y^0 \in A$, both such that $x_t^0 = x^0$, $y_t^0 = y^0$, $t \in [0, a]$, we consider the following initial value problems:

$$(1) \qquad \begin{cases} \dfrac{dx(t)}{dt} = f(x_t,t);\ t \in [0, a] \\ x(t) = x^0(t);\ t \in [-h, 0) \end{cases}$$

and

$$(2) \qquad \begin{cases} \dfrac{dy(t)}{dt} = G[y(t), t]; \; t \in [0, a] \\ y(0) = y^0. \end{cases}$$

Problem (1) contains, among others, the so called retarded functional differential equations, while (2) is an ordinary differential equation in $\mathcal{B}$.

We shall say that problems (1) and (2) have a solution if, respectively, there exist functions $x(t) \in \mathbf{R}^n$ and $y(t) \in A$, $t \in [0, a]$ such that:

$$(1^*) \qquad \begin{cases} x(t) = x^0(0) + \int_0^t f(x_s, s)\, ds; \; t \in [0, a] \\ x(t) = x^0(t); \; t \in [-h, 0) \end{cases}$$

and

$$(2^*) \qquad \begin{cases} y(t) = y^0 + \int_0^t G[y(s), s]\, ds; \; t \in [0, a] \\ y(0) = y^0 \end{cases}$$

where, of course, the integrals are supposed to make sense in some definite way, say, for example, as Riemann integrals.

We want to find when problems (1) and (2) are equivalent. More precisely, given f how should one construct G such that solutions of (1) and (2) are in one to one correspondence.

2. The equivalence

Let the function $f: A \times [0, a] \to \mathbf{R}^n$ be given and define a function G with domain in $A \times [0, a]$ and range in a function space by

$$(3) \qquad G(y, t)(\tau) = \begin{cases} 0; \; \tau \in [-h, t) \\ f(y_t, t); \; \tau \in [t, a] \end{cases}$$

for all $(y, t) \in A \times [0, a]$. Here $G(y, t)(\tau) \in \mathbf{R}^n$ denotes the value at τ of the function $G(y, t)$.

Assumption 1. The space $\mathcal{B}$ contains the right continuous step functions. Under this assumption we are sure that $G(y, t) \in \mathcal{B}$ for all $(y, t) \in A \times [0, a]$.

From now on problems (1) *and* (2) *will be considered with G related to f as in* (3).

If $x(t) \in \mathbf{R}^n$, $t \in [0, a]$, is a solution of problem (1) we define a function $y(t) \in A$, $t \in [0, a]$, by the relation:

$$(4) \qquad y(t) = x_t, \quad t \in [0, a].$$

On the other hand, if $y(t) \in A$, $t \in [0, a]$, is a solution of problem (2) we define a function $x(\tau) \in \mathbf{R}^n$, $\tau \in [-h, a]$, by

$$x(\tau) = \begin{cases} y(0)(\tau); \ \tau \in [-h, 0) \\ y(\tau)(\tau); \ \tau \in [0, a]. \end{cases} \tag{5}$$

Assumption 2. For all functions $y(t) \in A$, $t \in [0, a]$, that are either solutions of problem (2), or come from solutions of problem (1) by way of relation (4), we assume the existence, in some definite way, of the integrals $\int_0^t G[y(s), s]\, ds \in \mathcal{B}$ and $\int_0^t G[y(s), s](\tau)\, ds \in \mathbf{R}^n$ for all $t \in [0, a]$, $\tau \in [-h, a]$, and that

$$\left(\int_0^t G[y(s), s]\, ds\right)(\tau) = \int_0^t G[y(s), s](\tau)\, ds \tag{6}$$

holds for all $\tau \in [-h, a]$, $t \in [0, a]$.

From here on we take Assumptions 1 *and* 2 *to be valid. Later on we will specify a case in which they can actually be deduced.*

Before going to our main result we give a pair of auxiliary propositions.

PROPOSITION 1. *Let y be a solution of problem* (2), *then*:

$$y(t)(\tau) = y(\tau)(\tau); \ \tau \in [-0, t]$$
$$y(t)(\tau) = y(t)(t); \ \ \tau \in [t, a]$$

for all $t \in [0, a]$.

