SOME GENERALIZATIONS OF THE HAUSDORFF SEPARATION AXIOM

POR A. GARCÍA-MÁYNEZ

1. Introduction

In the last few years, a considerable attention has been paid to separation axioms in topological spaces lying between T_1 (the Frèchet axiom) and T_2 (the Hausdorff axiom). The KC-axiom: *every compact subset is closed* is perhaps the most useful. In [3], A. Wilansky proposes a definition of k-space for not necessarily Hausdorff spaces and proves that a KC-space is a k-space if and only if its one point compactification is a KC-space. We modify slightly Wilansky's definition of k-space, but both definitions turn out to be equivalent in KC-spaces. We propose also a definition of paracompactness for not necessarily regular spaces which of course reduces to the classic definition under the assumption of regularity. Finally, we introduce the concept of KS-space and by a series of examples we exhibit connections between locally compact, KC, KS, T_2 and k-spaces. Our two main results, (2.6) and (2.9), describe, respectively, a sufficient condition for a T_1-space to be Hausdorff and a necessary and sufficient condition for a KC, KS-space to have a KS one point compactification.

2. Basic definitions

A subset A of a topological space is *compactly closed* if for each closed compact set K in the space, $A \cap K$ is compact. X is a *k-space* if every compactly closed subset is closed. A compact space X is *shrinkable* if for each finite open cover $\{V_1, V_2, \cdots, V_n\}$ of X there exists a compact cover $\{K_1, K_2, \cdots, K_n\}$ of X such that $K_i \subset V_i$ for each i. A topological space is KC (resp. KS) if every compact (resp. compact closed) subset is closed (resp. shrinkable). X is a *US-space* if every convergent sequence in X has exactly one limit. X is *paracompact* if every open cover of X has a closed locally finite refinement. X is *locally compact* (resp. *locally paracompact*) if every point of X has a compact (resp. a closed paracompact) neighborhood.

As a consequence of these definitions, we obtain the following results. (2.1), (2.2) and the first part of (2.3) can be proved easily. We give a reference for the remaining statements.

(2.1) Every KC-shrinkable space is paracompact. Conversely, every compact and paracompact space is shrinkable.

(2.2) Every compact T_2-space is paracompact.

(2.3) $T_2 \Rightarrow KC \;\&\; KS \Rightarrow KC \Rightarrow US \Rightarrow T_1$ and no converse implication holds. (See examples (3.2), (3.4), (3.5) and (3.6) below).

(2.4) A KC-space which is locally compact or first countable is a k-space. (See, for instance, [2], Chap. VII, Th. 13).

(2.5) The one point compactification of a KC-space is a US-space. It is a KC-space if and only if X is a k-space. (See theorems 3 and 5 in [3]).

We prove now our first result.

(2.6) Every locally paracompact T_1-space is Hausdorff.

Proof. Let $a, b \in X$, $a \neq b$. Let H be a closed paracompact neighborhood of a. If $b \notin H$, then int (H) and $X \sim H$ are disjoint open sets about a and b, respectively, and there is nothing else to prove. So assume $b \in H$. For each $x \in H$, $a \neq x \neq b$, let V_x be an open set about x disjoint from $\{a, b\}$. Let V_a, V_b be open sets such that $a \in V_a$, $b \in V_b$, $a \notin V_b$ and $b \notin V_a$. The cover $\mathcal{G} = \{V_x \cap H \mid x \in H\}$ of H has a closed locally finite refinement $\{W_\alpha \mid \alpha \in I\}$. For each $\alpha \in I$ select $x(\alpha) \in H$ such that $W_\alpha \subset V_{x(\alpha)}$ and let $H_x = \bigcup_{x(\alpha)=x} W_\alpha$. Then $\{H_x \mid x \in H\}$ is also a closed locally finite refinement of $\mathcal{G}$. Finally,

$$R = \text{int}\,(H) \cap (X \sim \textstyle\bigcup_{x\neq a} H_x) \qquad \text{and} \qquad S = X \sim \textstyle\bigcup_{x\neq b} H_x$$

are disjoint open sets about a and b, respectively, and the proof is complete.