Proof. Take a fixed $t \in [0, a]$.

a) If $\tau \in [-0, t]$, then since y is a solution of (2) we have, using (6), that

$$\begin{aligned} y(t)(\tau) &= y(0)(\tau) + \left(\int_0^t G[y(s), s]\, ds\right)(\tau) = y(0)(\tau) + \int_0^t G[y(s), s](\tau)\, ds \\ &= y(0)(\tau) + \int_0^\tau G[y(s), s](\tau)\, ds = y(0)(\tau) + \left(\int_0^\tau G[y(s), s]\, ds\right)(\tau) \\ &= y(\tau)(\tau) \end{aligned}$$

since $\int_\tau^t G[y(s), s](\tau)\, ds = 0$ as is seen from (3).

b) If $\tau \in [t, a]$, then as above

$$y(t)(\tau) = y(0)(\tau) + \int_0^t G[y(s), s](\tau)\, ds = y(0)(t) + \int_0^t G[y(s), s](t)\, ds = y(t)(t)$$

again using (3). Q.E.D.

PROPOSITION 2. *If either x is a solution of problem* (1) *and y is defined by way of* (4), *or y is a solution of problem* (2) *and x is defined by way of* (5) *then*:

$$y(t)_t = x_t$$

for all $t \in [0, a]$.

Proof. The first instance follows directly from (4) since $(x_t)_t = x_t$. The second instance is deduced immediately from (5) and Proposition 1. In fact, Proposition

1 allows one to define x starting from y in such a way that $x_t = y(t)$ and (5) is the specific description of x. Q.E.D.

We now state our main result.

THEOREM 1. *If x is a solution of problem* (1) *then y defined by* (4) *is a solution of problem* (2) *and conversely, if y is a solution of problem* (2) *then x defined by* (5) *is a solution of problem* (1).

Proof. Suppose x is a solution of (1), then for all $t \in [0, a]$:

i) if $\tau \in [0, t]$ we have

$$y(t)(\tau) = x_t(\tau) = x(\tau) = x(0) + \int_0^\tau f(x_s, s)\, ds = y(0)(0) + \int_0^\tau G[y(s), s](\tau)\, ds$$
$$= y(0)(\tau) + \int_0^t G[y(s), s](\tau)\, ds = y(0)(\tau) + \left(\int_0^t G[y(s), s]\, ds\right)(\tau)$$

using (3), Proposition 2 and Assumption 2.

ii) Similarly if $\tau \in [t, a]$ we have:

$$y(t)(\tau) = x_t(\tau) = x(t) = x(0) + \int_0^t f(x_s, s)\, ds = y(0)(\tau) + \int_0^t G[y(s), s](t)\, ds$$
$$= y(0)(\tau) + \int_0^t G[y(s), s](\tau)\, ds = y(0)(\tau) + \left(\int_0^t G[y(s), s]\, ds\right)(\tau)$$

iii) if $\tau \in [-h, 0]$ then the desired relation follows directly from (4).

From i), ii) and iii) together there follows that

$$y(t) = y(0) + \int_0^t G[y(s), s]\, ds$$

for all $t \in [0, a]$, that is, y is a solution of problem (2).

Suppose now that y is a solution of problem (2) then for all $t \in [0, a]$:

$$x(t) = y(t)(t) = y(0)(t) + \left(\int_0^t G[y(s), s]\, ds\right)(t) = y(0)(0) + \int_0^t G[y(s), s](t)\, ds$$
$$= x(0) + \int_0^t f(y(s)_s, s)\, ds = x(0) + \int_0^t f(x_s, s)\, ds,$$

that is, x is a solution of (1). Q.E.D.

3. A case where assumptions 1 and 2 hold

Consider the space $L_1[-h, a, \mathbf{R}^n]$ of classes of Lebesgue integrable functions from $[-h, a]$ into $\mathbf{R}^n$. Consider a selector function φ from L_1 into the set of functions defined on $[-h, a]$ with values also in $\mathbf{R}^n$ which associates with each class in L_1 a representative function in the following way: If a class contains a right continuous function (which is obviously unique), let this function be the representative. In all other cases choose any function from the class as a representative. Let us make $\varphi(L_1)$ into a vector space defining the operations as follows: if $\tilde{f}_1, \tilde{f}_2 \in L_1$, $\alpha \in \mathbf{R}$, then $\varphi(\tilde{f}_1) + \varphi(\tilde{f}_2) = \varphi(\tilde{f}_1 + \tilde{f}_2)$ and $\alpha\varphi(\tilde{f}_1) = \varphi(\alpha\tilde{f}_1)$. Obviously, these operations concide with the respective pointwise operations on $[-h, a]$ for right continuous functions. Using the integral norm as in L_1 we see that $\varphi(L_1)$ can be made into a Banach space.

THEOREM 2. *Let $A \subset \varphi(L_1)$ be such that $x \in A$ implies that x is right continuous in $[-h, 0)$, continuous in $[0, a]$ and that $f(x_s, s)$ is defined and piecewise continuous for $s \in [0, a]$ (i.e. has a finite number of discontinuities only, all of the first kind). Then assumptions* 1 *and* 2 *are fulfilled using Riemann integrals.*

Proof. Assumption 1 holds by construction of $\varphi(L_1) = \mathcal{B}$. If $x(t)$ is a solution of (1) and $y(t)$ is defined by (4) then $G[y(s), s](\tau) = f(x_s, s)$ for $\tau \in [s, a]$ so that $G[y(s), s](\tau)$ is a piecewise continuous function of $s \in [0, a]$ for any fixed $\tau \in [-h, a]$ and the function $s \mapsto G[y(s), s]$ is also piecewise continuous in $[0, a]$. This implies the existence, in the Riemann sense, of both integrals in Assumption 2 when $y(t)$ is defined from a solution of (1). Now let $y(t) \in A$ be a solution of (2), i.e., $y(t) = y(0) + \int_0^t G[y(s), s]\, ds$ where the integral is taken in the Riemann sense. Approximating it by means of Riemann sums and using the properties of A we easily get $y(t)(\tau) = y(t)(t)$ for $t \le \tau \le a$ and $[y(t) - y(\tau)](\tau) = [\int_\tau^t G[y(s), s]\, ds](\tau) = 0$ for $0 \le \tau \le t \le a$. Thus $x \in A$ defined by (5) satisfies $x_t = y(t)$ for $t \in [0, a]$ and $G[y(s), s](s) = f(x_s, s)$ is piecewise continuous which implies that $\int_0^t G[y(s), s](\tau)\, ds$ exists in the Riemann sense for $t \in [0, a]$, $\tau \in [-h, a]$.

Finally, relation (6) can be proved as follows:

For some fixed $t \in [0, a]$ let Δ_n, $n = 1, 2, 3, \cdots$, be a sequence of partitions of the interval $[0, t]$ with norm $|\Delta_n| \to 0$ as $n \to \infty$, then

$$\int_0^t G[y(s), s]\, ds = \lim_{n\to\infty} \sum_{\Delta_n} G[y(s_i), s_i]\Delta s_i,$$

that is

$$\int_{-h}^a \Big| \int_0^t G[y(s), s]\, ds(\tau) - \sum_{\Delta_n} G[y(s_i), s_i]\Delta s_i(\tau) \Big|\, d\tau \to 0$$

as $n \to \infty$. This implies the existence of a subsequence of Δ_n, which we again denote Δ_n, such that

$$\int_0^t G[y(s), s]\, ds(\tau) - \sum_{\Delta_n} G[y(s_i), s_i]\Delta s_i(\tau) \to 0$$

as $n \to \infty$ almost everywhere for $\tau \in [-h, a]$. But clearly

$$\sum_{\Delta_n} G[y(s_i), s_i]\Delta s_i(\tau) = \sum_{\Delta_n} G[y(s_i), s_i](\tau)\Delta s_i$$

and

$$\sum_{\Delta_n} G[y(s_i), s_i](\tau)\Delta s_i \to \int_0^t G[y(s), s](\tau)\, ds$$

when $n \to \infty$. Therefore (6) holds almost everywhere for $\tau \in [-h, a]$. On the other hand, it follows from (3) that

$$\int_0^t G[y(s), s](\tau)\, ds = \begin{cases} 0; & \tau \in [-h, 0] \\ \int_0^\tau G[y(s), s](t)\, ds; & \tau \in [0, t] \\ \int_0^t G[y(s), s](t)\, ds; & \tau \in [t, a] \end{cases}$$

which proves that $\int_0^t G[y(s), s](\tau)\, ds$ is a continuous function of $\tau \in [-h, a]$, for fixed t, therefore by the choice of functions for $\varphi(L_1)$ we have that $(\int_0^t G[y(s), s] ds)(\tau)$ is continuous and (6) holds for all $\tau \in [-h, a]$. Q.E.D.

4. Example

Consider the equation

$$(7) \qquad \begin{cases} \dfrac{dx(t)}{dt} = x(t-1) \\ x(t) = 0 \quad \text{for} \quad t \in [-1, -\frac{1}{2}] \\ x(t) = 1 \quad \text{for} \quad t \in [-\frac{1}{2}, 0] \end{cases}$$

and its solution in $[-1, 2]$. Take for $\mathcal{B}$ the space $\varphi(L_1)$ as in section 3, for A the set of right continuous functions on $[-1, 2]$ which are piecewise continuous in $[-1, 0]$ and continuous in $[0, 2]$. Obviously the solution of problem (7) is an element of A and (7) is equivalent, by Theorem 1, to equation (2) with $G[y, t](\tau) = 0$ for $\tau \in [-1, t)$ and $G[y, t](\tau) = y(t-1)$ for $\tau \in [t, 2]$ where $y \in A$ and $t \in [0, 2]$.

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