(2.6.1) COROLLARY. *Even paracompact T_1-space X is normal and every locally paracompact T_1-space is completely regular.*

Proof. Assume first X is paracompact and T_1. Since every paracompact space is locally paracompact, X is Hausdorff. To prove X is regular, let $a \in X$ and let H be a closed set such that $a \notin H$. For each $x \in H$, there is an open set V_x such that $x \in V_x$ and $a \notin V_x^-$. Since X is paracompact and

$$\mathcal{H} = \{X \sim H\} \cup \{V_x \mid x \in H\}$$

is an open cover of X, $\mathcal{H}$ has an indexed closed locally finite refinement $\{T\} \cup \{T_x \mid x \in H\}$, where $T \subset X \sim H$ and $T_x \subset V_x$ for each $x \in H$. (See [1], Chap. VIII, Th. 1.4). Then $X \sim T$ and $X \sim \bigcup_{x\in H} T_x$ are disjoint open sets containing H and a, respectively.

To prove that X is normal we reason exactly as before replacing a by a closed set A disjoint from H. Assume now that X is locally paracompact and T_1. Since every paracompact T_1-space is normal and normality implies complete regularity, every point of X has a completely regular closed neighborhood. This clearly implies that X is completely regular.

(2.6.2) COROLLARY. *Every compact shrinkable KC-space is T_2.*

Proof. This is a consequence of (2.1) and (2.6).

(2.6.3) COROLLARY. *Every locally compact, KC, KS-space is T_2.*

We exhibit now a consequence of (2.5) and (2.6.2).

(2.7) A *KC*, k-space X is locally compact and Hausdorff if and only if the one point compactification X^+ of X is shrinkable.

Proof. If X is locally compact and T_2, then X^+ is T_2 and hence shrinkable by (2.2) and (2.1). If X^+ is shrinkable, then by (2.6.2) and (2.5), X^+ is Hausdorff. Hence X is Hausdorff and locally compact.

We will find the next lemma very useful in proving our second main result (2.9).

(2.8) LEMMA. *A subset* $C \subset X$ *is compactly closed if and only if* $C \cup \{\infty\}$ *is compact in* X^+.

Proof. Assume first C is compactly closed and let $\{W_\alpha \mid \alpha \in I\}$ be a cover of $C \cup \{\infty\}$ with open sets in X^+. Choose $\alpha_0 \in I$ such that $\infty \in W_{\alpha_0}$. Then $X^+ \sim W_{\alpha_0}$ is a closed compact subset of X. Since C is compactly closed, $C \cap (X^+ \sim W_{\alpha_0})$ is compact and $\{X \cap W_\alpha \mid \alpha \in I\}$ is a cover of this set with open sets in X. Therefore there exist $\alpha_1, \cdots, \alpha_n \in I$ such that $C \cap (X^+ \sim W_{\alpha_0}) \subset W_{\alpha_1} \cup \cdots \cup W_{\alpha_n}$. Therefore $\{W_{\alpha_0}, W_{\alpha_1}, \cdots, W_{\alpha_n}\}$ is a subfamily of $\{W_\alpha \mid \alpha \in I\}$ covering $C \cup \{\infty\}$ and this set is compact. Conversely, if $C \cup \{\infty\}$ is a compact subset of X^+ and K is a closed compact subset of X, then K is also closed in X^+ by the definition of the topology in X^+. Therefore $K \cap (C \cup \{\infty\}) = K \cap C$ is compact.

(2.9) THEOREM. *Let X be a KC, KS-space. Then X^+ is a KS-space if and only if the following condition holds*:
$*$) *If $K \subset V \subset X$, where K is compact and V is open, then there exist $S_1, S_2 \in 2^X$ such that $K \subset S_1 \subset S_2 \subset V$, S_2 is compact, and $X \sim S_1$ is compactly closed.*

Proof. Assume condition $*$) holds. Since every compact shrinkable space is a KS-space, it suffices to prove that X^+ is shrinkable. Let $V_1, V_2, \cdots, V_n$ be open sets in X^+ such that $X^+ = V_1 \cup \cdots \cup V_n$. We may assume, without loss of generality, that $\infty \in V_n$ but $\infty \notin V_i$ for $i < n$. The set $L = X^+ \sim V_n$ is compact and is contained in $V_1 \cup \cdots \cup V_{n-1}$. By the $*$) condition, there exist subsets S_1, S_2 of X such that $L \subset S_1 \subset S_2 \subset V_1 \cup \cdots \cup V_{n-1}$, where S_2 is compact and $X \sim S_1$ is compactly closed. Since S_2 is shrinkable, there exist compact sets $L_1, \cdots, L_{n-1}$ in X such that $S_2 = L_1 \cup \cdots \cup L_{n-1}$, where $L_i \subset V_i$, $i = 1, 2, \cdots, n-1$. Now, by (2.8), $L_n = X^+ \sim S_1$ is a compact subset of X^+. Then

$$X^+ = L_1 \cup L_2 \cup \cdots \cup L_n,$$

where L_i is compact in X^+ and $L_i \subset V_i$ for $i = 1, 2, \cdots, n$. Conversely, assume X^+ is shrinkable and let $K \subset V \subset X$, where K is compact and V is open. Since X^+ is shrinkable and $X^+ = (X^+ \sim K) \cup V$, there exist compact sets T_1, T_2 in X^+ such that $X^+ = T_1 \cup T_2$, $T_1 \subset X^+ \sim K$ and $T_2 \subset V$. If we define $S_1 = V \cap (X^+ \sim T_1)$, $S_2 = T_2 \cup K$, then $K \subset S_1 \subset S_2 \subset V$. Besides S_2 is compact since it is the union of two compact sets. Finally, by (2.8), $X \sim S_1$ is compactly closed since $X^+ \sim S_1 = T_1 \cup (X^+ \sim V)$ is compact.

3. Examples and counterexamples

In this section, we will prove, among other things, that the three hypotheses in corollary (2.6.3) are independent. We will say that a topological space is KF if every compact subset is finite. KF T_1-spaces are automatically KC and KS, so they are very useful to provide examples.

(3.1) *Example.* There exists a T_2, KF-space which is not a k-space.

Proof. Let Ω be the first uncountable ordinal and let $X = \Omega + 1$, i.e., X consists of all ordinals $\leq \Omega$. Define a topology τ in X as follows: $V \in \tau$ if and only if $\Omega \notin V$ or $\Omega \in V$ and for some ordinal $\alpha < \Omega$, $V \supset \{x \mid \alpha < x \leq \Omega\}$. It can be proved easily that X is T_2, KF, and non-discrete. But X is not a k-space since every subset of X is compactly closed.

(3.2) *Example.* There exists a compact shrinkable US-space which is not KC.

Proof. Let Y be the one point compactification of the space described in (3.1). By (2.5), Y is a US-space which is not KC; by (2.9), Y is shrinkable.

(3.3) THEOREM. *Every T_2, KF, k-space X is discrete.*

Proof. By (2.7) and (2.9), X is locally compact. But every locally compact T_2 KF-space is discrete.

(3.4) *Example.* There exists a KC, KS-space which is not T_2.

Proof. Let X be an uncountable set and let τ be the topology for X consisting of the empty set and all subsets of X with countable (finite or infinite) complements. It can be proved easily that every infinite subset of X has an open cover with no finite subcover, and hence X is KF. Now every non-empty open subset of X is dense in X, so X cannot be Hausdorff.

(3.5) *Example.* There exists a compact KC-space which is not shrinkable.

Proof. Let X be a T_2 first countable space which is not locally compact (for instance, the rationals in the line with the usual topology). By (2.4), X is a KC k-space and by (2.5) and (2.7), X^+ is a compact KC-space which is not shrinkable.

(3.6) *Example.* There exists a compact T_1-space which is not US.

Proof. Let X be an infinite set with the almost indiscrete topology, that is, the topology τ consisting of the empty set and all subsets of X with finite complements. (X, τ) is then a compact T_1-space such that every sequence in X which has no constant subsequence converges to each point of X, so X cannot be US.

We finish this paper with an open question:

Is every normal T_1-space locally paracompact?

CENTRO DE INVESTIGACIÓN DEL IPN

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